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0. Introduction

The local structure of a finite dimensional Lie group is determined by the structure of the corresponding Lie algebra through the exponential map (which is a local bijection at 0) and the C-H-D formula. For infinite dimensional Lie groups this connect ion in general fails, since the exponential map is usually not a local bijection at 0 (this situation occurs e.g. for the group of all C diffeomorphisms of a compact manifold cf. [3]). Also the analytic description of the group multiplication via C-H-D formula is not possible in general cf. [2] .

Nevertheless it seems to us that there exists in quite a general situation the possibility of transmitting the structure of a Lie algebra to the corresponding "Lie group" in analytic way. This possibility is based on the concepts of "Polynomial group of a topological group" and "Polynomial group of a Lie algebra". The main result of this note - Theorem 15 - establishes the isomorphism of this two objects and potentialy gives such analytic description of a group in the terms of its Lie algebra.

1. Polynomial groups

Let G be a topological group (all the topological group we deal with in this note are assumed to be Hausdorff). By $C(\mathbb{R},G)$ we denote the topological group of all continuous G-valued functions on the real line $\mathbb{R}$ with the pointwise multiplication and the compact-open topology. For the elements of $C(\mathbb{R},G)$ multiplication by real numbers is also defined, according to the formula $(s,f) \longrightarrow sf$ where $sf(t) = f(st)$ for $s \in R$ and $f \in C(R,G)$. Clearly this multiplication is a jointly continuous operation from $R \times C(R,G)$ into $C(R,G)$.

Let $\Lambda(G)$ denote the family of all one-parameter subgroups of G, i.e. the family of all continuous homomorphisms of the additive group of reals into G. $\Lambda(G)$ is a closed subset of $C(R,G)$.

Let $P(G)$ be the subgroup of $C(R,G)$ generated by $\Lambda(G)$. We shall call the elements of $P(G)$ polynomials, and $P(G)$ itself the polynomial group of G.

There are three aspects of the structure of $P(G)$: it is a group, it admits multiplication by real numbers (restricted from $C(R,G)$ and it is generated by its subset $\Lambda(G)$ composed of elements for which $na = a^n$ for any positive integer n.

We start with examining this situation in an abstract setting.

2. Free R-groups

Definition 1. A set W with a base point e is called an $\mathbb{R}$-set if a map $\mathbb{R} \times W \longrightarrow W : (s,w) \longrightarrow sw$ is defined, in such a way that for $s_1, s_2 \in \mathbb{R}$ and $w \in W$

$$(1) \qquad \begin{aligned} &\text{(i)} \quad s_1(s_2 w) = (s_1 s_2)w \\ &\text{(ii)} \quad 0w = e \\ &\text{(iii)} \quad 1w = w . \end{aligned}$$

A group H is said to be an $\mathbb{R}$-group if H is an $\mathbb{R}$-set with the unit e as the base point, and moreover for each $s \in \mathbb{R}$ and $h_1, h_2 \in H$

$$(2) \qquad s(h_1 h_2) = (sh_1)(sh_2)$$

In the obvious way one introduces the notions of $\mathbb{R}$-map $\mathbb{R}$-homomorphism, $\mathbb{R}$-subgroup etc.

Let A be an $\mathbb{R}$ set and G be a group. We shall call a map $j : A \rightarrow G$ exponential if for each $a \in A$ the function $\varphi_a : R \rightarrow G$ where $\varphi_a(s) = j(sa)$ is a one parameter subgroup of G.

Proposition 2. Let A be an R-set with the base point e. There exists unique R-group $F(A)$ such that

(a) There exist an exponential R-map $i : A \rightarrow F(A)$.

(b) For each exponential R-map $\alpha : A \rightarrow H$ where H is an R-group there exists unique R-homomorphism $\beta : F(A) \rightarrow H$ such that $\alpha = \beta \circ i$.

The group $F(A)$ will be called the free R-group over A.

