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On the conformal relation between twistors and Killing spinors


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1. Introduction.

We consider a Riemannian spin manifold \((M^n, g)\) of dimension \(n \geq 3\) and denote by \(S\) the spinor bundle. The kernel of the Clifford multiplication \(T \otimes S \rightarrow S\) is a subbundle of \(T \otimes S\) and there exists a projection of \(T \otimes S\) onto this bundle given by the formula

\[
p(X \otimes \psi) = X \otimes \psi + \frac{1}{n} \sum_{\alpha=1}^{n} e_{\alpha} \otimes e_{\alpha} \cdot X \cdot \psi,
\]

where \(X \cdot \psi\) denotes the Clifford multiplication of the vector \(X\) by \(\psi\). The twistor operator \(\mathcal{D}\) is defined as the composition of the covariant derivative \(\nabla\) and the projection \(p\)

\[
\mathcal{D} = p \circ \nabla : \Gamma(S) \xrightarrow{\nabla} \Gamma(T \otimes S) \xrightarrow{p} \Gamma(T \otimes S)
\]

(see [1]). Let \(D\) by the Dirac operator acting on sections of the bundle \(S\). Then we have the following formula for the operator \(\mathcal{D}\)

\[
\mathcal{D}\psi = \sum_{\alpha=1}^{n} e_{\alpha} \otimes (\nabla e_{\alpha} \psi + \frac{1}{n} e_{\alpha} \cdot D\psi).
\]

The kernel of the twistor operator is given by the equation

\[
\nabla_X \psi + \frac{1}{n} X \cdot D\psi = 0 \quad (1.1.)
\]

for any vector \(X \in T\). A more symmetric and equivalent form of this equation is

\[
X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi = \frac{2}{n} g(X, Y) D\psi.
\]

\(\mathcal{D}\) is a conformally invariant operator. In particular, if \(\bar{g} = \lambda g\) is a conformal change of the metric and \(\bar{S} \rightarrow S\)
denotes the natural isomorphism of the spin bundles, then $\psi$ belongs to the kernel of $\mathcal{D}$ if
\[ \lambda^{\frac{1}{4}} \psi \in \text{Kernel of } \mathcal{D} \] (see [2], [15]). On the other hand, the equation for Killing spinors is given by
\[ \nabla_X \psi + \frac{a}{n} X \cdot \psi = 0 \] (1.2.)
where $a \neq 0$ is a complex number. It is well known (see [7]) that if a Riemannian manifold has a non-trivial Killing spinor, then it must be an Einstein space with scalar curvature $R = \frac{4(n-1)}{n} a^2$. If $a$ is a real (imaginary) number, we call $\psi$ a real (imaginary) Killing spinor. Any Killing spinor is a twistor spinor, i.e. it belongs to the kernel of the twistor operator. In small dimensions we know many spaces with real Killing spinors (see [6],[7],[8],[9],[10],[11], [12],[13]), and there is a classification of complete Riemannian manifolds with imaginary Killing spinors (see [3], [4],[5]).

On the space $\text{Ker}(\mathcal{D})$ of all twistor spinors we have an invariant of order two, namely
\[ C_\psi := \text{Re} \langle D\psi, \psi \rangle \]
(see [14]). In this paper we observe that
\[ Q_\psi := |\psi|^2 |D\psi|^2 - C_\psi^2 - \sum_{\alpha=1}^{n} \left( \text{Re} \langle D\psi, e^{\alpha} \psi \rangle \right)^2 \geq 0 \]
is an invariant of order four on $\text{Ker}(\mathcal{D})$, too. Using the first integral on $\text{Ker}(\mathcal{D})$ we show in particular that a Riemannian manifold $(M^n, g)$ with a nowhere vanishing twistor spinor $\psi$ is conformally equivalent to a space $(M^n, \tilde{g})$ with non-negative scalar curvature
\[ \tilde{R} = \frac{4(n-1)}{n} (C_\psi^2 + Q_\psi) . \]
Moreover, we study the set $N_\psi = \{ m \in M^n : \psi(m) = 0 \}$ of all zeros of a twistor. It turns out that $N_\psi$ is a discrete subset of $M^n$. Finally we investigate the question under which conditions a twistor spinor can be conformally deformed into a Killing spinor. For example, $\psi \in \text{Ker}(\mathcal{D})$ can be conformally
deformed into a real Killing spinor if and only if $Q_{\psi} = 0$ and $C_{\psi} \neq 0$. Similar characterizations we obtain in the imaginary case, too.

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2. The first integral $Q_{\psi}$ on $\text{Ker}(\bar{\nabla})$.

