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ON A CONSTRUCTION CONNECTING LIE ALGEBRAS WITH GENERAL ALGEBRAS

P.Michor, W.Ruppert, K.Wegenkittl

Abstract: In this paper we introduce a general construction which associates an algebra $A(\mathfrak{L},b)$ with every pair $(\mathfrak{L},b)$, where $\mathfrak{L}$ is a Lie algebra and b is an invariant symmetric bilinear form on $\mathfrak{L}$. By virtue of this construction several well-known (associative and non-assiciative) algebras can be dealt with under a unified view. We give characterizations of those pairs $(\mathfrak{L},b)$ which generate associative algebras $A(\mathfrak{L},b)$ and of those algebras which can be represented in the form $A(\mathfrak{L},b)$.

1. Passing from Lie algebras to algebras

1.1. Definition Let $\mathfrak{L}$ be a Lie algebra over a (commutative) field k and let $b: \mathfrak{L} \times \mathfrak{L} \to k$ be an invariant (i.e. $b([X,Y],Z) = b(X,[Y,Z])$ for all $X,Y,Z \in \mathfrak{L}$) symmetric bilinear form on $\mathfrak{L}$. Then we define an algebra $A(\mathfrak{L},b)$ associated with the pair $(\mathfrak{L},b)$ as follows: as a vector space, $A(\mathfrak{L},b)$ is just the direct sum $\mathfrak{L} \oplus k$. The multiplication of $A(\mathfrak{L},b)$ is defined by the formula:

$$(X,s)(Y,t) := ([X,Y] + sY + tX,\ st + b(X,Y)).$$

Obviously, $A(\mathfrak{L},b)$ is an algebra and $(0,1)$ is its identity.

1.2. Proposition

(i) If char $k \neq 2$, then the algebra $A(\mathfrak{L},b)$ is commutative if and only if $\mathfrak{L}$ is abelian. If char $k = 2$, then $A(\mathfrak{L},b)$ is always commutative.

(ii) Suppose that char $k \neq 2$. Then $(\mathfrak{L},b)$ is isomorphic to $(\mathfrak{L}',b')$ (i.e. there is a Lie algebra isomorphism $\phi: \mathfrak{L} \to \mathfrak{L}'$ with $b(X,Y) = b(\phi(X),\phi(Y))$) if and only if $A(\mathfrak{L},b)$ is isomorphic to $A(\mathfrak{L}',b')$. For char $k = 2$ there are non-isomorphic pairs $(\mathfrak{L},b)$ and $(\mathfrak{L}',b')$ generating isomorphic algebras $A(\mathfrak{L},b)$ and $A(\mathfrak{L}',b')$.

(iii) $A(\mathfrak{L},b)$ is always flexible, i.e. we have $x(yx) = (xy)x$ for all $x,y \in A(\mathfrak{L},b)$. In particular, $A(\mathfrak{L},b)$ is always power-associative,

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i.e. $xx^2 = x^2x$ for all $x \in A(\mathfrak{L},b)$.
(iv) $A(\mathfrak{L},b)$ is always Lie admissible, i.e. the algebra $A(\mathfrak{L},b)^-$ defined on the same vector space, but with multiplication $[x,y] = xy - yx$, is a Lie algebra.
(v) $A(\mathfrak{L},b)$ is always Jordan admissible, i.e. the algebra $A(\mathfrak{L},b)^+$ defined on the same vector space, but with multiplication $x\circ y = xy + yx$, is a Jordan algebra.
(vi) We write $Ass(x,y,z)$ for the associator $x(yz) - (xy)z$ of three elements x,y,z. In $A(\mathfrak{L},b)$ we have

$$Ass((X,s),(Y,t),(Z,u)) = (\alpha_b(X,Y,Z),0),$$

where

$$\alpha_b(X,Y,Z) = -b(X,Y)Z + b(Y,Z)X + [[Z,X],Y].$$

In particular, $A(\mathfrak{L},b)$ is associative if and only if $\alpha_b(X,Y,Z) = 0$ for all $X,Y,Z \in \mathfrak{L}$.
(vii) The map α_b satisfies the identity

