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CORRESPONDENCE BETWEEN MAXIMAL IDEALS
IN ASSOCIATIVE ALGEBRAS AND LIE ALGEBRAS

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I. Correspondence between maximal ideals

The investigation of the Lie algebra $\mathfrak{X}(V)$ of C^∞-vector fields on a C^∞-manifold V, considered as the Lie algebra of derivations on the associative algebra $C^\infty(V)$ of C^∞-functions, leads naturally to the following definition (see [1]).

1. Definition : Let A be a commutative associative algebra with a unit element over a field K of characteristic zero, and let Der(A) denote the Lie algebra of derivations of A, which has a natural A-module structure. Let $L \subset Der(A)$ be a subalgebra and an A-submodule. The couple (A,L) will be called Lie bimodule. A Lie bimodule (A,L) will be called admissible if the following condition is satisfied :

$$LA = A .$$

(Let us remark that in [1] there are three more conditions. Two of them are in our setting automatically satisfied, the third one we do not need.)

Let J be an ideal in the associative algebra A, and $\mathcal{L}$ an ideal in the Lie algebra L. We introduce the following natations :

(1) $J^L = \{ f \in J \ ; \ Y_k(Y_{k-1}(\dots(Y_1 f)\dots)) \in J$ for any $Y_1,\dots,Y_k \in L$ and any $k = 1,2,\dots \}$,

(2) $L_J = \{ X \in L \ ; \ XA \subset J \}$,

(3) $L_J^\infty = \{ X \in L \ ; \ XA \subset J^L \}$,

(4) $P(\mathcal{L}) = \{ X \in \mathcal{L} \ ; \ AX \subset \mathcal{L} \}$,

(5) $P_X(\mathcal{L}) = \{ f \in A \ ; \ fX \in P(\mathcal{L}) \}$, $X \in L$,

(6) $I(\mathcal{L}) = \bigcap_{X \in L} P_X(\mathcal{L})$.

It can be shown (see [1]) that $J^L, P_X(\mathcal{L})$ for any $X \in L$, and $I(\mathcal{L})$

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are ideals in A, $P(\mathcal{L})$ is an A-submodule of L, L_J is a subalgebra in L, and L_J^∞ is an ideal in L.

In [1] the following theorem is proved :

2. Theorem : Let (A,L) be a Lie bimodule, and let $\mathcal{L} \subset L$ be an ideal. Then the ideal $I(\mathcal{L}) \subset A$ has the following two properties :

(i) $I(\mathcal{L})L \subset \mathcal{L}$

(ii) For any prime ideal $J \supset I(\mathcal{L})$ there is $\mathcal{L} \subset L_J^\infty$.

From now on we shall assume that (A,L) is an admissible Lie bimodule. We shall start with

3. Lemma : Let $J \subset A$ be an ideal. Then $I(L_J^\infty) = J^L$.

Proof : Let $f \in I(L_J^\infty)$. Then for any $X \in L$ we have $fX \in L_J^\infty$. Thus for any $g \in A$ we get $(fX)g \in J^L$ or equivalently $f \cdot Xg \in J^L$. Because $1 \in A = LA$ we conclude that $f \in J^L$.

Conversely let $f \in J^L$. We must prove that for any $X \in L$ there is $f \in P_X(L_J^\infty)$ or equivalently that $fX \in P(L_J^\infty) = L_J^\infty$. But for any $g \in A$ we have $(fX)g = f \cdot Xg \in J^L$ because J^L is an ideal in A. This shows that $fX \in L_J^\infty$. We have thus proved that $f \in I(L_J^\infty)$.

4. Definition : An ideal $J \subset A$ is called invariant ideal if $LJ \subset J$.

5. Lemma : Let $J \subset A$ be an ideal. Then J^L is an invariant ideal.

Proof is obvious.