First we collect some formulas that are valid for any twistor spinor $\psi \in \text{Ker}(\bar{\nabla})$. A general reference is the paper [14]. Any twistor spinor satisfies

$$D^2 \psi = \frac{n}{4(n-1)} R \psi$$

(2.1.)

and

$$\nabla X (D \psi) = \frac{n}{2(n-2)} (\frac{R}{2(n-1)} X - \text{Ric}(X)) \cdot \psi$$

(2.2.)

where $\text{Ric}: T \rightarrow T$ is the Ricci tensor of the space. Moreover, if $u = |\psi|^2$ denotes the square of the length of $\psi$, we have

$$\frac{n}{2} \Delta u = \frac{n}{4(n-1)} u - \langle D \psi, D \psi \rangle.$$  

(2.3.)

We denote by $S$ the $(1,1)$-tensor

$$S(X) := \frac{1}{n-2} (\frac{R}{2(n-1)} X - \text{Ric}(X))$$

and we consider the vector bundle $E = S \oplus S$ as well as the connection $\nabla^E$ in $E$ defined by the formula

$$\nabla^E_X := \nabla_X + \left( \begin{array}{cc} 0 & \frac{1}{n} X \\ -\frac{n}{2} S(X) & 0 \end{array} \right)$$

The twistor equation (1.1.) and formula (2.2.) show that

$$\nabla^E_X (D \psi) = 0$$

holds for any solution of the twistor equation.

Conversely, if

$$\nabla^E_X (\psi) = 0,$$

then $\varphi = D \psi$ and $\psi$ is a twistor spinor.

Consequently, the twistor spinors $\psi \in \text{Ker}(\bar{\nabla})$ correspond to the
\( E \)-parallel sections of the bundle \( E \) and we obtain in particular

**Proposition 1:** Let \((M^n, g)\) be a connected Riemannian manifold of dimension \( n \geq 3 \). The kernel of the twistor operator \( \mathcal{J} \) is a finite-dimensional space,

\[
\dim \ker(\mathcal{J}) \leq 2^{\lfloor \frac{n}{2} \rfloor + 1}.
\]

In particular, a twistor spinor \( \psi \) is defined by its values \( \psi(m_0), D\psi(m_0) \) at some point.

**Remark:** We understand the Weyl-tensor \( W \) of the Riemannian manifold in the usual way as a 2-form with values in the bundle \( \text{End}(S) \):

\[
W(x, y) \cdot \psi = \sum_{i, j} g(W(x, y) e_i, e_j) e_i \cdot e_j \cdot \psi.
\]

An easy computation yields now the following formula for the curvature tensor \( R^E \) of the connection \( \nabla^E \):

\[
R^E(x, y)(\psi) = \left( \frac{1}{4} W(x, y) \cdot \psi, \frac{1}{4} W(x, y) \cdot \psi + \frac{n}{2} ((\nabla Y S)(x) - (\nabla X S)(y)) \cdot \psi \right).
\]

**Example 1:** We consider a 3-dimensional Riemannian manifold \((M^3, g)\). The Weyl tensor vanishes, \( W = 0 \), and consequently we obtain the integrability condition

\[
(\nabla_X S)(Y) - (\nabla_Y S)(X) = 0,
\]

i.e. the space is locally conformally flat (see [16]). Therefore, if a 3-dimensional Riemannian manifold \((M^3, g)\) admits a non-trivial solution of the twistor equation, then \((M^3, g)\) is locally conformally flat.

**Example 2:** Denote by \( \Delta_n \) the usual \( \text{Spin}(n) \)-module and consider a twistor spinor \( \psi: \mathbb{R}^n \rightarrow \Delta_n \) on the flat Euclidean space. According to equation (2.2.) we have \( \nabla(D\psi) = 0 \), i.e.

\( D\psi = \psi \) is constant. Now we integrate the twistor equation

\[
0 = \nabla_T \psi + \frac{1}{n} T \cdot D\psi = \nabla_T \psi + \frac{1}{n} T \cdot \psi_1
\]

along the line \( \{sx: 0 \leq s \leq 1\} \) and obtain
\[ \psi(x) - \psi(0) = -\frac{1}{n} x \cdot \psi_1. \]

Consequently, the kernel of the twistor equation is given by the spinors \( \psi : \mathbb{R}^n \rightarrow \Delta_n \)
\[ \psi(x) = \psi_0 - \frac{1}{n} x \cdot \psi_1, \quad x \in \mathbb{R}^n, \]
with \( \psi_0, \psi_1 \in \Delta_n \). In particular we have
\[ \dim_{\mathbb{C}} \ker(\tilde{\psi}) = 2\left[\frac{n}{2}\right] + 1. \]

**Proposition 2:** Let \((M^n, g)\) be a connected Riemannian manifold and \( \psi \not\equiv 0 \) a twistor spinor. Then \( N_\psi = \{ m \in M^n : \psi(m) = 0 \} \) is a discrete subset of \( M^n \).