$$\alpha_b(X,Y,Z) + \alpha_b(Y,Z,X) + \alpha_b(Z,X,Y) = 0.$$

(viii) If char $k \neq 2,3$ and $A(\mathfrak{L},b)$ is alternative (i.e. $x(xy) = x^2y$ and $(xy)y = xy^2$), then it is associative.
Proof Assertion (i) follows from the identity $(X,s)(Y,t) - (Y,t)(X,s) = (2[X,Y],0)$.
(ii) Obviously, any isomorphism ϕ: $(\mathfrak{L},b) \to (\mathfrak{L}',b')$ induces an isomorphism $A(\mathfrak{L},b) \to A(\mathfrak{L}',b')$, $(X,s) \to (\phi(X),s)$. Suppose now that char $k \neq 2$ and that ψ: $A(\mathfrak{L},b) \to A(\mathfrak{L}',b')$ is an isomorphism. Let $X \in \mathfrak{L}\backslash\{0\}$ and write $\psi(X,s) = (X',s')$. Since ψ preserves units, $X' \neq 0$. From $\psi((X,0)^2) = (\psi(X,0))^2$ we conclude that $2s'X' = 0$ and $b(X,X) = s^2 + b'(X',X')$. Thus we get the isomorphism we need by defining ψ^*: $\mathfrak{L} \to \mathfrak{L}'$, $\psi^*(X) = X'$ if $X \neq 0$ and $\psi^*(0) = 0$.
To construct a counterexample in case char $k = 2$, let $k = \mathbb{Z}/2\mathbb{Z}$ and choose a basis for k^2, say $\langle X,Y \rangle$. Then we take $\mathfrak{L}$ to be k^2 with trivial Lie structure and $b = 0$; for $\mathfrak{L}'$ we take k^2 with the Lie structure defined by $[X,Y] = X + Y$; b' is defined by stipulating $b'(X,X) = b'(Y,Y) = b'(X,Y) = 1$. Then $\mathfrak{L}$ is not isomorphic to $\mathfrak{L}'$, but $A(\mathfrak{L},b) \cong A(\mathfrak{L}',b')$ via the morphism ψ: $A(\mathfrak{L},b) \to A(\mathfrak{L}',b')$ given by $\psi(X,0) = (X,1)$, $\psi(Y,0) = (Y,1)$, $\psi(X,1) = (X,0)$ and $\psi(Y,1) = (Y,0)$.
The proof of assertions (iii) - (vii) rests on simple calculations and is therefore left to the reader.
(viii) By Bourbaki [2], p.612, an algebra is alternative if and only if its associator is skew-symmetric. Thus if $A(\mathfrak{L},b)$ is alternative, then α_b is skew-symmetric and hence (vii) takes the form $3\alpha_b(X,Y,Z) = 0$, so (vi) implies the assertion. □
Remark Note that in the proof of (vii) and (viii) we did not use the

assumption that b is symmetric.

If we require b only to be bilinear and char $k \neq 2$, then invariance and symmetry of b are equivalent to the flexibility of $A(\mathfrak{L},b)$.

1.3. Notation We write $\varkappa$ for the Cartan-Killing form, $\varkappa(X,Y) = \mathrm{trace}(\mathrm{ad}X \cdot \mathrm{ad}Y)$. The set $\{ X \in \mathfrak{L}: b(X,\mathfrak{L}) = 0 \}$ is denoted by $\mathfrak{L}^{\perp}$, and $\{ X \in \mathfrak{L}: b(X,Y) = 0 \}$ by $Y^{\perp}$.

Throughout the rest of this section we always assume that char $k = 0$ and that $\mathfrak{L}$ is finite dimensional.

1.4. Lemma Assume that $A(\mathfrak{L},b)$ is associative. Then

(i) $\varkappa(X,Y) = (n-1)b(X,Y)$, where $n = \dim \mathfrak{L}$.

(ii) every commutative subalgebra $\mathfrak{C}$ of $\mathfrak{L}$ with $\dim \mathfrak{C} > 1$ lies in the ideal $\mathfrak{L}^{\perp}$.

(iii) $[\mathfrak{L}^{\perp},[\mathfrak{L},\mathfrak{L}]] = 0$.

(iv) $(\mathrm{ad}U)^2 V = b(U,U)V$ for all $U \in \mathfrak{L}$, $V \in \mathfrak{L}^{\perp}$.

Proof We infer from 1.2.(vi) that

(*) $\quad [X,[Y,Z]] = b(X,Y)Z - b(Z,X)Y \qquad$ for all $X,Y,Z \in \mathfrak{L}$.

