Jaroslav Hrubý

Turbulence via gauge and supersymmetry theory

In: Jarolím Bureš and Vladimír Souček (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1990. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 22. pp. [97]–102.

Persistent URL: http://dml.cz/dmlcz/701464

TURBULENCE VIA GAUGE AND SUPERSYMMETRY THEORY

J. Hrubý

The turbulence is one of the most enigmatic phenomena of physics. It appears as a seemingly chaotic behaviour of flows of fluids when the relevant and only dimensionless parameter, the Reynold's number, is sufficiently large:

$$R_e = \frac{v_0 L}{\nu} \gg 1 \quad ,$$

where v_0 is a typical velocity of the flow, L is a characteristics length scale and ν means the molecular viscosity.

It is important to bear in mind that practically all flows in nature and even in small-scale laboratory condition have very high Reynold's number, i.e. are turbulent.

The phenomenon of turbulence is widely believed to be adequately described by the Navier-Stokes equation, which for incompressible fluids has the form:

$$\dot{\vec{\Omega}} + (\vec{v}\vec{\nabla})\vec{\Omega} - (\vec{\Omega}\vec{\nabla})\vec{v} = \nu \Delta \vec{\Omega} \quad , \qquad (1)$$
$$\vec{\Omega} = \mathrm{rot}\, \vec{v} \quad .$$

The situation in the theory of turbulence now is that all theories of turbulence have hitherto turned out to be unsuccessful in the long run. The contrast between the theoretical achievements and the tremendous advance of experimentation and of simple empirical theories of turbulence, have greatly discuraged further work on fundamental theory.

"This paper is in final form and no version of it will be submitted for publication elsewhere."

Noneless it doesn t discurage the mathematical physicist from seeking new views of turbulence.

The study of turbulence problem also have inspired other branches of physics or it has developed the same tools independently.

One of this is the method of generating functionals and path integrals,which has been quite independently formulated in the theory of turbulence and gauge field theories.

The last point has an intimate connection with the new way for the solution of the turbulence problem,which was proposed (MIGDAL A.A.).

It is known,that the turbulence problem is usually described by the eq.(1),where the variable is the velocity field $\vec{v}(\vec{x},t)$,which has the random behaviour in the turbulent flow and it makes invalid differential eq. as (1) and the number of degree of freedom makes invalid also supercomputers for solving the full turbulence problem.

A.A.Migdal has used an analogical situation in gluon gauge theory,where gauge field has also the random behaviour and proposed Fokker-Planck eq. in loop space for the description of turbulence.

In this description the analoque of the force line is the turbulent line and analoque of the Willson loop functional will be the corresponding functional for the turbulent flow line.

Our goal is incorporating the supersymmetry technique in this theory via the intimate way proposed by Parisi and Sourlas (PARISI G. and SOURLAS N.) and show interesting coincidence with supersymmetric quantum mechanics (SSQM) and a new point of view on the turbulent flows via the supersymmetry (SUSY).

To show the role of SUSY we shall start from the Migdal that Navier-Stokes eq. can be interpreted as Langevin eq. for the flux lines.

We shall work in (1+1) dim.,where Langevin eq. has the following form:

$$\frac{dc}{dt} = v(c) + f(c,t), \tag{2}$$

$C(t)$ is a time dependent flux line and $f(c,t)$ is a random force.

We shall assume the Gaussian distribution:

$$P(f(c,t)) = \int Df(c,t) \exp\left[-\frac{1}{2Q(c-c')}\int dt\, f^2(c,t)\right] \tag{3}$$

and $\langle f \rangle = 0$, $\langle f(c,t), f(c',t')\rangle = \delta(t-t')\, Q(c-c')$,

where $Q(c-c')$ is a correlation function conected with the viscosity, namely $Q(0) = 2\nu$. In (3) we assume the functional integral all function f.

Now we shall go to the integration over the flux lines and the Jacobian $\det \left|\frac{\partial f(c,t)}{\partial c}\right|$ will be written by using the anticommuting (grassmanian) variables.

On this place we remark that it is usual to incorporate the anticommuting variables for the calculation of the Jacobian in Feyman's path integral formulation of the gauge theory.

It is known that in this gauge theory, when we change variables, the functional determinant appears. The calculation of this determinant via using the anticommuting variables leaves invariant the form of the action in the path integral.

For showing this now I repeat some basic facts of Berezin mathematics:

$$\{\theta,\theta\} = \{\theta,\bar\theta\} = 0 \quad , \quad \int d\theta = 0 \quad , \quad \int \theta d\theta = 1 .$$

From the anticommutativity follows:

$$e^{-\bar\theta\theta} = 1 - \bar\theta\theta$$

and the following relation are valid:

$$\int \bar\theta\, e^{-\bar\theta\theta}\, d\bar\theta d\theta = \int \theta\, e^{-\bar\theta\theta} d\bar\theta d\theta = 0 , \tag{4a}$$

$$\int e^{-\bar\theta\theta} d\bar\theta d\theta = \int (1-\bar\theta\theta)\, d\bar\theta d\theta = -1 . \tag{4b}$$

In the simplest case let us assume there is a random function f with the distribution $P(f)$, such that

$$\int df\, P(f) = 1 .$$

Let X be another random function connected with f as follows

$$\mu(x) = f \tag{5}$$

and we are interested in the average value of a function $F(x)$:

$$\langle F(x) \rangle = \int F[x(f)]\, P(f)\, df \quad , \tag{6}$$

but we don't know the inverse formula to the eq.(5),i.e.$X(f)$.