**Proof:** Suppose \( \psi(m) = 0 \). Using formula (2.2.) we have
\[ \nabla(D\psi)(m) = 0. \]

With respect to
\[ (YXu)(m) = 2(Y(\nabla_X \psi, \psi))(m) = -\frac{2}{n} (Y(X \cdot D\psi, \psi))(m) = \]
\[ = \frac{2}{n^2} (X \cdot D\psi, Y \cdot D\psi)(m) = \frac{2}{n^2} g(X,Y) |D\psi(m)|^2 \]
we see that the Hessian of the function \( u = |\psi|^2 \) at the point \( m \in M^n \) is given by
\[ \text{Hess}_m u(X,Y) = \frac{2}{n^2} g(X,Y) |D\psi(m)|^2. \]

In case \( D\psi(m) \neq 0 \), \( m \) is a non-degenerate critical point of \( u \) and consequently an isolated zero point of \( \psi \). In case \( D\psi(m) = 0 \), we obtain \( \psi \equiv 0 \) by proposition 1.

We consider now a geodesic \( \gamma(t) \) in \( M^n \) and a twistor spinor \( \psi \). Denote by \( u(t), v(t) \) the functions \( u(\gamma(t)), |D\psi|^2(\gamma(t)) \).

Moreover, we introduce the functions
\[ f_1(t) = g(S(\dot{\gamma}(t)), \dot{\gamma}(t)) \]
\[ f_2(t) = \frac{n^2}{2} |S(\dot{\gamma}(t))|^2. \]

Using the twistor equation as well as formula (2.2.) we obtain
\[ \frac{d^2 u(t)}{dt^2} = f_1(t)u(t) + \frac{n^2}{2} v(t) \]
\[ \frac{d^2 v(t)}{dt^2} = f_2(t)u(t) + \frac{n^2}{2} \frac{df_1(t)}{dt} \frac{du(t)}{dt} + f_1(t)v(t) \]

(2.4.)

**Proposition 3:** Let \( \psi \not\equiv 0 \) be a twistor spinor and denote by \( \chi: [0,T] \to M^n \) a geodesic joining of two zero points of \( \psi \). Then

a.) \( \text{Ric}(\chi) \) is parallel to \( \dot{\chi} \).
b.) \( \text{grad} \ u \) is parallel to \( \dot{\chi} \).
c.) \( \frac{dv}{dt} = \frac{n^2}{2} g(S(\chi), \dot{\chi}) \frac{du}{dt} \).
d.) \( u \cdot v = \frac{n^2}{4} \left( \frac{du}{dt} \right)^2 \).

**Proof:** Using the notation introduced before we have

\[ u(0) = \frac{du}{dt} (0) = \frac{dv}{dt} (0) = 0, \quad v(0) > 0 \]
\[ u(T) = \frac{du}{dt} (T) = \frac{dv}{dt} (T) = 0, \quad v(T) > 0. \]

Since \( u(t) \) and \( v(t) \) satisfy the equations (2.4.), we obtain

\[ \frac{d}{dt} \left( \frac{dv}{dt} - \frac{n^2}{2} f_1 \frac{du}{dt} \right) = (f_2 - \frac{n^2}{2} f_1^2) u. \]

If \( f_2 - \frac{n^2}{2} f_1^2 \not\equiv 0 \) on the interval \([0,T]\), we have

\[ 0 = \frac{dv}{dt} (T) - \frac{n^2}{2} f_1(T) \frac{du}{dt} (T) = \int_0^T (f_2 - \frac{n^2}{2} f_1^2) > 0 \]

because \( f_2 - \frac{n^2}{2} f_1^2 \geq \frac{n^2}{2} (|S(\chi)|^2 - g(S(\chi), \chi)^2) > 0 \),

a contradiction. In case \( f_2 - \frac{n^2}{2} f_1 \equiv 0 \), \( \text{Ric}(\chi) \) is parallel to \( \dot{\chi} \) and \( \frac{dv}{dt} = \frac{n^2}{2} g(S(\chi), \dot{\chi}) \frac{du}{dt} \).

Moreover, we calculate

\[ \frac{d}{dt}(u \cdot v - \frac{n^2}{4} \left( \frac{du}{dt} \right)^2) = \frac{du}{dt} v + u \frac{dv}{dt} - \frac{n^2}{2} \frac{du}{dt} \frac{d^2 u}{dt^2} = \]
\[ = \frac{du}{dt} v + \frac{n^2}{2} f_1 u \frac{du}{dt} - \frac{n^2}{2} \frac{du}{dt} \left( f_1 u + \frac{2}{n^2} v \right) = 0, \]
i.e. $uv = \frac{n^2}{4} (\frac{du}{dt})^2$. Since $\psi$ is a twistor spinor vanishing at some point, we have

$$uD\psi = \frac{n}{2} \text{grad} u \cdot \psi$$

This implies $u \cdot v = \frac{n^2}{4} |\text{grad} u|^2$ and consequently

$$|\text{grad} u|^2 = (\frac{du}{dt})^2,$$

i.e. the gradient of $u$ is parallel to $\dot{\gamma}$.