Thus $\varkappa(X,Y) = \mathrm{Trace}(\mathrm{ad}X \cdot \mathrm{ad}Y) = \mathrm{Trace}(b(X,Y)\mathrm{id} - b(X,.)Y) = nb(X,Y) - b(X,Y) = (n-1)b(X,Y)$, which establishes (i). If in (*) we put $X = Y = U$, $Z = V$, then we get (iv).

(ii) Let A,B be two linearly independent elements of $\mathfrak{C}$. Then by (*) we have for any $X \in \mathfrak{L}$

$$0 = [X,[A,B]] = b(X,A)B - b(B,X)A$$

and hence $b(X,A) = b(X,B) = 0$; that is, $A,B \in \mathfrak{L}^{\perp}$. Thus $\mathfrak{C} \subseteq \mathfrak{L}^{\perp}$.

(iii) The right hand side of (*) vanishes whenever $X \in \mathfrak{L}^{\perp}$, thus $[\mathfrak{L}^{\perp},[\mathfrak{L},\mathfrak{L}]] = 0$. □

1.5. Lemmma Suppose that $A(\mathfrak{L},b)$ is associative. Then the following assertions hold:

(i) $\mathfrak{L}$ is either solvable or simple of rank 1.

(ii) If $0 \neq \mathfrak{L}^{\perp} \neq \mathfrak{L}$, then $\mathfrak{L}^{\perp} = [\mathfrak{L},\mathfrak{L}] = [\mathfrak{L},[\mathfrak{L},\mathfrak{L}]]$ and $\mathfrak{L}^{\perp}$ is commutative. Moreover, $X \in \mathfrak{L}^{\perp}$ if and only if $b(X,X) = 0$.

(iii) If $\mathfrak{L}$ is solvable, then $\dim \mathfrak{L}/\mathfrak{L}^{\perp} \leq 1$.

Proof The assertions are obvious for $\dim \mathfrak{L} \leq 1$, so let us assume that $n = \dim \mathfrak{L} > 1$. Then we have $b = \frac{1}{n-1} \varkappa$, by 1.4.(i), and hence $\mathfrak{L}^{\perp} = 0$ if and only if $\mathfrak{L}$ is semisimple.

(i) If $\mathfrak{L}$ is semisimple, then by 1.4.(ii) every Cartan-subalgebra of $\mathfrak{L}$ has dimension 1, so $\mathfrak{L}$ is actually simple of rank 1. Assume now that $\mathfrak{L}$ is not semisimple. Then by our assumption above, $\mathfrak{L}^{\perp} \neq 0$.

Suppose that $\mathfrak{S}$ is a semisimple subalgebra of $\mathfrak{L}$. Since $\mathfrak{S} = [\mathfrak{S},\mathfrak{S}] \subseteq [\mathfrak{L},\mathfrak{L}]$, 1.4.(iii) yields that $[\mathfrak{L}^{\perp},\mathfrak{S}] = 0$. Now any non-zero $Y \in \mathfrak{L}^{\perp}$ together with any linearly independent $S \in \mathfrak{S}$ generates a two-dimensional commutative Lie subalgebra $\mathfrak{C}$ of $\mathfrak{L}$, which by 1.4.(ii) is contained in $\mathfrak{L}^{\perp}$, so $[S,\mathfrak{S}] \subseteq [\mathfrak{L}^{\perp},\mathfrak{S}] = 0$, a contradiction. This establishes (i).

(ii) Assume that $0 \neq Z \in \mathfrak{L}^{\perp}$. Then formula (*) of the proof of 1.4. implies that $[X,[Y,Z]] = b(X,Y)Z$ for all $X,Y \in \mathfrak{L}$. By 1.4.(iii) $[Y,Z] = 0$, and hence $b(X,Y) = 0$, whenever $Y \in [\mathfrak{L},\mathfrak{L}]$, $X \in \mathfrak{L}$. Thus $[\mathfrak{L},\mathfrak{L}] \subseteq \mathfrak{L}^{\perp}$. Conversely, let $X,Y \in \mathfrak{L}$ with $b(X,Y) \neq 0$. Then $Z = b(X,Y)^{-1}[X,[Y,Z]] \in [\mathfrak{L},[\mathfrak{L},\mathfrak{L}]]$. Thus $[\mathfrak{L},\mathfrak{L}] \subseteq \mathfrak{L}^{\perp} \subseteq [\mathfrak{L},[\mathfrak{L},\mathfrak{L}]] \subseteq [\mathfrak{L},\mathfrak{L}]$; the commutativity of $\mathfrak{L}^{\perp}$ follows from 1.4.(iii).