6. Lemma : Let $J \subset A$ be an ideal. Then $J^L = L_J^\infty A$.

Proof : The inclusion $L_J^\infty A \subset J^L$ is obvious from the definition of L_J^∞. Taking $\mathcal{L} = L_J^\infty$ in Th. 2 and using the equality $I(L_J^\infty) = J^L$ of Lemma 3 we obtain

$$J^L L \subset L_J^\infty$$
$$J^L LA \subset L_J^\infty A$$
$$J^L A \subset L_J^\infty A$$
$$J^L \subset L_J^\infty A ,$$

which finishes the proof.

7. Lemma : Let $J \subset A$ be an ideal. Then $L_J^\infty = L_{J^L}$.

Proof : The previous lemma shows that $L_J^\infty \subset L_{J^L}$. The converse inclusion $L_{J^L} \subset L_J^\infty$ is obvious from the definitions of L_{J^L} and L_J^∞ .

8. Corollary : If $J \subset A$ is an invariant ideal, then $L_J^\infty = L_J$.

9. Proposition : Let $\mathcal{L} \subsetneqq L$ be an ideal. Then there exists an invariant ideal $J \subsetneqq A$ such that $\mathcal{L} \subset L_J$.

Proof : Th. 2(i) shows that $I(\mathcal{L}) \subsetneqq A$ (otherwise $L = AL \subset \mathcal{L}$). Thus there exists a prime ideal $\tilde{J}$ such that $\tilde{J} \supset I(\mathcal{L})$, $\tilde{J} \subsetneqq A$. By virtue of Th. 2(ii) there is $\mathcal{L} \subset L_{\tilde{J}}^\infty$. We set $J = \tilde{J}^L \subsetneqq A$. Then using lemma 7 we get

$$\mathcal{L} \subset L_{\tilde{J}}^\infty = L_{\tilde{J}^L} = L_J.$$

10. Lemma : Let J_1, J_2 be two invariant ideals, $J_1 \neq J_2$. Then $L_{J_1} \neq L_{J_2}$.

Proof : Let us assume that $L_{J_1} = L_{J_2}$. Then by Corollary 8 we have $L_{J_1}^{\infty} = L_{J_2}^{\infty}$, and consequently $I(L_{J_1}^{\infty}) = I(L_{J_2}^{\infty})$. Using Lemma 3 we get $J_1^L = J_2^L$. But because J_1 and J_2 are invariant we have $J_1 = J_2$, which is a contradiction.

11. Definition : An invariant ideal $J \subsetneq A$ is called maximal invariant ideal if

$J' \subsetneq A$ is an invariant ideal, $J' \supset J \Rightarrow J' = J$.

We shall introduce the following notation :

$\mathrm{Specm}_I A$ = the set of all maximal invariant ideals in A ,

Specm L = the set of all maximal ideals in L

12. Theorem : The correspondence $J \longrightarrow L_J$ defines a bijection $\iota : \mathrm{Specm}_I A \longrightarrow \mathrm{Specm}\, L$.

Proof : Lemma 10 shows that ι is injective. Let $\mathfrak{M} \in \mathrm{Specm}\, L$. By virtue of Prop. 9 there exists an invariant ideal $J \subsetneq A$ such that $\mathfrak{M} \subset L_J$. Obviously $L_J = L_J^{\infty}$ is an ideal, and $L_J \subsetneq L$ (otherwise $A = LA = L_J A \subset J$). Thus $\mathfrak{M} = L_J$, and J is a maximal invariant ideal by Lemma 10.

II. Maximal invariant ideals in $C^{\infty}(V)$.

Let us consider now a connected paracompact C^{∞}-manifold V, dim V = m. Our goal is to describe all maximal invariant ideals in the real algebra $C^{\infty}(V)$ of all C^{∞}-functions on V. First we recall the following definition.

13. Definition : A nonempty family $\mathcal{F}$ of closed sets of V is called a z-filter on V if

(i) $\emptyset \notin \mathcal{F}$,

(ii) $Z, Z' \in \mathcal{F} \Rightarrow Z \cap Z' \in \mathcal{F}$,

(iii) $Z \in \mathcal{F}$, $Z \subset Z'$, Z' is a closed subset of V $\Rightarrow Z' \in \mathcal{F}$.