**Proposition 4:** Let $(M^n, g)$ be a complete connected Riemannian manifold and suppose that the $(1,1)$-tensor

$$S := \frac{1}{n-2} \left( \frac{R}{2(n-1)} - \text{Ric} \right)$$

is non-negative. Then any twistor spinor $\psi \neq 0$ vanishes at most at one point.

**Proof:** Suppose $u(p_1) = 0 = u(p_2)$, $p_1 \neq p_2$ and consider a geodesic $\gamma : [0,T] \to M^n$ from $p_1$ to $p_2$. Then

$$\frac{d^2}{dt^2} u(t) = f_1(t)u(t) + \frac{2}{n} v(t) \geq 0$$

since $S$ is non-negative. With respect to $u(0) = u(T) = 0$ and $\frac{du}{dt}(0) = \frac{du}{dt}(T) = 0$ we conclude $u(T) \equiv 0$ on $[0,T]$, i.e. $\psi$ vanishes on the curve $\dot{\gamma}(t)$, a contradiction to proposition 2.

**Example 2:** The condition $S \geq 0$ is satisfied in particular if $(M^n, g)$ is an Einstein space with scalar curvature $R \leq 0$. On the Euclidean space $\mathbb{R}^n$ and on the hyperbolic space $H^n$ there exist twistor spinors vanishing at some point (see example 2).

We denote by $(\psi, \phi)$ the real part $\text{Re}<\psi, \phi>$ of the Hermitian product of two spinors. Given an arbitrary spinor $\psi \in \Gamma(S)$ we define the function $Q_\psi$ by the formula

$$Q_\psi = |\psi|^2 |D\psi|^2 - (D\psi, \psi)^2 - \frac{1}{n} \sum_{\alpha=1}^{n} (D\psi, e_\alpha \cdot \psi)^2.$$  

Denote by $V_\psi$ the real subspace of $S$ given by

$$V_\psi = \{X \cdot \psi : X \in T\}.$$ 

Then we have

$$Q_\psi = u \cdot \text{dist}^2(D\psi, \text{Lin}_R(\psi, V_\psi)).$$
Proposition 5: If \( \psi \in \text{Ker}(\mathcal{D}) \) is a twistor spinor, then \( Q_\psi \) is constant.

Proof: Since \((D\psi, \psi)\) is constant for \( \psi \in \text{Ker}(\mathcal{D}) \) (see [14]), we have

\[
\nabla_X(Q_\psi) = 2(\nabla_X(\psi),\psi)D\psi + 2u(\nabla_X(D\psi),D\psi) - 2 \sum_{\alpha=1}^{n}(D\psi,e_{\alpha} \cdot \psi)(\nabla_X(D\psi),e_{\alpha} \cdot \psi) - 2 \sum_{\alpha=1}^{n}(D\psi,e_{\alpha} \cdot \psi)(D\psi,e_{\alpha} \cdot \nabla_X \psi).
\]

Using the twistor equation (1.1.) and formula (2.2.) we obtain, with respect to

\[
\sum_{\alpha=1}^{n}(e_{\alpha} \cdot \psi,D\psi)(e_{\alpha} \cdot X \cdot D\psi,D\psi) = -(X \cdot \psi,D\psi)|D\psi|^2
\]

\[
\sum_{\alpha=1}^{n}(e_{\alpha} \cdot \psi,D\psi)(e_{\alpha} \cdot \psi,Z \cdot \psi) = (Z \cdot \psi,D\psi)|\psi|^2,
\]

that \( \nabla_X(Q_\psi) = 0 \) immediately.

Remark: For any twistor spinor \( \psi \) let us introduce the vector field

\[
T_\psi = 2 \sum_{\alpha=1}^{n}(\psi,e_{\alpha} \cdot D\psi)e_{\alpha}.
\]

Then we have

\[
T_\psi = -n \text{grad } u
\]

(see [14]) and an elementary calculation provides the formula

\[
|C_\psi \cdot \psi - uD\psi - \frac{1}{2} T_\psi \cdot \psi|^2 = u Q_\psi.
\]

In particular, if \( \psi \) is a twistor spinor such that \( C_\psi = 0 = Q_\psi \), then

\[
uD\psi = \frac{n}{2} \text{grad}(u) \cdot \psi
\]

holds.