To show the second part of (ii), suppose that $b(X,Y) \neq 0$, but $b(X,X) = 0$. Then $[X,[X,Y]] = -b(Y,X)X$, hence $X \in [\mathfrak{L},\mathfrak{L}] = \mathfrak{L}^{\perp}$, a contradiction.

(iii) Suppose that $\mathfrak{L}$ is solvable and that there are elements $X,Y \in \mathfrak{L}$ such that $X + \mathfrak{L}$ and $Y + \mathfrak{L}$ are linearly independent in $\mathfrak{L}/\mathfrak{L}^{\perp}$. Then we get

$$b(X,X)Y - b(Y,X)X = [X,[X,Y]] \in [\mathfrak{L},\mathfrak{L}] = \mathfrak{L}^{\perp}$$

Thus $b(X,X) = 0$ and therefore, by (ii), $X \in \mathfrak{L}^{\perp}$, a contradiction. □

<u>1.6. Theorem</u> Suppose that char $k = 0$ and $\mathfrak{L}$ is finite-dimensional. Then $A(\mathfrak{L},b)$ is associative if and only if one of the following assertions hold:

(i) $\mathfrak{L}$ is a simple Lie algebra of rank 1 and $b = \frac{1}{n-1}\varkappa$, where $n = \dim \mathfrak{L}$.

(ii) $\mathfrak{L}$ is nilpotent of step 2 (i.e. $[\mathfrak{L},[\mathfrak{L},\mathfrak{L}]] = 0$) and $b = 0$.

(iii) $\dim \mathfrak{L} \leq 1$ and b is arbitrary.

(iv) $\mathfrak{L}^{\perp} = [\mathfrak{L},\mathfrak{L}]$ and there is an element $X \in \mathfrak{L}$ such that $\mathfrak{L}$ is the split extension $\mathfrak{L}^{\perp} \odot kX$ of $\mathfrak{L}^{\perp}$ with the one-dimensional subspace kX. Moreover, $\mathfrak{L}^{\perp}$ is commutative and $(\mathrm{ad}X)^2 Y = b(X,X)Y$ for all $Y \in [\mathfrak{L},\mathfrak{L}]$; $b = \frac{1}{n-1}\varkappa$.

<u>Proof:</u> Suppose first that $A(\mathfrak{L},b)$ is associative and that $\dim \mathfrak{L} > 1$. If $\mathfrak{L}^{\perp} = 0$, then assertion (i) holds, by 1.4.(i) and 1.5.(i). If $\mathfrak{L}^{\perp} \neq 0$ then, by 1.4.(iii),(iv) and 1.5.(ii),(iii) either $\mathfrak{L}^{\perp} = \mathfrak{L}$ (which implies (ii)) of $\dim \mathfrak{L}/\mathfrak{L}^{\perp} = 1$ and hence (iv) holds.

Conversely, it is immediate that each of the assertions (ii) - (iv) implies that the condition in 1.2.(vi), $\alpha_b = 0$, is satisfied, so that $A(\mathfrak{L},b)$ is associative (Note that in case (iv) every product $[A,[B,C]]$ vanishes unless A and B, or A and C, are contained in $kX\setminus\{0\}$. In the case of (i), we first remark that we may assume $k =$

$\mathbb{C}$, since the condition of 1.2.(vi) naturally extends to the complexification $(\mathfrak{L}\otimes\mathbb{C}, b_{\mathbb{C}})$, and $A(\mathfrak{L},b)$ can be considered as a subalgebra of the algebra $A(\mathfrak{L}\otimes\mathbb{C}, b_{\mathbb{C}})$, taken as an algebra over k (cf. Bourbaki [3], p.21). Thus we are left to show that $A(\mathfrak{sl}(2,\mathbb{C}),\frac{1}{2}\varkappa)$ is associative; this will be done in example 2.5. of the next section.□

2. Examples

2.1. The trivial cases:

If dim $\mathfrak{L} = 0$, then $b = 0$ and $A(0,0) \cong k$.

