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PONTRYAGIN ALGEBRA OF A TRANSITIVE LIE ALGEBROID

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INTRODUCTION

The Chern-Weil homomorphism h^P of a principal fibre bundle (pfb) P has been known for some forty years [Ch]. On the other hand, in analogy to the theory of Lie groups and Lie algebras, each pfb P has its algebraic equivalent: a transitive Lie algebroid (tLa) $A(P)$ - constructed on the basis of the right-invariant vector fields on P . $A(P)$ is simply a vector bundle equipped with some structures (of an algebraic nature) like a structure of a Lie algebra in the module of sections.

It turns out that the Chern-Weil homomorphism of P is a notion of the Lie algebroid of this pfb! This means that, knowing only the Lie algebroid $A(P)$ of $P=P(M,G)$, one can uniquely reproduce the ring of invariant polynomials $(Vg^*)_I$ and the Chern-Weil homomorphism $h^P: (Vg^*)_I \longrightarrow H(M)$ (g denotes the Lie algebra of G).

We pay our attention to the fact that this holds although in the Lie algebroid $A(P)$ there is no direct information about the structural Lie group of P !

This paper is in final form and no version of it will be submitted for publication elsewhere.

In addition, we must point out two things:

1) A tLa is - in some sense - a simpler structure than a pfb. Namely, nonisomorphic pfb's can possess isomorphic Lie algebroids. For example, there exists a nontrivial pfb for which the Lie algebroid is trivial (the nontrivial $Spin(3)$ -structure of the trivial pfb $[RP(5) \times SO(3)]$ [Kub]₂).

2) There exist "nonintegrable" tLa's, ie tLa's which cannot be realized as the Lie algebroids of pfb's. First examples were constructed by R.Almeida and P.Molino [Al-Mol]₁₋₂ (see also [Mol]) basing themselves on transversally complete foliations. The tLa of the foliation of a compact simply connected Lie group by the left cosets of a connected and nonclosed subgroup is an example of a nonintegrable tLa [Mol].

In connection with the above, it seems important to construct the Chern-Weil homomorphism of a tLa A in such a way that it will agree with the Chern-Weil homomorphism of any pfb P for which A is its Lie algebroid. In addition, this homomorphism will probably be useful to investigate some nonintegrable tLa's.

Originally, the notion of a Lie algebroid was invented in connection with the study of differential groupoids [J.Pradines in [Pral]₁₋₂ introduced the so-called Lie functor which assigns a Lie algebroid to any differential groupoid]. Since each pfb P determines a differential groupoid (the so-called Lie groupoid PP^{-1} of Ehresmann [Ehr]), therefore each pfb P defines - in an indirect manner - a tLa $A(P)$. P.Libermann noticed [Lib] that the vector bundle of this tLa $A(P)$, $P=P(M,G)$, is canonically isomorphic to the vector bundle TP/G (investigated earlier by M.Atiyah in the context of the problem of the existence of a complex connection in a complex pfb [At]). The construction of the Lie functor for pfb's with the omission of the indirect step of differential groupoids was made independently by K.Mackenzie [Mac] and by the author [Kub]₁.

LIE FUNCTOR FOR PFB'S

We begin with giving the fundamental (for our considerations) definition of a tLa and with a construction of the Lie functor. We assume that all the manifolds considered are C^∞ and Hausdorff, and that M - the base of tLa's - is connected.

Definition. [Pra]₁₋₂. By a transitive Lie algebroid (tLa) on a manifold M we shall mean a system

$$(1) \quad A = (A, [\cdot, \cdot], \gamma)$$

consisting of a vector bundle A over M and mappings

$$[\cdot, \cdot]: \text{Sec} A \times \text{Sec} A \rightarrow \text{Sec} A, \quad \gamma: A \rightarrow TM,$$

such that

- (a) $(\text{Sec} A, [\cdot, \cdot])$ is an $\mathbb{R}$ -Lie algebra,
- (b) γ is an epimorphism of vector bundles,
- (c) $\text{Sec} \gamma: \text{Sec} A \rightarrow \mathfrak{X}(M)$ is a homomorphism of Lie algebras,
- (d) $[[\xi, f\eta]] = f \cdot [[\xi, \eta]] + (\gamma \circ \xi)(f) \cdot \eta$ for $f \in C^\infty(M)$, $\xi, \eta \in \text{Sec} A$.

Let (1) and $(A', [\cdot, \cdot]', \gamma')$ be two Lie algebroids on the same manifold M . By a homomorphism between them we mean a strong homomorphism $H: A \rightarrow A'$ of vector bundles, such that

- (a) $\gamma' \circ H = \gamma$,
- (b) $\text{Sec} H: \text{Sec} A \rightarrow \text{Sec} A'$ is a homomorphism of Lie algebras.