Proposition 6: Let \((M^n,g)\) be a Riemannian manifold with a twistor spinor \( \psi \) such that \( C_\psi = 0 = Q_\psi \) and suppose that \( \psi \) does not vanish at any point. Then \((M^n,g)\) is conformally equivalent to a Ricci-flat space \((\tilde{M}^n,\tilde{g})\) with parallel spinor.
Proof: Consider the metric \( \bar{g} = \frac{1}{u^2} g, \) \( u = |\psi|^2. \) Using the identification \( \bar{S} \rightarrow S \) of the spin bundles we have (see [2])

\[
\bar{\nabla}_t \left( \frac{1}{\sqrt{u}} \bar{\psi} \right) = \frac{u \nabla_t \left( \frac{1}{\sqrt{u}} \psi \right)}{\sqrt{u}} + \frac{1}{2} t \cdot \text{grad}(u) \cdot \frac{1}{\sqrt{u}} \psi + \frac{1}{2} du(t) \frac{1}{\sqrt{u}} \bar{\psi} = \frac{1}{\sqrt{u}} \left\{ u \nabla_t \psi + \frac{1}{2} t \cdot \text{grad}(u) \cdot \psi \right\}.
\]

According to \( C_{\psi} = 0 = Q_{\psi} \) we can apply equation (2.6.) and then, from the twistor equation

\[
0 = \nabla_t \psi + \frac{1}{n} t \cdot D\psi = \nabla_t \psi + \frac{1}{2u} t \cdot \text{grad}(u) \cdot \psi,
\]

it results that \( \nabla_t \left( \frac{1}{\sqrt{u}} \bar{\psi} \right) = 0, \) i.e. \( \frac{1}{\sqrt{u}} \bar{\psi} \) is a parallel spinor with respect to the metric \( \bar{g}. \)

Corollary 1: Let \( (M^n, g) \) be a Riemannian manifold that is not conformally equivalent to a space \( (M^n, \bar{g}) \) with parallel spinor. Then a twistor spinor \( \psi \in \text{Ker}(\bar{\mathcal{D}}) \) vanishes at some point if and only if \( C_{\psi} = 0 = Q_{\psi} \).

For any twistor spinor \( \psi \) we introduce the function

\[
H_{\psi} = \text{dist}^2(i\psi, V_{\psi})
\]

defined on the set \( \{ m \in M^n : \psi(m) \neq 0 \}. \)

Proposition 7: Let \( \psi \) be a twistor spinor satisfying \( C_{\psi} = 0 = Q_{\psi} \). Then

\[
\frac{H_{\psi}}{d}
\]

is constant.

Proof: The derivative of the function \( f = \sum_{\alpha=1}^{n} (i\psi, e_\alpha \cdot \psi)^2 \) is given by

\[
\text{df}(X) = -\frac{4}{n} \sum_{\alpha=1}^{n} (i\psi, e_\alpha \cdot \psi)(i\psi, e_\alpha \cdot X \cdot D\psi).
\]

Since \( C_{\psi} = 0 = Q_{\psi} \), we have \( uD\psi = \frac{n}{2} \text{grad}(u) \cdot \psi \) and consequently

\[
\text{df} = \frac{2}{u} f \, du.
\]

Finally we obtain
Let $f: M^n \rightarrow \mathbb{C}$ be a complex valued function on $M^n$ and consider the equation
\[ \nabla_X \psi + \frac{f}{n} X \cdot \psi = 0. \]
A. Lichnerowicz (see [14]) proved that if $\psi \neq 0$ and $\text{Re}(f) \neq 0$, then $\text{Re}(f)$ is constant and $\psi$ is a real Killing spinor. We consider now the case $f = ib$.

**Proposition 8:** If $\nabla_X \psi + \frac{ib}{n} X \cdot \psi = 0$ with a real function $b: M^n \rightarrow \mathbb{R}$, then
a.) $u \cdot H_\psi$ is constant
b.) $Q_\psi = b^2 u H_\psi$.

**Proof:** Suppose $\nabla_X \psi + \frac{ib}{n} X \cdot \psi = 0$. Then $D\psi = ib\psi$ and we obtain $Q_\psi = b^2 u H_\psi$ by definition of $Q_\psi$. Since
\[ u H_\psi = u^2 - \sum_{\alpha=1}^{n} (i\psi, e_\alpha \cdot \psi)^2 \]
we calculate
\[ \nabla_X(u H_\psi) = 4u(\nabla_X \psi, \psi) - 2 \sum_{\alpha=1}^{n} (i\psi, e_\alpha \cdot (i \nabla_X \psi, e_\alpha \cdot \psi) \]
\[ - 2 \sum_{\alpha=1}^{n} (i\psi, e_\alpha \cdot \psi)(i\psi, e_\alpha \cdot \nabla_X \psi) = \]
\[ = - \frac{4b}{n} u (iX \cdot \psi, \psi) - \frac{2b}{n} \sum_{\alpha=1}^{n} (i\psi, e_\alpha \cdot \psi)(X \cdot \psi, e_\alpha \cdot \psi) \]
\[ + \frac{2b}{n} \sum_{\alpha=1}^{n} (i\psi, e_\alpha \cdot \psi)(i\psi, e_\alpha \cdot X \cdot \psi) = \]
\[ = \frac{4b}{n} u (i\psi, X \cdot \psi) - \frac{2b}{n} (i\psi, X \cdot \psi) - \frac{2b}{n} (i\psi, X \cdot \psi) = \]
\[ = 0, \]
i.e. $|\psi|^2 H_\psi$ is constant.