If dim $\mathfrak{L} = 1$, then $\mathfrak{L} \cong k$. Let $b(X,Y) := \alpha XY$ for some $\alpha \in k$. Then $A(\mathfrak{L},b) \cong {}^{k[X]}/\langle X^2 - \alpha\rangle$ (the isomorphism is given by $(1,0) \to X$).

If $k = \mathbb{R}$, we get for

(i) $\alpha < 0$ the algebra of complex numbers.

(ii) $\alpha = 0$ the commutative associative algebra generated by 1 and δ with $\delta^2 = 0$, sometimes called the algebra of dual numbers.

(iii) $\alpha > 0$ the commutative associative algebra generated by 1 and ε with $\varepsilon^2 = 1$.

These are all quadratic algebras over $\mathbb{R}$ in the sense of Bourbaki.

2.2. Let $\mathfrak{L} = \mathfrak{so}(3,\mathbb{R})$ and let $b = \varkappa$, its Cartan-Killing form. Let $\mathbb{E}^3$ be the oriented Euclidean 3-space with inner product $\langle .,.\rangle$ and normed determinant function det. Define a cross product "x" in $\mathbb{E}^3$ by stipulating $\langle X\times Y, Z\rangle = \det(X,Y,Z)$. Then $\mathfrak{so}(3,\mathbb{R})$ is isomorphic to $(\mathbb{E}^3,\times)$ in such way that $[X,Y] = X\times Y$ and $\varkappa(X,Y) = -2\langle X,Y\rangle$. To see this, put

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

and notice that $[X_i, X_{i+1}] = X_{i+2}$, where we compute the indices modulo 3. The product formula in $A(\mathfrak{so}(3,\mathbb{R}),\frac{1}{2}\varkappa)$ is then

$$(X,s)(Y,t) = (X\times Y + sY + tX,\ st - \langle X,Y\rangle),$$

which yields exactly the algebra $\mathbb{H}$ of quaternions: choose a positively oriented orthonormal basis i,j,k in $\mathbb{E}^3$ and check that the multiplication table is:

	(i,0)	(j,0)	(k,0)
(i,0)	(0,−1)	(k,0)	(−j,0)
(j,0)	(−k,0)	(0,−1)	(i,0)
(k,0)	(j,0)	(−i,0)	(0,−1)

Then obviously in the algebra $A(\mathfrak{so}(3,\mathbb{R}),\alpha\varkappa)$, $\alpha \in \mathbb{R}$, we get the multiplication table:

	$(i,0)$	$(j,0)$	$(k,0)$
$(i,0)$	$(0,-2\alpha)$	$(k,0)$	$(-j,0)$
$(j,0)$	$(-k,0)$	$(0,-2\alpha)$	$(i,0)$
$(k,0)$	$(j,0)$	$(-i,0)$	$(0,-2\alpha)$

This is associative if and only if $\alpha = \frac{1}{2}$.

2.3. Let $\mathfrak{L} = so(3,\mathbb{C})$ and let $b = \varkappa_{\mathbb{C}}$ be again its (complex) Cartan-Killing form. Then $\mathfrak{L} \cong \mathbb{C}^3$, $[X,Y] = X\times_{\mathbb{C}}Y$ (the "complexified vector product" with the same coordinate formula as the real one), and $\varkappa_{\mathbb{C}}(X,Y) = -2\ \Sigma_{i=1}^{3} X^iY^i$. As we just take the product formula of 2.2. with complex scalars, we get $A(so(3,\mathbb{C}),\frac{1}{2}\ \varkappa_{\mathbb{C}}) \cong \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ (cf. 2.5.). Likewise the algebra $A(so(3,\mathbb{C}),\alpha\varkappa_{\mathbb{C}})$ for $\alpha \in \mathbb{C}$ is given by the second multiplication table of 2.2., but now over $\mathbb{C}$. $A(so(3,\mathbb{C}),\alpha\varkappa_{\mathbb{C}})$ is associative if and only if $\alpha = \frac{1}{2}$.

