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ON MUTATIONAL DEFORMATION RETRACTS

José M. R. Sanjurjo

Let $X$ be a closed subset of a metrizable space $X'$ considered as a closed subset of an ANR($\mathcal{K}$)-space $P$. The family $\mathcal{U}(X',P)$ of all open neighborhoods of $X'$ in $P$ is called the complete neighborhood system of $X'$ in $P$. By a mutational deformation retraction of $X'$ to $X$ we mean a mutational retraction (see [5]) $r: \mathcal{U}(X',P) \rightarrow \mathcal{U}(X,P)$ such that for every $U' \in \mathcal{U}(X',P)$ and for every $r \in \mathcal{R}$ with range $U'$ there exists $V' \in \mathcal{U}(X',P)$ contained in $U'$ and in the domain of $r$ such that $r|_{V'} = i$ (the identity) in $U'$. If the homotopy can be chosen stationary on $X$ we say that $r$ is a stationary mutational deformation retraction. A mutational retraction $r: \mathcal{U}(X',P) \rightarrow \mathcal{U}(X,P)$ is said to be regular if for every $U' \in \mathcal{U}(X',P)$ and for every $r, r' \in \mathcal{R}$ with range $U'$ there exists $V' \in \mathcal{U}(X',P)$ such that $r|_{V'} = r'|_{V'}$ (rel $X$) in $U'$. The notion of regular mutational retraction is a generalization of Dydak's notion of regular fundamental retraction [2] in which a more restrictive condition is imposed on homotopies.

The problem whether every $W$-shape deformation retract is stationary has been raised by K. Borsuk in his book [1] (p. 190, Problem 4.15) and, up to the author's knowledge, is open even in the compact case. In the present note we give a partial answer to the analogous problem in Fox shape theory [4]. The reader is referred to [1], [3] and [6] for information about theory of shape.

Theorem 1. Let $r: \mathcal{U}(X',P) \rightarrow \mathcal{U}(X,P)$ be a deformation mutational retraction. Then $r$ is stationary if and only if $r$ is regular.

Proof. The part "only if" is trivial, we are going to prove the converse. Let $U' \in \mathcal{U}(X',P)$ and consider $r \in \mathcal{R}$ with range $U'$ and domain $U'_0 \in \mathcal{U}(X',P)$. Since $r$ is a mutational deformation retraction there exists an open neighborhood $V' \in \mathcal{U}(X',P)$ of $X'$ in $P$ such that

(1) $r|_{V'} = i$ in $U'$.

Since $U' \in \mathcal{U}$ ANR it is easy to see, by using the homotopy extension theorem, that there exist a map $s: V' \rightarrow U'$ and an open neighborhood

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U ⊂ V' of X in P such that
(2) s(x)=x for every x∈U
(3) r|V'=s|V', (rel X) in U'.

Since r is regular, there exists r'|W' such that r':W' → U where
W' ⊂ V' is an open neighborhood of X' in P and such that
(4) r'=r|W', (rel X) in U'.

Let us now define a map ϕ:K=W×{0}∪W×{1} → V' by
ϕ(x,0)=x, ϕ(x,1)=r'(x) for every x∈W'
and ϕ(x,t)=x for (x,t)∈X×I.

Since ϕ(x,1)∈U it follows from (2) that
sϕ(x,0)=s(x), sϕ(x,1)=r'(x) for every x∈W'
and sϕ(x,t)=x for (x,t)∈X×I.

It follows from (1) and (3) that ϕ=sϕ in U'. Moreover sϕ is homo-
topic in U' to the map ψ:K → U' defined by
ψ(x,0)=ψ(x,1)=r'(x) for every x∈W'
and ψ(x,t)=x for (x,t)∈X×I.

To see it consider a homotopy χ:W×I → U' such that χ(x,0)=s(x),
χ(x,1)=r'(x) for x∈W' and χ(x,t)=x for (x,t)∈X×I. We define a map
F:K×I → U' by
F((x,o),t')=χ(x,t'), F((x,1),t')=r'(x) for x∈W', t'∈I
and F((x,t),t')=x for x∈X and t,t'∈I.

Obviously, F((x,t),0)=sϕ(x,t) and F((x,t),1)=ψ(x,t) for (x,t)∈K.
hence ϕ=sϕ=ϕ. Since U'∈ANR and ψ can be extended to W'×I (by the
map ψ(x,t)=r'(x)) then, in virtue of the homotopy extension theorem,
ϕ can also be extended to a map ϕ:W'×I → U' which realizes a homoto-
py between i and r' stationary on X. Since r|W'=r' (rel X) in U' we
conclude that r|W'=i (rel X) in U' and, consequently, r is statio-
nary.

Corollary. Let X be an MANR [5] with compact components. If X is
a mutational deformation retract of a metrizable space X' lying in
P∈ANR(M), then X is a stationary mutational deformation retract of
X'.

Proof. By Corollary 3.11 of [5] X=⨁{X_i,i∈I}, where {X_i,i∈I}
is the family of all components of X. Since X_i∈ANR for every i∈I
it follows from Dydak's Corollary 1, [2], that each X_i is a regular
mutational retract of one of its neighborhoods in P. Hence, there
exists a neighborhood W of X in P which can be represented as a to-
topological sum W=⨁{W_i,i∈I}, where W_i is a neighborhood of X_i in P
and X_i is a regular mutational retract of W_i for i∈I. Consequently,
there exists a regular mutational retraction r:U(W,P) → U(X,P).
Since X is a mutational retract of X' there exists a map s:X' → W
such that \( s(x) = x \) for every \( x \in X \). Let \( s: U(X',,P) \to U(W,P) \) be a mutation generated by \( s \). Then \( r' = \tilde{r} \circ s \) is a regular mutational retraction and, since \( X \) is a mutational deformation retract of \( X' \), we can easily get from Theorem 1 that \( r' \) is a stationary mutational deformation retraction.

REFERENCES


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