**Corollary 2:** If $\psi$ is a solution of the equation $\nabla_X \psi + \frac{ib}{n} X \cdot \psi = 0$ and $Q_\psi \neq 0$, then $b$ is constant and $\psi$ is an imaginary Killing spinor.

**Corollary 3:** If $\psi$ is a non-trivial solution of the equation $\nabla_X \psi + \frac{ib}{n} X \cdot \psi = 0$ and $Q_\psi = 0$, then $\frac{1}{iu} \psi$ is a parallel
spinor with respect to the metric $\bar{g} = \frac{1}{u^2} g$.

**Proposition 9:** Let $\psi \in \text{Ker}(\mathcal{D})$ be a twistor spinor and denote by $u$ the square of the length of $\psi$, $u = |\psi|^2$. Then $u$ is a solution of the following equation

$$\frac{nR}{4(n-1)} u^2 = C_\psi^2 + Q_\psi + \frac{n}{2} u^2 \Delta (\ln u) + \frac{n(n-2)}{4} u^2 |\text{grad}(\ln u)|^2.$$

**Proof:** We consider the vector field

$$T_\psi = 2 \sum_{\alpha=1}^{n} (\psi, e_\alpha \cdot D\psi) e_\alpha.$$

Then we have

$$Q_\psi = u |D\psi|^2 - C_\psi^2 - \frac{1}{4} |T_\psi|^2$$

and

$$\text{grad} u = -\frac{1}{n} T_\psi$$

(see [14]). Consequently we obtain by equation (2.3.)

$$\Delta (\ln u) = \frac{1}{u^2} |\text{grad} u|^2 + \frac{1}{u} \Delta(u) = \frac{1}{u^2 n^2} |T_\psi|^2 + \frac{R}{2(n-1)} - \frac{2}{n \cdot u} |D\psi|^2$$

$$|\text{grad}(\ln u)|^2 = \frac{1}{u^2 n^2} |T_\psi|^2.$$

Finally we have

$$\frac{n}{2} u \Delta (\ln u) + \frac{n(n-2)}{4} u |\text{grad}(\ln u)|^2 =$$

$$= \frac{1}{4u} |T_\psi|^2 + \frac{nR}{4(n-1)} u - |D\psi|^2 =$$

$$= \frac{1}{4u} |T_\psi|^2 + \frac{nR}{4(n-1)} u - \frac{C_\psi^2 + Q_\psi}{u} - \frac{1}{4u} |T_\psi|^2 =$$

$$= \frac{nR}{4(n-1)} u - \frac{C_\psi^2 + Q_\psi}{u}$$

and this is the equation we claimed.

**Theorem 1:** Let $(M^n, g)$ be a Riemannian spin manifold of dimension $n \geq 3$ with a nowhere vanishing twistor spinor $\psi$. The Riemannian metric

$$\bar{g} = \frac{1}{|\psi|^4} g$$
has constant and non-negative scalar curvature

$$\bar{R} = \frac{4(n-1)}{n}(c_n^2\bar{Q})$$.

Proof: Denote by $h := u^{-n/2+1}$. Then $h^{n-2} = \frac{1}{u^2}$ and the metrics $\bar{g}$ and $g$ are related by

$$\bar{g} = h^{n-2}g.$$ 

Then the scalar curvatures are related by the formula

$$\frac{4}{\bar{R}} h^{n-2} = \frac{4(n-1)}{n-2} \frac{\Delta h}{h} + R.$$ 

The result follows now by a direct calculation using the formula of proposition 9.

3. The conformal deformation of twistor spinors into Killing spinors

Consider a Riemannian spin manifold $(M^n, g)$ and a twistor spinor $\psi \in \text{Ker}(\mathcal{D})$. We say that $\psi$ is conformally equivalent to a Killing spinor if there exists a conformal change of the metric $\bar{g} = \lambda g$ such that $\lambda^4 \psi$ is a Killing spinor with respect to the metric $\bar{g}$. We introduce the function $f = \frac{1}{2} \lambda^{-\frac{1}{2}}$. Then the equation

$$\nabla_X(\lambda^4 \psi) + \frac{a}{n} X \cdot (\lambda^4 \psi) = 0$$

becomes equivalent to

$$a \psi - 2fD \psi + n \text{grad}(f) \cdot \psi = 0.$$ (3.1.)