By a z-ultrafilter we shall mean a maximal z-filter, i.e. one not contained in any other z-filter.

For $f \in C^{\infty}(V)$ and a z-filter $\mathcal{F}$ on V we shall denote

$Z_n(f) = \{p \in V \; ; \; j_p^n(f) = 0\}$, $0 \leq n \leq \infty$,

$Z^{\leftarrow}[\mathcal{F}] = \{g \in C^{\infty}(V) \; ; \; Z_0(g) \in \mathcal{F}\}$,

$\mathcal{Z}^{\leftarrow}[\mathcal{F}] = \{g \in C^{\infty}(V) \; ; \; Z_n(g) \in \mathcal{F}$ for $0 \leq n < \infty\}$,

where $j_p^n(f)$ denotes n-th jet of the function f at the point p. It is obvious that both $Z^{\leftarrow}[\mathcal{F}]$ and $\mathcal{Z}^{\leftarrow}[\mathcal{F}]$ are ideals in $C^{\infty}(V)$.

14. Theorem : Let $M \subset C^{\infty}(V)$ be a maximal ideal. Then there exists a unique z-ultrafilter $\mathcal{A}$ on V such that $M = Z^{\leftarrow}[\mathcal{A}]$.

Proof is easy (see [3]).

15. Lemma : Let $J \subset C^\infty(V)$ be an invariant ideal. Let $f \in J$ and $0 \leq n < \infty$. Then there exists $g \in J$ such that $Z_0(g) = Z_n(f)$.

Proof : $\dim V = m$ and therefore (see [2]) we can find $m+1$ families $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_m$ of open subsets in V

$\mathcal{U}_i = \{U_{i\alpha} ; \alpha \in \Sigma_i\}$, $0 \leq i \leq m$

with the following properties

(i) $\bigcup_{i=0}^{m} \bigcup_{\alpha \in \Sigma_i} U_{i\alpha} = V$

(ii) For any $0 \leq i \leq m$, and any $\alpha, \beta \in \Sigma_i$, $\alpha \neq \beta$ there is $U_{i\alpha} \cap U_{i\beta} = \emptyset$.

(iii) Each $U_{i\alpha}$ is a domain of a chart $(x_1^{(i\alpha)}, \ldots, x_m^{(i\alpha)})$.

Furthermore we can find open subsets $V_{i\alpha}$, $0 \leq i \leq m$, $\alpha \in \Sigma_i$ such that

(iv) $\overline{V}_{i\alpha} \subset U_{i\alpha}$

(v) $\bigcup_{i=0}^{m} \bigcup_{\alpha \in \Sigma_i} V_{i\alpha} = V$.

There exist vector fields $X_{ij} \in \mathfrak{X}(V)$, $0 \leq i \leq m$, $1 \leq j \leq m$ such that for any $\alpha \in \Sigma_i$ and $p \in V_{i\alpha}$ there is

$$X_{ij}(p) = (\partial / \partial x_j^{(i\alpha)})(p).$$

We set

$$g = \sum_{i=0}^{m} \sum_{k=0}^{n} \sum_{1 \leq j_1, \ldots, j_k \leq m} (X_{ij_1} \ldots X_{ij_k} f)^2 .$$

Obviously $g \in J$ and $Z_0(g) = Z_n(f)$.

16. Lemma : Let $\mathcal{A}$ be a z-ultrafilter on V. Then $\mathcal{Z}^{\leftarrow}[\mathcal{A}]$ is a maximal invariant ideal.