Proof. The proof is standard, and we shall only briefly sketch it. Let G be the free group over $A \setminus \{e\}$. Extending the canonical embedding $k : A \setminus \{e\} \to G$ to $\bar{k} : A \to G$ by letting $\bar{k}(e)$ be the unit element of G we obtain an R-group structure on G, with $s(a_1 \ldots a_n) = (sa_1)\ldots(sa_n)$ for $s \in R$ and $a_1 \ldots a_n \in A$. Let I be the normal subgroup of G generated by the subset $\{ \lambda a \cdot \mu a \cdot [(\lambda+\mu)a]^{-1} : \lambda, \mu \in R,\ a \in \bar{k}(a)\}$. Since I is R-subgroup of G the quotient group $F(A) = G/I$ is an R-group and the quotient homomorphism $\pi : G \to F(A)$ is an R-map. We define $i = \pi \circ \bar{k}$.

Remark 3. The mapping $i : A \to F(A)$ is injective. In fact splitting $A \setminus \{0\}$ into disjoint "lines" i.e. subsets of the form $[a] = \{sa : s \in \mathbb{R}\}$ and picking one representant a^* from each "line" $[a]$ and letting X be the linear space with a base formed by so chosen representants we may define an injective exponential $\mathbb{R}$-map $\alpha : A \to X$ putting $\alpha(a) = s\alpha(a^*)$ where $a = sa^*$ and $\alpha(a^*)$ denotes a^* as an element of X. Clearly injectivity of α implies injectivity of i.

3. Algebraic properties of F(A)

To abbreviate the notation we shall not distinguish between A and $i(A)$ (which is allowed by Remark 3) and we shall write a instead of $i(a)$. We shall also abbreviate $aba^{-1}b^{-1}$ to $\{a,b\}$ and inductively we shall write $\{a_1,\ldots,a_k\}$ instead of $\{a_1\{a_2,\ldots,a_k\}\}$. As usually $H^{(n)}$ will denote the smallest subgroup of a group H containing all the terms $\{h,\tilde{h}\}$ with $h \in H$ and $\tilde{h} \in H^{(n-1)}$, where $H^{(1)} = H$.

We shall omit simple proofs of the following two lemmas

Lemma 4. The group $F(A)^{(n)}$ is generated by the elements $\{a_1,\ldots,a_k\}$, with $a_i \in A$ $i = 1,\ldots,k$ and $k \geq n$.

Lemma 5.

(a) Let $a \in F(A)^{(n)}$ or $b \in F(A)^n$ then $\{a,b\} = \{b^{-1},a\}$ mod $F(A)^{(n+2)}$

(b) Let $a \in F(A)^{(n)}$ or b and c belong to $F(A)^{(n)}$ then $\{a,bc\} = \{a,b\}\{a,c\}$ mod $F(A)^{(n+2)}$.

(c) Let a,b $F(A)^{(n)}$, then $a^n b^n = (ab)^n$ mod $F(A)^{(n+1)}$.

The fact that $\mathcal{E}$ is a continuous homomorphism is valid for any topological group G. To prove that $\mathcal{E}$ is open let us observe that for G a Banach Lie group the restriction of $\mathcal{E}$ to $\Lambda(G)$ coincides with the exponential map $\mathrm{Exp}\,\Lambda(G) \to G$ thus $\mathcal{E}$ is locally open at $e \in P(G)$. Since $\mathcal{E}$ is a homomorphism it is an open map.

5. The polynomial group of a Lie algebra

Another important example of a Lie R-groups is provided by the following construction.