With each tLa (1) we associate a short exact sequence

$$(2) \quad 0 \rightarrow g \hookrightarrow A \xrightarrow{\gamma} TM \rightarrow 0$$

$(g = \text{Ker} \gamma)$ called the Atiyah sequence of (1). g is a Lie algebra bundle if in each fibre $g|_x$ the Lie algebra structure is defined by: $[v, w] := [[\xi, \eta]](x)$, $\xi, \eta \in \text{Sec} A$, $\xi(x) = v$, $\eta(x) = w$ (see [Al-Mo]₁, [Mac] and [Kub]₂).

For the construction of the Lie functor, we take any pfb $P=P(M,G)$ with the projection $\pi:P\rightarrow M$ and the action $R:P\times G\rightarrow P$, and define another pfb $TP(TM,TG)$ with the projection $\pi_*:TP\rightarrow TM$ and the action $R_*:TP\times TG\rightarrow TP$. We can treat G as a closed subgroup of TG ($G\cong\langle\vartheta_a; a\in G\rangle$, ϑ_a being the null tangent vector at a). The restriction of R_* to G is then equal to $R^T:TP\times G\rightarrow TP$, $(v,a)\mapsto(R_a)_*v$, R_a being the action of a on P . We put

$$A(P)=TP/G$$

- the space of all orbits of R^T , and denote by $\pi^A:TP\rightarrow A(P)$, $v\mapsto[v]$, the natural projection. By [Ko-No], we see that the structure of a Hausdorff C^∞ -manifold, such that π^A is a submersion, exists in $A(P)$. In the end, we define the projection $p:A(P)\rightarrow M$, $[v]\mapsto\pi z$, if $v\in T_zP$. For each point $x\in M$, in the fibre $p^{-1}(x)$ there exists exactly one $\mathbb{R}$ -vector space structure such that $[v]+[w]=[v+w]$ if $\pi_p(v)=\pi_p(w)$, $\pi_p:TP\rightarrow P$ being the projection. The system $(A(P), p, M)$ is a vector bundle [Mac], [Kub]₁. Let $\mathfrak{X}^R(P)$ denote the $C^\infty(M)$ -module of all C^∞ global right-invariant vector fields on P .

Proposition. [Mac], [Kub]₁. For each cross-section $\eta\in\text{Sec}A(P)$, there exists exactly one C^∞ right-invariant vector field $\eta'\in\mathfrak{X}^R(P)$ such that $[\eta'(z)]=\eta(\pi z)$. The mapping

$$(3) \quad \text{Sec}A(P)\rightarrow\mathfrak{X}^R(P), \quad \eta\mapsto\eta',$$

is an isomorphism of $C^\infty(M)$ -modules. ■

Now, we define some $\mathbb{R}$ -Lie algebra structure $[\cdot, \cdot]$ in the $\mathbb{R}$ -vector space $\text{Sec}A(P)$ by demanding that (3) be an isomorphism of $\mathbb{R}$ -Lie algebras. We also take the mapping $\gamma:A(P)\rightarrow TM$, $[v]\mapsto\pi_*v$.

Theorem. [Mac], [Kub]₁. The object

$$A(P) = (ACP), [\cdot, \cdot], \gamma$$

is a transitive Lie algebroid. A homomorphism $F = (F, \mu): P(M, G) \rightarrow P'(M, G')$ of pfb's [$\mu: G \rightarrow G'$ - a homomorphism of Lie groups, $F(za) = F(z) \cdot \mu(a)$] determines a mapping $dF: ACP \rightarrow ACP'$, $[v] \mapsto [F_* v]$, which is a homomorphism of Lie algebroids. The correspondence $P \mapsto ACP$, $F \mapsto dF$, is a covariant functor (called the Lie functor for pfb's). ■

AN INTERPRETATION OF SECTIONS OF THE LIE ALGEBROID OF THE LIE GROUPOID $GL(f)$

Let f be any vector bundle over M and $GL(f)$ - the Lie groupoid of all linear isomorphisms between fibres of f . Ngo-Van-Que [NVQ] discovered an operator interpretation of sections of the Lie algebroids $ACGL(f)$ (see also [Kum] and [Mac]). We describe it in a little different manner. For a section $\sigma \in \text{Sec } f$ and for $x \in M$, we put $\tilde{\sigma}_x: GL(f)_x \rightarrow f|_x$, $h \mapsto h^{-1}(\sigma(\beta h))$. Then we have the following