2.4. Let $\mathfrak{L} = sl(2,\mathbb{R})$ and let $b = \varkappa$, the Cartan-Killing form. Then $\mathfrak{L}$ is the Lie algebra of traceless 2×2 - matrices. Choose the following basis of $\mathfrak{L}$:

$$X_0 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad X_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad X_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then $[X_0,X_1] = X_2$, $[X_1,X_2] = -X_0$, $[X_2,X_0] = X_1$, and $\frac{1}{2}\varkappa(\Sigma x^iX_i,\Sigma y^iY_i) = -x^0y^0 + x^1y^1 + x^2y^2$. Now let $\mathbb{L}^3$ be the Lorentzian 3-space with inner product $\langle .,.\rangle_{\mathbb{L}^3}$, with signature +,-,-. Define the Lorentzian vector product $\times_L$ on $\mathbb{L}^3$ by $\langle X\times_L Y,Z\rangle = -\det(X,Y,Z)$. For the standard basis e_0,e_1,e_2 on $\mathbb{L}^3$ we get

$$e_0\times_L e_1 = e_2 \qquad e_1\times_L e_2 = -e_0 \qquad e_2\times_L e_0 = e_1.$$

Thus $(sl(2,\mathbb{R}),[.,.],\frac{1}{2}\varkappa)$ is isomorphic to $(\mathbb{L}^3,\times_L,-\langle .,.\rangle_L)$ and the multiplication formula of 1.1. becomes on $\mathbb{L}^3\times\mathbb{R}$:

$$(X,s)(Y,t) = (X\times_L Y + sY + tX,\ st - \langle X,Y\rangle_L)$$

This gives an associative algebra, sometimes called the algebra of pseudoquaternions (see Yaglom,[11]): check the multiplication table

	$(e_0,0)$	$(e_1,0)$	$(e_2,0)$
$(e_0,0)$	$(0,-1)$	$(e_2,0)$	$(-e_1,0)$
$(e_1,0)$	$(-e_2,0)$	$(0,1)$	$(-e_0,0)$
$(e_2,0)$	$(e_1,0)$	$(e_0,0)$	$(0,1)$

But in fact this algebra is isomorphic to the full algebra of 2×2 - matrices:

$$(0,1) \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma_0 \qquad (e_0,0) \to \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = i\sigma_2$$

$$(e_1,0) \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 \qquad (e_2,0) \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3$$

gives the same multiplication table for the matrix multiplication. Here the σ_i are the Pauli matrices, very dear to physicists. Thus $A(sl(2,\mathbb{R}),\frac{1}{2}\varkappa) \cong L(\mathbb{R}^2,\mathbb{R}^2)$, the algebra of all 2×2 - matrices.

$A(\mathfrak{sl}(2,\mathbb{R}),\alpha\varkappa)$ gives the same multiplication table, but with $(0,-2\alpha)$, $(0,2\alpha)$, $(0,2\alpha)$ in the main diagonal, which is associative if and only if $\alpha = \frac{1}{2}$.

2.5. Let $\mathfrak{L} = \mathfrak{sl}(2,\mathbb{C})$, $\varkappa_{\mathbb{C}}$ its Cartan-Killing form. Then we can apply the discussion of 2.4. with complex scalars and conclude that $A(\mathfrak{sl}(2,\mathbb{C}),\frac{1}{2}\varkappa_{\mathbb{C}}) \cong A(\mathfrak{sl}(2,\mathbb{R}),\frac{1}{2}\varkappa) \otimes_{\mathbb{R}} \mathbb{C}$ equals the algebra of complex 2x2 matrices. This is well known to physicists via the formula $\sigma_i\sigma_j = \delta_{ij} + \sqrt{-1}\,\varepsilon_{ijk}\sigma_k$ for the Pauli matrices.

2.6. Let $\mathfrak{L}$ be the real 2-dimensional Lie algebra with generators X,Y satisfying $[X,Y] = X$ (This is the Lie algebra of the "ax+b" - group). Then the Cartan-Killing form $\varkappa$ is given by $\varkappa(X,\mathfrak{L}) = 0$ and $\varkappa(Y,Y) = 1$. This gives an associative algebra $A(\mathfrak{L},\varkappa)$ which is isomorphic to the real algebra of all upper triangular 2x2 matrices:

$$(0,1) \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (X,0) \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad (Y,0) \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

gives the correct multiplication table.

2.7. The algebra of Cayley numbers is not of the form $A(\mathfrak{L},b)$ since it is alternative, but not associative (cf. 1.2.(viii)). But it can be represented in a similar form: we use the isomorphism $\mathfrak{so}(3,\mathbb{C}) \cong (\mathbb{C}^3,\times_{\mathbb{C}})$ of 2.3. and consider the usual hermitian inner product $(.,.)$ on $\mathbb{C}^3$. Then $\mathbb{C}^3 \times \mathbb{C}$, with multiplication

$$(X,s)(Y,t) := (\overline{X \times_{\mathbb{C}} Y} + sY + \bar{t}X,\ st - (X,Y))$$

is the algebra of Cayley numbers (see Greub, [4]).
If char k = 2, the Cayley numbers are associative.