Let $\mathfrak{g}$ be a Lie algebra. Denote by T the tensor algebra of the linear space $\mathfrak{g}$, and let $\hat{T}$ be the Magnus algebra of $\mathfrak{g}$, i.e. the infinite product $\prod_{n=0} T_n$, where T_n $n=0,1,2,3$ denotes the homogeneous component of order n of T. (The elements of $\hat{T}$ may be viewed as formal series $f = \sum_{n=0} f_n$ with $f_n \in T_n$. The algebra $\hat{T}$ may be obtained as the complution of T with respect to the metric ρ where

$$\rho(f,g) = \sum_{n=0} \frac{1}{2^n} \frac{\rho_n(f_n,g_n)}{1+\rho_n(f_n,g_n)} \quad \text{for } f = \sum f_n , \quad g = \sum g_n \text{ and}$$

ρ_n is for $n = 1,2,\ldots$ a discrete metric on T_n. Let $\hat{L}$ be the closed Lie subalgebra of $\hat{T}$ generated by $\mathfrak{g} = T_1$, and let $\hat{M}$ be the closed two-sided ideal of $\hat{T}$ generated by $\mathfrak{g} = T_1$. For any $a \in \hat{M}$ the series $\sum_{n=0} \frac{a^n}{n!}$ is convergent and it defines the exponential map $\exp : \hat{M} \to 1 + \hat{M}$. It is known c.f. [1], [4] that this map is a bijection with the inverse map $\log : 1+\hat{M} \to \hat{M}$ defined by the series $\log b = \sum_{n=1} (-1)^{n+1} \frac{(b-1)^n}{n}$, and the set $G = \exp(\hat{L})$ is a subgroup of the group of all invertible elements of $\hat{T}$.

Let $\mathcal{L}(\mathfrak{g})$ be the set of all G-valued functions on R which are finite pointwise products of exponential functions

$$\mathcal{L}(\mathfrak{g}) = \left\{ f(t) = \exp tx_1 \ldots \exp tx_n : x_i \in \mathfrak{g} \quad \text{for } 1 \leq i \leq n,\ n=1,2,\ldots \right\}$$

$\mathcal{L}(\mathfrak{g})$ with the pointwise multiplication and multiplication by real numbers, defined by the formula $(sf)(t) = f(st)$ for $s \in R$ and $f \in \mathcal{L}(\mathfrak{g})$, is an R-group.

Proposition 6. Let $a \in F(A)^{(n)}$ and k be a positive integer. Then

(3) $$ka = a^{k^n} \mod F(A)^{(n+1)}$$

Proof. Assume first, that $a = \{x_1,\dots,x_n\}$ with $x_i \in A$ $i=1,\dots,n$. If $n = 1$ the equality (3) results from the condition (a) of Proposition 2. Reasoning by induction, suppose that (3) holds for all the elements of the form $\{x_1,\dots,x_{n-1}\}$. Applying Lemma 5(b) we get $k\{x_1,\dots,x_n\} = \{kx_1, k\{x_2,\dots,x_n\}\} = \{x_1^k, \{x_2,\dots,x_n\}^{k^{n-1}}\} =$ $= \{x_1^k, \{x_2,\dots,x_n\}\}^{k^{n-1}} = \{x_1,\dots,x_n\}^{k^n}$ (all the equalities mod $F(A)^{n+1}$).

Passing to the general case, let $a \in F(A)^{(n)}$. Then by Lemma 4 $a = a_1 \dots a_s$ with $a_i = \{x_{i,1},\dots,x_{i,m(i)}\}$ and with no loss of generality we may assume that $m(i) = n$ $i = 1,\dots,s$. Now, by the first part of our proof and Proposition 5(c) we get

$$ka = ka_1 \dots ka_s = a_1^{k^n} \dots a_s^{k^n} = (a_1 \dots a_s)^{k^n} = a^{k^n} .$$

Corollary 7. Let $a \in F(A)$ and k be a positive integer. Let $a_1 = a$ and define inductively $a_n = ka_{n-1} \cdot a_{n-1}^{-k^{n-1}}$ for $n = 2,3,\dots$ Then $a_n \in F(A)^{(n)}$.

Proposition 8. Let H be an $\mathbb{R}$-subgroup of $F(A)$ such that $F(A)^{(n)} \subset H$. Let X be a linear space and $f : H \to X$ be a group homomorphism such that $f(sh) = s^n f(h)$ for $h \in H$ and each positive $s \in \mathbb{R}$.

Then the restriction of f to $F(A)^{(n)}$ uniquely determines f.