Indeed, with respect to $\nabla_X \psi + \frac{1}{n} X \cdot D \psi = 0$ we use only the well-known formulas describing the change of the covariant derivative (see [2]) in order to derive (3.1.). Consequently, $\psi$ is conformally equivalent to a Killing spinor iff (3.1.) has a positive solution $f$ for some constant $0 \neq a \in \mathbb{C}$.

Theorem 2: Let $(M^n, g)$ be a Riemannian spin manifold and $0 \neq \psi \in \text{Ker}(\mathcal{D})$ a twistor spinor. Then $\psi$ is conformally
equivalent to a real Killing spinor if and only if $C\psi \neq 0$ and $Q\psi = 0$. In this situation there exists up to a constant precisely one metric

$$\tilde{g} = \frac{1}{|\psi|^4} g$$

with respect to which $\psi$ becomes a Killing spinor.

Proof: Let $0 \neq a$ be a real number and suppose $f$ is a solution of (3.1.). Then

$$a|\psi|^2 - 2f C\psi = 0$$

and, consequently, $C\psi \neq 0$. Moreover, in this case we have

$$\text{dist}(D\psi, \text{Lin}_R(\psi, \nu_\psi)) = 0,$$

i.e. $Q\psi = 0$. Conversely, suppose $C\psi \neq 0$ and $Q\psi = 0$. Again we consider the vector field $T^\psi$ defined by

$$T^\psi = 2 \sum_{\alpha=1}^{n} (\psi, e_\alpha \cdot D\psi) e_\alpha.$$

With respect to (2.5.) we have

$$C\psi \psi - u D\psi - \frac{1}{2} T^\psi \psi = 0,$$

and since $\psi$ is a twistor spinor, we know

$$T^\psi = - \nabla \text{grad}(u).$$

Consequently

$$C\psi \psi - u D\psi + \frac{n}{2} \text{grad}(u) \cdot \psi = 0,$$

i.e. $f = \frac{u}{2}$ is a solution of (3.1.). Finally we remark that any solution $f^*$ of (3.1.) is proportional to $u$ in case $a \in \mathbb{R}$ since $f^*$ must satisfy the relation $a \cdot u - 2f^* C\psi = 0$.

For an arbitrary spinor field $\psi$ we introduce the 1-form

$$\eta_\psi(X) = \frac{1}{i} \langle X \cdot \psi, \psi \rangle = \text{Im} \langle X \cdot \psi, \psi \rangle.$$

**Theorem 3**: Let $(M^n, g)$ be a Riemannian spin manifold and $0 \neq \psi$ a twistor spinor with $Q\psi = 0$. Then $\psi$ is conformally equivalent to an imaginary Killing spinor if and only if

a.) $C\psi = 0$, $H\psi \equiv 0$
b.) \( \frac{\eta_{\psi}}{|\psi|^4} \) is the differential of a positive function. In this situation, for any function \( k > 0 \) with \( \frac{\eta_{\psi}}{|\psi|^4} = dk \), the twistor spinor becomes a Killing spinor in the metric 
\[ \tilde{g} = \frac{1}{|\psi|^4k^2} g. \]

Proof: By equation (2.5.) \( Q_{\psi} = 0 \) implies
\[ u D\psi = C_{\psi} \psi + \frac{n}{2} \text{grad}(u) \cdot \psi. \]

Suppose now that
\[ \pm i\psi - 2f D\psi + n \text{grad}(f) \cdot \psi = 0 \]
has a solution \( f > 0 \). Then \( -2f(D\psi, \psi) = 0 \), i.e. \( C_{\psi} = 0 \) and
\[ u D\psi = \frac{n}{2} \text{grad}(u) \cdot \psi. \]
This implies \( D\psi \in V_{\psi} \) and, finally, \( i\psi \in V_{\psi} \). Thus we have the necessary condition \( H_{\psi} \equiv 0 \).

Furthermore, we calculate \( \eta_{\psi} \) and obtain
\[ \eta_{\psi}(x) = -\langle i X \cdot \psi, \psi \rangle = \pm \langle 2f X \cdot D\psi - nX \cdot \text{grad}(f) \cdot \psi, \psi \rangle = \pm \frac{\eta_{\psi}}{u^2} \]
\[ = \pm \frac{n}{u^2} \left\{ -f \text{du}(x) + df(x) \right\} \]
\[ \frac{\eta_{\psi}}{u^2} = \pm \frac{n}{u^2} df(x), \]
i.e. \( \pm \frac{\eta_{\psi}}{u^2} \) is the differential of a positive function.

Conversely, suppose \( Q_{\psi} = 0 \), \( C_{\psi} = 0 \), \( H_{\psi} \equiv 0 \) and \( \pm \frac{\eta_{\psi}}{u^2} = dk \).