2.8 Let $\mathfrak{L}$ be a nilpotent Lie algebra of step 2. Then $\mathfrak{L} = V \oplus W$ as a vector space, and $[\mathfrak{L},W] = 0$, $[X,Y] =: \omega(X,Y) \in W$ for $X,Y \in V$, where $\omega: V \times V \to W$ is an arbitary skew symmetric bilinear map. If we want an associative algebra, then $b = 0$ and $A(\mathfrak{L},0) = V \times W \times k$ as a vector space with product

$$(v,w,0)(v',w',0) = (0,\omega(v,v'),0)$$

and $(0,0,1)$ as unit.

3. Passing from algebras to Lie algebras

3.1. Proposition Let A be an algebra with unit over a commutative field k. Then A is Lie admissible (cf. 1.2.(iv)) if and only if the associator $Ass(x,y,z) = x(yz) - (xy)z$ satisfies

$$(*) \qquad \Sigma_{\sigma\in\mathfrak{S}_3} \operatorname{sgn}(\sigma) Ass(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}) = 0$$

for all triplets x_1,x_2,x_3 of elements in A, where $\mathfrak{S}_3$ denotes the group of permutations of { 1,2,3 }. If char k ≠ 2,3 and A is alternative, then (*) implies that A is associative.

<u>Proof</u> The proof of the first assertion is an easy computation and therefore left to the reader. For the second we only have to note that by Bourbaki [2], p.612, A is associative if and only if Ass is skew symmetric; if Ass is skew symmetric then the left side of (*) is just 6 $\mathrm{Ass}(x_1x_2x_3)$.

<u>3.2. Remarks</u> (i) Often conditions stronger than (*) have been dealt with in the literature; such as (cf. Nijenhuis and Richardson [7])

$$\mathrm{Ass}(x,y,z) = \mathrm{Ass}(y,x,z)$$
$$\mathrm{Ass}(x,y,z) = \mathrm{Ass}(x,z,y)$$
$$\mathrm{Ass}(x,y,z) = \mathrm{Ass}(z,y,x)$$

None of these conditions is satisfied for all of the algebras $A(\mathfrak{L},b)$ of section 1.

(ii) Proposition 3.1. has an obvious generalization to graded algebras and graded Lie algebras.

<u>3.3 Definition</u> Let $\mathfrak{G}$ be a subgroup of $\mathfrak{S}_3$. Then an algebra A is called $\mathfrak{G}$ - assoctiative if

$$\Sigma_{\sigma\in\mathfrak{G}}\mathrm{sgn}(\sigma)\mathrm{Ass}(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}) = 0$$

<u>3.4. Remarks</u> (i) By 1.2.(v) every algebra $A(\mathfrak{L},b)$ is $\mathfrak{A}_3$ - associative, where $\mathfrak{A}_3$ denotes the alternating group in three elements. More generally, for char k ≠ 2 every flexible (cf 1.2.(iii)) Lie admissible algebra is $\mathfrak{A}_3$ - associative, since flexibility can be linearized to $\mathrm{Ass}(x,y,z) + \mathrm{Ass}(z,y,x) = 0$: this shows that flexibility is not a kind of $\mathfrak{G}$ - associativity.

(ii) By 3.1., $\mathfrak{S}_3$ - associativity is equivalent to Lie admissibility. The conditions in 3.2. correspond to $\mathfrak{G}$ - associative algebras, where $\mathfrak{G}$ is a two element subgroup of $\mathfrak{S}_3$.

(iii) The {1} - associative algebras are just the associative algebras.

(iv) If $\mathfrak{G} \leq \mathfrak{H}$, then every $\mathfrak{G}$ - associative algebra is also $\mathfrak{H}$ - associative.

(v) Note the formula

$$\mathrm{Ass}(x,y,z) + \mathrm{Ass}(y,z,x) + \mathrm{Ass}(z,x,y) = [x,yz] + [y,zx] + [z,xy]$$

Thus an algebra A is $\mathfrak{A}_3$ - associative if and only if

$$[x,yz] + [y,zx] + [z,xy] = 0 \qquad \text{for all } x,y,z \in A$$

Throughout the rest of this section let A denote a unital algebra over k such that $2\dim A \neq 0$ in k.