Proof. Let $h \in H$. Define h_j $j = 1,2,\dots,n$ as in Corollary 7.

Then $h_n \in F(A)^{(n)}$ and $f(h_j) = f(kh_{j-1} \cdot h^{-k^{j-1}}) =$ $= (k^n - k^{j-1}) f(h_{j-1})$ $j = 2,3,\dots$. Hence $f(h_n) = \lambda f(h)$ where $\lambda = \prod_{j=1}^{n-1} (k^n - k^j)$. Let $b(h) = \lambda^{-\frac{1}{n}} h_n$. Then $b(h) \in F(A)^{(n)}$ and $f(h) = f(b(h))$. Hence $f = \tilde{f} \circ b$ where $\tilde{f}$ is the restriction of f to $F(A)^{(n)}$.

Proposition 9. Let for $i = 1,2$ and $n = 1,2,\ldots$ $K_{i,n}$ be R-subgroup of $F(A)$, $X_{i,n}$ be linear spaces over R and $f_{i,n} : K_{i,n} \to X_{i,n}$ be group homomorphisms such that

(a) $K_{i,n+1} = \ker f_{i,n}$

(b) $F(A)^{(n)} \subset K_{i,n}$

(c) $K_{1,n+1} \cap F(A)^{(n)} \subset K_{2,n+1} \cap F(A)^{(n)}$

(d) $f_{i,n}(sh) = s^n f_{i,n}(h)$ for positive $s \in R$ and $h \in K_{i,n}$.

Then $K_{i,n} \subset K_{2,n}$ $\quad n = 1,2,\ldots$.

Proof. Observe that (b) implies $K_{1,1} = F(A) = K_{2,1}$. Reasoning by induction assume that $K_{1,k} \subset K_{2,k}$ and let $a \in K_{1,k+1}$ i.e. $a \in K_{1,k}$ and $f_{1,k}(a) = 0$. In particular $a \in K_{1,k}$ hence also a $K_{2,k}$. Let $b = (\sqrt[n]{\lambda})^{-1} a_n$ be defined as in Proposition 8. Then $b \in K_{1,k+1} \cap F(A)^{(k)}$ hence by (c) $b \in K_{2,k+1} \cap F(A)^{(k)}$ i.e. $f_{2,k}(b) = 0$. But $f_{2,k}(a) = f_{2,k}(b)$ by Proposition 8. Hence $a \in K_{2,k+1}$.

4. The polynomial group of a Lie group

Suppose now that G is a (finite dimensional) Lie group $\Lambda(G)$ may be then identified with the Lie algebra of G and the Lie algebra structure of $\Lambda(G)$ may be derived from the topological group structure of $P(G)$ via the Trotter formulas

$$(4)\quad \begin{aligned} &(a) \quad (\varphi_1 + \varphi_2)(t) = \lim_{n\to\infty} \left(\varphi_1(\tfrac{t}{n})\, \varphi_2(\tfrac{t}{n})\right)^n \\ &(b) \quad [\varphi_1, \varphi_2](t^2) = \lim_{n\to\infty} \left(\{\varphi_1, \varphi_2\}(\tfrac{t}{n})\right)^{n^2} \end{aligned}$$

Let us note that the formulas (4) may be extrapolated to the sequence of formulas

$$(5)\quad d_k f(t^k) = \lim_{n\to\infty} (f(\tfrac{t}{n}))^{n^k}$$

so that we obtain 4(a) for $k = 1$ and $f(t) = \varphi_1(t)\cdot\varphi_2(t)$ and 4(b) for $k = 2$ and $f(t) = \{\varphi_1, \varphi_2\}(t)$. Next observe that $k!\, d_k f$ may be interpreted geometrically as the k-th derivative of