Proposition. Let $\xi \in \text{Sec } ACGL(f)$. Then the mapping

$$\mathfrak{L}_\xi(\sigma): M \rightarrow f, \quad x \mapsto \xi_x(\tilde{\sigma}_x)$$

is a C^∞ -section of f , and $\mathfrak{L}_\xi: \text{Sec } f \rightarrow \text{Sec } f$, $\sigma \mapsto \mathfrak{L}_\xi(\sigma)$, is a differential operator of order ≤ 1 such that

$$(4) \quad \mathfrak{L}_\xi(f \cdot \sigma) = f \cdot \mathfrak{L}_\xi(\sigma) + X(f) \cdot \sigma, \quad f \in C^\infty(M), \quad \sigma \in \text{Sec } f, \quad \text{where } X = \gamma \circ \xi.$$

Conversely, for any differential operator $\mathfrak{L}$ of order ≤ 1 in the vector bundle f , such that (4) holds for some $X \in \mathfrak{X}(M)$, there exists exactly one section $\xi \in \text{Sec } ACGL(f)$ such that $\mathfrak{L} = \mathfrak{L}_\xi$ and $X = \gamma \circ \xi$. ■

REPRESENTATIONS OF LIE GROUPOIDS AND LIE ALGEBROIDS IN VECTOR BUNDLES

By a representation of a transitive Lie groupoid Φ in a vector bundle f (both over M) we mean a strong homomorphism $T: \Phi \rightarrow GL(f)$ of Lie groupoids; whereas by a representation of a transitive Lie algebroid A in a vector bundle f we mean a strong homomorphism $T': A \rightarrow A(GL(f))$ of tLa's. Of course, for a representation $T: \Phi \rightarrow GL(f)$, the differential $dT: A(\Phi) \rightarrow A(GL(f))$ is a representation of the Lie algebroid $A(\Phi)$ in f .

Definition.(a). For a representation $T: \Phi \rightarrow GL(f)$ of a Lie groupoid Φ in f , we define the vector space of invariant sections of f in the following way

$$(Secf)_I = \left\{ \sigma \in Secf : \bigwedge_{h \in \Phi} [T(h)(\sigma_{\alpha h}) = \sigma_{\beta h}] \right\}$$

(b). For a representation $T': A \rightarrow A(GL(f))$ of a transitive Lie algebroid A in f , we define analogously

$$(Secf)_{I^0} = \left\{ \sigma \in Secf : \bigwedge_{\xi \in Sec A} [\mathfrak{L}_{T' \circ \xi}(\sigma) = 0] \right\}.$$

The following facts play the fundamental role in our theory:

Theorem 1. Let $T: \Phi \rightarrow GL(f)$ be any representation of a Lie groupoid Φ in f , and $dT: A(\Phi) \rightarrow A(GL(f))$ its differential. Then

(a) $(Secf)_I \subset (Secf)_{I^0},$

(b) if Φ is connected, then $(Secf)_I = (Secf)_{I^0}.$ ■

For a representation $T: \Phi \rightarrow GL(f)$ and for $x \in M$ we take the induced representation $T_x: G_x \rightarrow GL(f|_x)$ of the isotropy Lie group G_x in the vector space $f|_x$. By $(f|_x)_I$ we denote the space of T_x -invariant vectors. Then we have

Theorem 2. For an arbitrary $v \in (f|_x)_I$, the section

$$\sigma_v: M \rightarrow f, \quad y \mapsto T(h)(v), \quad \text{where } h \in \mathbb{R} \text{ and } \alpha h = x, \quad \beta h = y,$$

is a correctly defined smooth invariant section, and the mapping

$$(f|_x)_I \rightarrow (Sec f)_I, \quad v \mapsto \sigma_v,$$

is an isomorphism of vector spaces. ■

In addition, we have the following fact: For an arbitrary representation $T': A \rightarrow A(GL(f))$ of a tLa A in f , each T' -invariant section $\sigma \in (Sec f)_I$ is uniquely determined by its value at any point.

Now, for a tLa A , having (2) as its Atiyah sequence, we define the adjoint representation

$$ad: A \rightarrow A(GLg)$$

in such a way that $\mathcal{R}_{ad \circ \xi}(\sigma) = [\xi, \sigma]$, $\sigma \in Secg$. ad induces the representation $ad^\vee: A \rightarrow A(GLVg^*)$ by the formula

$$\begin{aligned} \langle \mathcal{R}_{ad^\vee \circ \xi} \Gamma, \sigma_1 \vee \dots \vee \sigma_k \rangle &= (\gamma \circ \xi) \langle \Gamma, \sigma_1 \vee \dots \vee \sigma_k \rangle \\ &\quad - \sum_{i=1}^k \langle \Gamma, \sigma_1 \vee \dots \vee [\xi, \sigma_i] \vee \dots \vee \sigma_k \rangle. \end{aligned}$$