3.5. Definition Let $L_x: y \to xy$ and $R_x: y \to yx$ denote left and right multiplication by x. Then define

$$\tau_A: A \to k, \quad \tau_A(x) := \frac{1}{2\dim A}\,\mathrm{trace}(L_x + R_x)$$
$$\langle x,y\rangle_A := \tau_A(xy).$$

τ_A is said to be a Clifford trace if the complementary projection $\pi_A: A \to A$, $\pi_A(x) := x - \tau_A(x)1$ satisfies the Clifford equation

$$\pi_A(x)\pi_A(y) + \pi_A(y)\pi_A(x) = 2\langle\pi_A(x),\pi_A(y)\rangle_A 1$$

3.6. Lemma (i) If τ_A is a Clifford trace, then $\langle .,.\rangle_A$ is symmetric.
(ii) If $A = A(\mathfrak{L},b)$, then τ_A is a Clifford trace.
Proof (i) trivial
(ii) An easy computation shows that $\tau_A(X,s) = s$, $\pi_A(X,s) = (X,0)$, and that the Clifford equation holds. □

3.7. Theorem Let A be a unital algebra over k such that $2\dim A \neq 0$ in k. Then the following assertions are equivalent:
(i) A can be written in the form $A = A(\mathfrak{L},b)$ for some Lie algebra $\mathfrak{L}$ and invariant form b.
(ii) A is a flexible Lie admissible algebra and τ_A is a Clifford trace.
(iii) A is a flexible $\mathfrak{U}_3$ - associative algebra and τ_A is a Clifford trace.
Proof (i) ⇒ (ii) by 1.2.(iii), 1.2.(iv) and 3.6.(ii).
(ii) ⇔ (iii) by 3.1.,3.4.(i) and 3.4.(iv).
(ii) ⇒ (i) The commutator algebra $A^- = (A,[.,.]_A)$ introduced in 1.2.(iv) is a Lie algebra. If we consider k as one-dimensional (trivial) Lie algebra, then $\tau_A: A^- \to k$ is a Lie homomorphism. We define $\mathfrak{L}$ to be the Lie algebra $\ker \tau_A$, provided with the Lie bracket $[.,.]_{\mathfrak{L}} = \frac{1}{2}[.,.]_A$, and $b(X,Y) = \langle X,Y\rangle_A$ for all $X,Y \in \mathfrak{L}$. b is symmetric and invariant by 3.6.(i) and the remark to proposition 1.2. Let $\pi_A: A^- \to \ker \tau_A = \mathfrak{L}$ be the complementary projection, $\pi_A(x) = x - \tau_A(x)1$; π_A is also a Lie algebra homomorphism. Let $X,Y \in \mathfrak{L}$. Then (XY denoting the product in A)

$$XY = \frac{1}{2}(XY - YX) + \frac{1}{2}(XY + YX) = \frac{1}{2}[X,Y]_A + \frac{1}{2}(\pi_A(X)\pi_A(Y) + \pi_A(Y)\pi_A(X)) =$$
$$= \frac{1}{2}[X,Y]_A + \langle\Pi_A(X),\pi_A(Y)\rangle_A 1 = [X,Y]_{\mathfrak{L}} + b(X,Y)1.$$

For arbitrary $x,y \in A$ we have $x = \pi_A(x) + \tau_A(x)1$, $y = \pi_A(y) + \tau_A(y)1$, and we get

$$\begin{aligned} xy &= (\pi_A(x) + \tau_A(x)1)(\pi_A(y) + \tau_A(y)1) = \\ &= \pi_A(x)\pi_A(y) + \tau_A(x)\pi_A(y) + \tau_A(y)\pi_A(x) + \tau_A(x)\tau_A(y)1 = \\ &= [\pi_A(x),\pi_A(y)]_{\mathfrak{L}} + \tau_A(x)\pi_A(y) + \tau_A(y)\pi_A(x) + \tau_A(x)\tau_A(y)1 + \\ &\quad + b(\pi_A(x),\pi_A(y))1. \end{aligned}$$

Thus the map $A \to A(\mathfrak{L},b)$, $x \to (\pi(x),\tau(x))$ is the required isomorphism. □

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