$f \in P(G)$ at 0, provided the preceeding derivatives of f at 0 vanish. More exactly $d_k f$ is the unique one-parameter subgroup of G tangent at 0 to the curve $t \to f(\sqrt[k]{t})$. Such a group exists for each analytic curve with vanishing first $k-1$ derivatives. Thus $d_1 : P(G) \to \Lambda(G)$ is a well defined-homomorphism, and inductively d_k is defined on $\ker d_{k-1}$ and it is a homomorphism. The closer examination of $P(G)$ for G a Lie group suggests the following

<u>Definition 10</u>. An R-group K is said to be an $\mathbb{R}$-Lie group provided there exist a Lie algebra $\mathfrak{k}$, an R-map $\mathrm{Exp} : \mathfrak{k} \to K$ and a sequence of homomorphisms $d_0 = 0$ $d_k : \ker d_{k-1} \to \mathfrak{k}$ $k = 1,2,\ldots$ such that

(a) Exp is an exponential R-map

(b) $\mathrm{Exp}(\mathfrak{k})$ generates K

(c) $d_1 \circ \mathrm{Exp} = \mathrm{id}_{\mathfrak{k}}$

(d) $d_k(sk) = s^k d_k(k)$ for each positive $s \in R$ and $k \in \ker d_{k-1}$

(e) $K^{(k)} \subset \ker d_k$ and $d_k(\{x_1 \ldots x_k\}) = [x_1,\ldots,x_k]$ for $x_1,\ldots,x_k \in \mathfrak{k}$ $k = 1,2,\ldots$.

(f) $\bigcap_{k=1} \ker d_k = \{e\}$.

Our observations may be now summarized in the following form

<u>Proposition 11</u>. Let G be a finite dimensional (or more general Banach-Lie) group. The polynomial group $P(G)$ is a Lie R-group. Moreover the evaluation map

(6) $\varepsilon : P(G) \ni f \longrightarrow f(1) \in G$

is an open continuous homomorphism .

<u>Proof</u>. (a) and (b) follow from the definition of $P(G)$ for any topological group G. Defining d_k by the formula (5) we obtain (c) and (d). The fact that d_k is defined on $\ker d_{k-1}$ and that $d_k : \ker d_{k-1} \to \Lambda(G)$ is a homomorphism as well as (e) and (d) may be observed using C-H-D description of the group multiplication in G .

Applying pointwise the log map to $f \in \mathcal{L}(\mathfrak{g})$ we get the exponential form of f :

(7) $$F(t) = \log f(t) = \sum_{n=1} p_n(f)\cdot t^n$$

where $p_n(f) \in T_n \cap \hat{L}$ is a Lie polynomial for $n = 1,2,\ldots$. We shall need the following properties of the coefficients p_n :

<u>Proposition 12.</u> Let $f, f_1, f_2 \in \mathcal{L}(\mathfrak{g})$. Then

(a) $p_n(sf) = s^n p_n(f)$ for $s \in R$

(b) $p_1(f_1\cdot f_2) = p_1(f_1) + p_1(f_2)$.

(c) If $p_k(f_1) = 0$ for $k \leqslant n$ or $p_k(f_2) = 0$ for $k \leqslant n$ then $p_{k+1}(f_1\cdot f_2) = p_{k+1}(f_1) + p_{k+1}(f_2)$.

(d) $p_k(f_1\cdot f_2)$ depends only on $p_j(f_i)$ for $j \leqslant k$ $i=1,2$, and is expressed in the terms of $p_j(f_i)$ using only sum, multiplication by scalars and Lie bracket operations. In particular

(8) $$p_k(\{e^{tx_1},\ldots,e^{tx_k}\}) = [x_1,\ldots,x_k]$$

(e) $P_n = \bigcap_{k\leqslant n} \ker p_k$ is a normal subgroup of $\mathcal{L}(\mathfrak{g})$ for $n = 1,2,\ldots$.