In this case we have use the integration over X :

$$\langle F(x) \rangle = \int dx\, F(x)\, \exp(\ln P) \frac{df}{dx} \quad . \tag{7}$$

In this place we incorporate the anticommuting variables using (4):

$$\langle F(x) \rangle = \int dx\, d\bar{\theta}\, d\theta\, \exp\left(\ln P + \bar{\theta} \frac{d\mu}{dx} \theta\right) . \tag{8}$$

The expression in the exponent in (8) is invariant under SUSY transformation:

$$x \to x + \bar{\varepsilon}\theta + \bar{\theta}\varepsilon \quad , \tag{9a}$$

$$\theta \to \theta - \varepsilon \frac{d \ln P}{d\mu} \quad , \tag{9b}$$

$$\bar{\theta} \to \bar{\theta} - \bar{\varepsilon} \frac{d \ln P}{d\mu} \quad . \tag{9c}$$

For the Gaussian distribution $\ln P \sim \mu^2$ the expression in (8) is equal

$$\mu^2(x) + \bar{\theta} \mu'(x)\, \theta \quad ,$$

which has the familiar form of the superpotential in SSQM (WITTEN E.).

Let us return to the Langevin eq.(1) for the flux line $C(t)$ and we shall interest about the average value:

$$\langle c(t), c(t') \rangle \sim \int D f\, c(t) c(t')\, P\{f\} \quad . \tag{10}$$

In the relation (10) we shall use imaginary time τ as it is usual in SSQM.Then the correlation function (10) coincide with the Green function and by the same way as in (8) we get:

$$\langle c(\tau), c(\tau') \rangle = \int Dc\, D\psi\, D\bar{\psi}\, c(\tau)\, c(\tau') \times$$

$$\times \exp\left\{ -\frac{1}{2Q(c-c')} \int d\tau \left([\dot{c}_\tau + v(c)]^2 + \bar{\psi}(i\partial_\tau - v'(c))\psi \right) \right\} , \tag{11}$$

where we denote the anticommuting variables ψ ($\bar{\psi}$) and

$$\partial_\tau = \frac{\partial}{\partial \tau} \;,\quad v'(c) = \frac{\partial v}{\partial c} \;,\quad \dot{c}_\tau = \frac{\partial c}{\partial \tau} \;.$$

Likewise as the "surface-term" in the Lagrangian formalism vanishes, the term $\dot{C}_{\tau}\, v(C)$ vanishes here too.

We can see that in the exponent in (11) appears the Lagrangian of the SSQM, when $Q(C-C') \sim 1$.

The function $Q(C-C')$ determines the intensity of the measure of the turbulence and for the $C = C'$ it is $Q(0) = 2\nu$.

It is clear that for the $\nu = 0$ it corresponds to the case of ideal liquid; it means $f(C, \tau) = 0$.

In the work (MIGDAL A.A.) there is shown that the function Q has the analytic form :

$$Q(k) \sim const. \times exp\left(-\frac{k^2}{2}\right). \tag{12}$$

In the case of SUSY we have for the supersymmetric transformed flux line $C' = C + \bar{\varepsilon}\psi + \bar{\psi}\varepsilon$ the following expression:

$$Q(C-C') = Q(-\bar{\varepsilon}\psi - \bar{\psi}\varepsilon),$$

which means, that the intensity of the fluctuations in the SUSY case is the even function of the anticommuting variables.

Using the form of Q from (12), we can see from the anticommutativity ε, ψ that

$$Q(C-C') \sim 1,$$

which is needed that in the exponent in (11) the Lagrangian of SSQM appear.

We now show that the realization of the stacionary or non-stationary regime for the flux line $X(t)$ is connected with the SUSY breaking.

For this purpose we introduce as is usual the probability density $P(x(t), t)$ for a solution of the Langevin eq. (1).

This probability density satisfies the Fokker-Planck eq.:

$$\frac{\partial P}{\partial t} = \frac{\partial \{ v(x)\, P \}}{\partial x} + \frac{1}{2} Q(x-x') \frac{\partial^2 P}{\partial x^2}. \tag{13}$$

Eq.(13) has the time independent solution (it means the stationary solution):

$$P(x) \sim exp\left[-\frac{2}{Q(x-x')} \int_{-\infty}^{x} dy\, v(y) \right], \tag{14}$$

if and only if:

$$\int_{-\infty}^{+\infty} dx\, \rho(x) = 1 \quad . \tag{15}$$

In eq.(14) the function v playes the role of the super-potential and the different sign for the $v(x)$ for $x \to \pm\infty$ assures that (14) and (15) are not in contradiction.

But it is the case,when SUSY is not spontaneously broken.

It is known that the Fokker-Planck eq. (13) can be written as the Schrödinger eq.:

$$\frac{d}{dt} W(\rho) = - \hat{H} W(\rho) \quad , \tag{16}$$

where $W(\rho) = \rho(x,t) \exp\left(\frac{v(x)}{2}\right)$

and $\hat{H} = -\frac{1}{2}\left(\frac{d^2}{dx^2} + v^2(x) \pm v'(x)\right) \equiv H_\pm$,

what is the Hamiltonian of SSQM.

We conclude by the following:

we showed the incorporation of SUSY in the path integral description of turbulence proposed by A.A.Migdal is not only formally but it appears in the interesting coincidence with SSQM.Supersymmetry breaking is connected with the nonstationary flow.

From SSQM Hamiltonian there appears two Fokker-Planck eqs. correspond forward and backward Hamiltonians $H_\pm$.

We also showed that the stochastic and the supersymmetric formulation of the turbulence problem coincide.

REFERENCES

MIGDAL A.A."Nonlinear waves;Structures and bifurgation"in russian,Moscow "Nauka" (1987)159.

PARISI G.and SOURLAS N.,Nucl.Phys.B206 (1983) 321.

WITTEN E.,Nucl.Phys.B188 (1981)513.