In particular, we have

$$\begin{aligned} (Sec Vg^*)_I = \left\{ \Gamma \in Sec Vg^*: \right. & \left. \xi \in \bigwedge^s A, \sigma_1, \dots, \sigma_k \in Sec f \left((\gamma \circ \xi) \langle \Gamma, \sigma_1 \vee \dots \vee \sigma_k \rangle \right. \right. \\ & \left. \left. = \sum_{i=1}^k \langle \Gamma, \sigma_1 \vee \dots \vee [\xi, \sigma_i] \vee \dots \vee \sigma_k \rangle \right) \right\} \end{aligned}$$

In addition, if $\Gamma^s \in (Sec V^s g^*)_I$, $s=1, 2, \dots$, then

$$\Gamma^1 \vee \Gamma^2 \in (Sec^{k_1+k_2} V^{k_1+k_2} g^*)_I$$

so $\phi_k(\text{Sec}^k \mathfrak{g}^*)_{I^0}$ is a subalgebra of $\phi_k(\text{Sec}^k \mathfrak{g}^*)$.

K. Mackenzie [Mac] proved that if $A = A(\Phi)$, then ad is a differential of the adjoint representation $Ad: \Phi \rightarrow GL(\mathfrak{g})$ defined by: $Ad(h) = (\tau_h)_* \circ \tau_x$, $\tau_h: G_x \rightarrow G_y$, $a \mapsto hah^{-1}$, $x = ah$, $y = \beta h$.

THE CHERN-WEIL HOMOMORPHISM OF A TRANSITIVE LIE ALGEBROID

By a connection in a tLa $A = (A, [\cdot, \cdot], \gamma)$ we mean a splitting of the Atiyah sequence $0 \rightarrow \mathfrak{g} \hookrightarrow A \xrightarrow{\gamma} TM \rightarrow 0$ of A , i.e. a mapping $\lambda: TM \rightarrow A$ such that $\gamma \circ \lambda = id_{TM}$. For a connection λ in A , the uniquely determined morphism of vector bundles $\omega: A \rightarrow \mathfrak{g}$ fulfilling $\omega|_{\mathfrak{g}} = id$ and $\text{Ker} \omega = \text{Im} \lambda$ is called a connection form of λ . By a curvature base-form (or a curvature tensor) of a connection λ we shall mean the 2-form Ω_M on M with values in the vector bundle $\mathfrak{g}$, defined by the formula

$$\Omega_M(X, Y) = -\omega([\lambda \circ X, \lambda \circ Y]) - [\lambda \circ [X, Y] - [\lambda \circ X, \lambda \circ Y]].$$

Theorem. The mapping

$$\begin{aligned} \gamma^M: \phi_k(\text{Sec}^k \mathfrak{g}^*) &\longrightarrow \Omega^k(M) \\ \text{Sec}^k \mathfrak{g}^* \ni \Gamma &\longmapsto \frac{1}{k!} \langle \Gamma, \underbrace{\Omega_M \vee \dots \vee \Omega_M}_{k\text{-times}} \rangle \end{aligned}$$

is a homomorphism of algebras such that the form $\gamma_M(\Gamma)$ is closed when Γ is invariant. ■

The superposition

$$h^A: \phi_k(\text{Sec}^k \mathfrak{g}^*)_{I^0} \xrightarrow{\gamma} Z(M) \longrightarrow H(M)$$

is called the Chern-Weil homomorphism of A . Its image $\text{Im} h^A$ is a subalgebra of $H(M)$ called the Pontryagin algebra of A .

Theorem. The Chern-Weil homomorphism h^A of a tLa A is

independent of the choice of a connection. ■

Now, take any pfb $P(M, G)$ and let $A = A(P)$ be its Lie algebroid. Then, for the Lie groupoid of Ehresmann $\Phi = PP^{-1}$ and for the adjoint representation $Ad: \Phi \rightarrow GL(\mathfrak{g})$, we have, by Theorems 1 and 2, the commuting diagram:

$$\begin{array}{ccc}
 \oplus_k (\text{Sec } \nabla \mathfrak{g}^*)_I \circ & \xrightarrow{h^A} & H(M) \\
 \cong \uparrow & \searrow h^{\langle \Phi_x \rangle} & \\
 (V(\mathfrak{g}|_x)^*)_I & \xrightarrow{\quad} & H(M) \\
 \cong \uparrow & \nearrow h^P & \\
 (V\mathfrak{g}^*)_I & &
 \end{array}$$

from which we obtain that the Chern-Weil homomorphism h^P of a pfb P is an invariant of the Lie algebroid of P .

Remarks. 1/. It is possible to construct the characteristic homomorphism h^A of those nontransitive Lie algebroids A for which γ is of the constant rank (such Lie algebroids are called regular).

2/. There exists a characteristic homomorphism of flat (and of partially flat) regular Lie algebroids - an object analogous to that for flat (and for foliated) pfb's.

3/. The proofs of the above-mentioned theorems will appear in the next work by the author.

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