<u>Proof</u>. (a) (b) (c) and (d) are direct consequences of the formula (7) defining coefficients p_n and the Campbell-Hausdorff formula. To prove (e) observe that (b) and (c) imply that P_n is a subgroup of $\mathcal{L}(\mathfrak{g})$. Let $f(t) = \exp(F(t)) \in P_n$. Thus $F(t) = t^n h(t)$ where $h(t) \in \hat{L}$. Let us note that $\exp tx\cdot f(t)\cdot \exp(-tx) = \exp g(t)$ where

$$g(t) = \sum_{k=0} \frac{ad^k_{tx}(F(t))}{k!} = t^n\left(\sum_{k=0} \frac{ad^k_{tx}(h(t))}{k!}\right)$$

Hence $p_k(\exp g(t)) = 0$ for $k \leqslant n$ and thus $\exp g(t) \in P_n$. Since the functions $t \longrightarrow \exp tx$ with $x \in \mathfrak{g}$ generate $\mathcal{L}(\mathfrak{g})$, the group P_n is normal in $\mathcal{L}(\mathfrak{g})$ for $n = 1,2,\ldots$.

It is known (cf. [4]) that the Lie subalgebra L of T generated by $\mathfrak{g}$ is isomorphic with the free Lie algebra over $\mathfrak{g}$, i.e. each linear map from $\mathfrak{g}$ into a Lie algebra $\mathfrak{h}$ extends uniquely to a Lie algebra homomorphism from L to $\mathfrak{h}$. Let $j : L \to \mathfrak{g}$ be such homomorphism extending the identity map. For $n = 1,2,\ldots$ let $q_n = j \circ p_n$ be the composition map. Let $Q = \bigcap_{n=1} \ker q_n$. Clearly Q is a normal subgroup of $\mathcal{L}(\mathfrak{g})$.

Definition 13. Let $P(\mathfrak{g}) = \mathcal{L}(\mathfrak{g})/Q$. We shall call $P(\mathfrak{g})$ the polynomial group of a Lie algebra $\mathfrak{g}$.

Let $\pi : \mathcal{L}(\mathfrak{g}) \to P(G)$ be the quotient homomorphism. Let $d_1 : P(\mathfrak{g}) \to \mathfrak{g}$ be the homomorphism induced by q_1 i.e. such that $d_1 \circ \pi = q_1$.
Assuming that homomorphisms $d_1 \ldots d_k$ are defined in such a way that $\pi^{-1}(\ker d_j) = \ker q_j$ $j = 1,2,\ldots,k$; d_j is defined on $\ker d_{j-1}$ and $d_j \circ \pi_j = q_j$ where $\pi_j : \ker q_{j-1} \to \ker q_{j-1}/Q$ is the quotient homomorphism let us observe that $\pi^{-1}(\ker d_k) = (d_k \circ \pi_k)^{-1}\{e\} = \ker q_k$ and define $d_{k+1} : \ker d_k \to \mathfrak{g}$ to be induced by q_{k+1}, i.e. to satisfy the equality $d_{k+1} \circ \pi_{k+1} = q_{k+1}$. Define also the map $\mathrm{Exp} : \mathfrak{g} \to P(\mathfrak{g})$ as the composition $\mathrm{Exp} = \pi \circ i$ where $i(x) = e^{tx}$ for $x \in \mathfrak{g}$.

Proposition 14. $P(\mathfrak{g})$ together with the map Exp and homomorphisms d_k, $k = 1,2,\ldots$ satisfies the conditions (a) - (f) of the Definition 10, i.e. $P(\mathfrak{g})$ is an $\mathbb{R}$-Lie group.

Proof. (a) results from the identity $e^{(t_1+t_2)x} = e^{t_1x} \cdot e^{t_2x}$.
(b) and (f) are consequences of the definition of $P(\mathfrak{g})$.
(c) - (e) follow from Proposition 12.

6. Uniqueness theorem and functorial properties of polynomial group.

Theorem 15. Let H_i $i = 1,2$ be R-Lie groups with the corresponding Lie algebra $\mathfrak{h}_i$, let $\mathrm{Exp} : \mathfrak{h}_i \to H_i$ and homomorphisms $\{d_{i,n}\}_{n=1}^{\infty}$ $i = 1,2$ be as in Definition 10.