Proof : $\mathcal{Z}^{\leftarrow}[\mathcal{A}]$ is obviously an invariant ideal and $\mathcal{Z}^{\leftarrow}[\mathcal{A}] \subsetneq C^\infty(V)$. Thus it suffices to prove that it is a maximal invariant ideal. Let $f \in C^\infty(V)$, $f \notin \mathcal{Z}^{\leftarrow}[\mathcal{A}]$, and let us consider the invariant ideal J generated by f and $\mathcal{Z}^{\leftarrow}[\mathcal{A}]$. Because $f \notin \mathcal{Z}^{\leftarrow}[\mathcal{A}]$ there exists $0 \leq n < \infty$ such that $Z_n(f) \notin \mathcal{A}$. By virtue of Lemma 15 there exists $g \in J$ such that $Z_0(g) = Z_n(f)$. Because $\mathcal{A}$ is a z-ultrafilter there exists a closed subset $Z \in \mathcal{A}$ such that $Z \cap Z_0(g) = \emptyset$. Furthermore there exists $\bar{g} \in C^\infty(V)$ such that $Z_\infty(\bar{g}) = Z$, and $\bar{g}(p) \neq 0$ for any $p \in V - Z$. Obviously $\bar{g} \in \mathcal{Z}^{\leftarrow}[\mathcal{A}]$. We have thus $g^2 + \bar{g}^2 \in J$, $g^2 + \bar{g}^2 > 0$ on V. Consequently $J = C^\infty(V)$. This proves that $\mathcal{Z}^{\leftarrow}[\mathcal{A}]$ is a maximal invariant ideal.

17. Theorem : Let $M \subset C^\infty(V)$ be a maximal invariant ideal. Then there exists a unique z-ultrafilter $\mathcal{A}$ on V such that $M = \mathcal{Z}^{\leftarrow}[\mathcal{A}]$.

Proof : A maximal invariant ideal M is contained in a maximal ideal of $C^{\infty}(V)$, and thus by virtue of Th. 14 there exists a z-ultrafilter $\mathcal{A}$ such that $M \subset Z^{\leftarrow}[\mathcal{A}]$. Let $f \in M$, and let $0 \leq n < \infty$ be arbitrary. By virtue of Lemma 15 there exists $g \in M$ such that $Z_n(f) = Z_0(g)$. Thus $Z_n(f) \in \mathcal{A}$ for every $0 \leq n < \infty$, i.e. $f \in \mathcal{Z}^{\leftarrow}[\mathcal{A}]$. This shows that $M \subset \mathcal{Z}^{\leftarrow}[\mathcal{A}]$. But M is maximal invariant. Using Lemma 16 we obtain $M = \mathcal{Z}^{\leftarrow}[\mathcal{A}]$.

Now let $M = \mathcal{Z}^{\leftarrow}[\mathcal{A}] = \mathcal{Z}^{\leftarrow}[\bar{\mathcal{A}}]$, where $\mathcal{A}$, $\bar{\mathcal{A}}$ are z-ultrafilters, and let us assume that $\mathcal{A} \neq \bar{\mathcal{A}}$. Then there exist $Z \in \mathcal{A}$, $\bar{Z} \in \bar{\mathcal{A}}$ such that $Z \cap \bar{Z} = \emptyset$. Let $f, \bar{f} \in C^{\infty}(V)$ be such that $Z_{\infty}(f) = Z$, $Z_{\infty}(\bar{f}) = \bar{Z}$, $f(p) \neq 0$ for $p \notin Z$, $\bar{f}(p) \neq 0$ for $p \notin \bar{Z}$. Then $f, \bar{f} \in M$, and consequently $f^2 + \bar{f}^2 \in M$. On the other hand $f^2 + \bar{f}^2 > 0$ on V, which is a contradiction. This shows that $\mathcal{A} = \bar{\mathcal{A}}$.

18. <u>Remark</u> : Combining Ths. 12 and 17 we can reprove the theorem (see [3], Th. 9) characterizing maximal ideals in the Lie algebra $\mathfrak{X}(V)$.

REFERENCES

1 GRABOWSKI J. "Algebry Liego pól wektorowych na rozmanitościach", Thesis, University of Warsaw, 1981.

2 MUNKRES J.R. "Elementary differential topology", Ann. Math. Studies 54, Princeton, New Jersey, Princeton Univ. Press, 1966.

3 VANŽURA J. "Maximal ideals in the Lie algebra of vector fields", to appear in Comm. Math. Univ. Carolinae.

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