Then \( f = uk \) is a solution of equation (3.1.). Indeed, we have
\[ u D\psi = \frac{n}{2} \text{grad}(u) \cdot \psi \] (since \( C_{\psi} = 0 = Q_{\psi} \)) and, consequently, we obtain with \( f = uk \)
\[ -2f D\psi + n \text{grad}(f) \cdot \psi = n \cdot u \text{grad}(k) \cdot \psi = \]
\[ = n \cdot u \cdot \sum_{\alpha=1}^{n} \text{dk}(e_{\alpha})e_{\alpha} \cdot \psi = \]
\[ = \pm n \cdot \frac{1}{u} \sum_{\alpha=1}^{n} (ie_{\alpha} \psi, \psi)e_{\alpha} \psi = \]
\[ = \pm ni \psi. \]

The latter equation follows from \( H_{\psi} \equiv 0 \), i.e. \( i\psi \in V_{\psi} \).
Theorem 4: Let $M^n$, $g$ be a Riemannian spin manifold and $0 \neq \psi$ a twistor spinor with $Q_\psi \neq 0$. Then $\psi$ is conformally equivalent to an imaginary Killing spinor if and only if

$$c_\psi = 0$$

and

$$\text{dist}^2(D_\psi, \text{Lin}_R(i\psi, V_\psi)) = 0.$$ 

In this case there exists exactly one positive function $k$ with $\frac{\eta \psi}{u^2} = dk$ such that $\psi$ becomes a Killing spinor with respect to the metric

$$\tilde{g} = \frac{1}{|\psi|^4k^2}g.$$ 

Proof: Suppose $Q_\psi \neq 0$ and that equation (3.1.) has a positive solution $f$ for some imaginary number $a$. Then

$$\text{dist}^2(D_\psi, \text{Lin}_R(i\psi, V_\psi)) = 0$$

and $c_\psi = 0$, and we obtain the necessary conditions mentioned above. On the other hand, if $Q_\psi \neq 0$, $c_\psi = 0$ and

$$\text{dist}(D_\psi, \text{Lin}_R(i\psi, V_\psi)) = 0,$$

then there exist a function $A$ and a vector field $\xi$ such that $D_\psi = A\psi + \xi \cdot \psi$. Using the twistor equation $\nabla_x \psi = -\frac{1}{n} \xi \cdot D_\psi$ we obtain

$$\nabla_x (D_\psi) = \left\{ \frac{A^2}{n} X + \nabla_x \xi + \frac{2}{n} \langle \xi, X \rangle \xi - \frac{1}{n} |\xi|^2 X \right\} \psi.$$ 

With respect to $Q_\psi \neq 0$ we know that $i\psi$ is linearly independent (over $R^4$) of $V_\psi$. The latter formula as well as formula (2.2.) yield now

$$dA(X) = -\frac{2}{n} A \langle \xi, X \rangle.$$ 

Consider the function $f := \frac{1}{2} \frac{1}{|A|}$ (since $Q_\psi \neq 0$, $A$ cannot vanish). Then we have

$$\text{grad}(f) = \frac{1}{n} \frac{1}{|A|} \xi = \frac{1}{n} 2f \xi$$

and, consequently,

$$D_\psi = A\psi + \xi \cdot \psi = \text{sgn}(A) \frac{1}{2f} i\psi + \frac{n}{2f} \text{grad}(f) \cdot \psi,$$

i.e. $f$ is a solution of equation (3.1.) and the corresponding conformally equivalent metric is given by
\[ \bar{g} = A^2 g. \]

Furthermore, \( D\psi = Ai\psi + \xi \cdot \psi \) implies
\[ \frac{n}{2} du = A \gamma_\psi + u \xi = A \gamma_\psi - \frac{n}{2} u \frac{dA}{A} \]
and, finally,
\[ -\frac{n}{2} d\left( \frac{1}{Au} \right) = \frac{\eta_\psi}{u^2}. \]

This means that \( A^2 \) is given by \( A^2 = \frac{1}{u^2 k^2} \) for a unique function \( k > 0 \) satisfying \( dk = \frac{\eta_\psi}{u^2}. \)

On a manifold of dimension \( n = 3, 5 \) we have \( H_\psi \equiv 0 \) for an arbitrary spinor field. Therefore Theorem 3 and Theorem 4 provide the following Corollary:

Let \((M^n, g)\) be a Riemannian spin manifold of dimension \( n=3, 5 \) and let \( \psi \in \text{Ker}(\mathfrak{D}) \) be a twistor spinor. Then \( \psi \) is conformally equivalent to an imaginary Killing spinor if and only if

a.) \( Q_\psi = 0, C_\psi = 0 \)

b.) \( \pm \frac{\eta_\psi}{u^2} \) is the differential of a positive function.

REFERENCES


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