For each Lie algebra homomorphism $\varphi : \mathfrak{h}_1 \to \mathfrak{h}_2$ exists a unique $\mathbb{R}$-group homomorphism Φ extending φ i.e. such that

$$
(9) \quad
\begin{array}{lll}
\text{(i)} & \Phi \circ \mathrm{Exp} = \mathrm{Exp} \circ \varphi & \\
\text{(ii)} & \Phi(\ker d_{1,n}) \subset \ker d_{2,n} & n=1,2,\ldots \\
\text{(iii)} & \varphi \circ d_{n,1} = d_{n,2} \circ \Phi & n = 1,2,\ldots .
\end{array}
$$

Proof. Let $F(\mathfrak{h}_1)$ denotes the free R-group over the R-set $\mathfrak{h}_1$. Consider the commutative diagram

$$
(10) \qquad
\begin{tikzcd}
\mathfrak{h}_1 \arrow[rr, "\varphi"] \arrow[dr, "i"] \arrow[dd, "\mathrm{Exp}"'] & & \mathfrak{h}_2 \arrow[dd, "\mathrm{Exp}"] \\
& F(\mathfrak{h}_1) \arrow[dl, "T_1"'] \arrow[dr, "T_2"] & \\
H_1 \arrow[rr, dashed, "\Phi"] & & H_2
\end{tikzcd}
$$

where $T_i : F(\mathfrak{h}_1) \to H_i$ is for $i = 1,2$ the $\mathbb{R}$-group homomorphism induced by the $\mathbb{R}$-map $\mathrm{Exp} : \mathfrak{h}_1 \to H_1$ and $\mathrm{Exp} \circ \varphi : \mathfrak{h}_1 \to H_2$ correspondingly.

Since the condition 9 (i) determines Φ on $\mathrm{Exp}(\mathfrak{h}_1)$, and this subset generates H_1, Φ has to be unique if it exists, and has to be defined by the formula

$$
(11) \quad \Phi(\mathrm{Exp}x_1 \cdot \mathrm{Exp}x_2 \ldots \mathrm{Exp}x_n) =
$$
$$
= \mathrm{Exp}(\varphi(x_1)) \cdot \mathrm{Exp}(\varphi(x_2)) \ldots \mathrm{Exp}(\varphi(x_n))
$$
$$
\text{for } x_1, \ldots, x_n \in \mathfrak{h}_1, \quad n = 1,2,\ldots .
$$

From the diagram (10) we conclude that the necessary and sufficient condition for Φ to be well defined by (11) is the inclusion $\ker T_1 \subset \ker T_2$. To prove it, put $K_{i,1} = F(\mathfrak{h}_1)$ for $i=1,2$ and $K_{i,n+1} = T_i^{-1}(\ker d_{i,n})$ for $n = 1,2,\ldots$ $i = 1,2$. Observe that by (f) of Definition 10

$$
(12) \qquad \ker T_i = \bigcap_{n=1}^{\infty} K_{i,n} \qquad i = 1,2
$$

Let for $n = 1,2,\ldots$ and $i = 1,2$ $f_{i,n} : K_{i,n} \to \mathfrak{h}_i$ be the

R-group homomorphism defined by the formula $f_{i,n} = d_{i,n} \circ T_i$. It is easy to check that the groups $K_{i,n}$ and homomorphisms $f_{i,n}$ satisfy assumptions (a) - (d) of Proposition 9. In particular the condition (c) results from the fact that

$$(13) \qquad f_{2,n}(h) = \varphi \circ f_{1,n}(h) \quad \text{for } h \in F(\mathfrak{h}_1)^{(n)}$$

which is derived from the equalities $f_{1,n}(\{x_1,\dots,x_n\}) = [x_1,\dots,x_n]$ $f_{2,n}(x_1,\dots,x_n) = [\varphi(x_1),\dots,\varphi(x_n)]$ valid for $n = 1,2,\dots$ and each n-tuple $x_1,\dots,x_n$ of the elements of $\mathfrak{h}_1$.

Applying Proposition 9 we obtain inclusions $K_{1,n} \subset K_{2,n}$ $n = 1,2,\dots$ and hence by (12) the inclusion $\ker T_1 \subset \ker T_2$. To prove 9 (ii) observe that for $n = 1,2,\dots$

$$\phi(\ker d_{1,n}) = \phi(T_1(K_{1,n})) = T_2(K_{1,n}) \subset T_2(K_{2,n}) = \ker d_{2,n}$$

To prove 9 (iii) observe that by (13) for $n = 1,2,3,\dots$

$$d_{2,n} \circ \phi \circ T_1 = d_{2,n} T_2 = f_{2,n} = \varphi \circ f_{1,n} = \varphi \circ d_{1,n} T_1 \text{ for } h \in F(\mathfrak{h}_1)^{(n)}$$

Then Proposition 8 implies that

$$d_{2,n} \circ \phi \circ T_1 = \varphi \circ d_{1,n} \circ T_1 \qquad \text{on } T^{-1}(\ker d_{n-1})$$

hence $d_{2,n} \circ \phi = \varphi \circ d_{1,n}$ on $\ker d_{1,n}$ $n = 1,2,\dots$
This concludes the proof.

Theorem 15 easily implies the following uniqueness

<u>Corollary 16</u>. For each real Lie algebra $\mathfrak{g}$ the polynomial group $P(\mathfrak{g})$ of the algebra $\mathfrak{g}$ is the unique R-Lie group associated with $\mathfrak{g}$. In particular for a Banach-Lie group G with the Lie algebra $\mathfrak{g}$ the R-Lie groups $P(G)$ and $P(\mathfrak{g})$ are isomorphic.

<u>Corollary 17</u>. Let L be a Lie algebra. The group $P(L)$ has the following universal property: for each Lie algebra homomorphism $\psi : L \to \mathfrak{g}$ where $\mathfrak{g}$ is the Lie algebra of a connected Lie group G, there exists unique group homomorphism $\Psi : P(L) \to G$ such that $\operatorname{Exp} \circ \psi = \Psi \operatorname{Exp}$.

<u>Proof</u>. Let $P(G)$ be the polynomial group of G and let ε be

the evaluation map defined in (6). Let $\Phi : P(L) \to P(G)$ be the R-group homomorphism from Theorem 15, and define $\Psi = \varepsilon \circ \Phi$.

Concluding remarks

The universal property of the group P(L) stated in Corollary 17 may be viewed as the analogue of universal property of the enveloping algebra U(L) of L. The only difference is that the class of Lie groups in not well defined in the general setting. This justifies the title of this note.

In general one would like to say that a topological group G is Lie, provided its polynomial group P(G) is an $\mathbb{R}$-Lie group with the attached Lie algebra $\mathbb{R}$-bijective with $\mathcal{L}(G)$. The question whether this structure may be derived from some simpler axioms imposed on G is a separate problem which we shall treat elsewhere.

The Theorem 15 suggests that the category of Lie algebras over $\mathbb{R}$ is equivalent to the "category of $\mathbb{R}$-Lie groups". Unfortunately we dont know the answer to the basic question how to formulate an "inner" definition of $\mathbb{R}$-Lie group.

We purposedly left aside a variety of topological questions arising with connection of polynomial groups. We end with the following proposition

Proposition 18. Let $\mathfrak{g}$ be a topological Lie algebra. The group $P(\mathfrak{g})$ has the natural topology W of a topological R-group. It is the weakest of R-group topologies on $P(\mathfrak{g})$ for which all the maps d_n are continuous.

Proof. The topology W may be obtained via the injective map $P(\mathfrak{g}) \ni a \xrightarrow{i} \{d_n(a)\} \in \prod_{n=1} \mathfrak{g}$ of $P(\mathfrak{g})$ into the topological product of the countable number of copies of $\mathfrak{g}$. The fact that W is the R-group topology results easily from (a) (b) and (d) of Proposition 12.

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