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TOWARDS ONE CONJECTURE ON COLLAPSING OF THE SERRE SPECTRAL SEQUENCE

MARTIN MARKL

INTRODUCTION

Consider a fibration $F \overset{\iota}{\hookrightarrow} E \overset{p}{\to} B$ of simply connected spaces having the homotopy lifting property with respect to polyhedra. The cohomology of all the spaces under consideration are classically related by the Serre spectral sequence (the definition, as well as other results mentioned in this paragraph, are to be found for example in [2] or [7]) which starts from $E_2^{p,q} = H^p(B; \mathcal{H}^q(F; R))$ and converges to $H^*(E; R)$, here R is an integral domain and $\mathcal{H}^*(F; R)$ is the local system of the fiber cohomology. If such a fibration consists of simply connected spaces having the rational cohomology of finite type, then it is also the "rational fibration" in the sense of [5] (see [10, I.5.(4)]), this fact will be useful in the sequel.

The Serre spectral sequence is a rather complicated object, hence from the computational point of view most important are namely the cases, when our spectral sequence is not far from being trivial or even when it collapses. The following classical theorem shows that especially the collapsing of the Serre spectral sequence is closely related with the cohomological properties of the fiber inclusion $\iota\colon F \to E$.

THEOREM. *Suppose that R is a field. Then the following contidions on the fibration $F \overset{\iota}{\hookrightarrow} E \overset{p}{\to} B$ are equivalent:*

1. *the map $\iota\colon F \hookrightarrow E$ induces an epimorphism $\iota^*\colon H^*(E; R) \to H^*(F; R)$,*
2. *the Serre spectral sequence collapses at the E_2-level (i.e., the higher differentials d_r are trivial for $r \geq 2$) and the fibration is orientable (i.e., the fundamental group $\pi_1(B)$ acts trivially on the cohomology $H^*(F; R)$ of the fiber).*

We accept here the usual definition and say that our fibration $F \overset{\iota}{\hookrightarrow} E \overset{p}{\to} B$ is *totally non-cohomologous to zero* (abbreviated by TNCZ) if one (and hence both) of equivalent contitions of the previous theorem is satisfied. Under some mild finiteness assumptions, the collapsing of the Serre spectral sequence enables us to determine the ring structure of the cohomology of total space as in the following statement (again R has to be a field).

This paper is in final form and no version of it will be submitted for publication elsewhere

THEOREM. *Suppose that both $H^*(B;R)$ and $H^*(F;R)$ are free R-modules of finite type. If our fibration $F \xrightarrow{\iota} E \xrightarrow{p} B$ is TNCZ, then $H^*(E;R)$ and $H^*(B;R) \otimes_R H^*(F;R)$ are isomorphic as graded $H^*(B;R)$-modules. Here the action of $H^*(B;R)$ on $H^*(E;R)$ is given by $b.e \longmapsto p^*(b) \cup e$, for $b \in H^*(B;E)$ and $e \in H^*(E;R)$, while the action on the tensor product $H^*(B;R) \otimes_R H^*(F;R)$ is induced by the multiplication on the first factor. Consequently, the map p^* is a monomorphism and there exists an isomorphism*

$$H^*(F;R) \cong H^*(E,R)/(p^*(H^+(B;R))$$

of graded rings, here $H^+(B;R)$ denotes the maximal ideal of $H^*(B;R)$ generated by elements of positive degrees.

It is worth to mention that the algebra structure of $H^*(E,R)$ does not coincide in general with the natural algebra structure on the tensor product $H^*(B;R) \otimes_R H^*(F;R)$ induced by the multiplication on the factors.

1. MOTIVATIONS

The conjecture we aim to discuss says, roughly speaking, that the Serre spectral sequence collapses for comparatively large class of fibrations. Before going to precise formulations, we would like to write down some indications, supporting this hypothesis. From now on, we shall work exclusively with the singular cohomology having the coefficients in a fixed field $\mathbf{k}$ of characteristic zero, although some of the following statements are, after suitable modifications, valid also for fields of an arbitrary characteristic. For brevity we will omit the coefficients in our formulas.

The first indication is the following theorem of A. Borel [1].

THEOREM. *For an equal-rank pair $G \supset U$ of compact Lie groups the Serre spectral sequence of the universal fibration*

$$G/U \longrightarrow BU \longrightarrow BG$$

collapses.

Digression. We would like to show here how the previous theorem enables us to compute the cohomology of the homogeneous space G/U for an equall-rank pair $G \supset U$ of compact Lie groups; this computation will be needed in the sequel. The proofs of the forthcomming statements can be found for example in [1].

At first, fix a common maximal torus $T = T^r \subset U \subset G$, where r is the common rank. Denote by $W(G) \stackrel{\text{def.}}{=} N_G T/T$ and $W(U) \stackrel{\text{def.}}{=} N_U T/T$ the Weil groups of G and U respectively. Notice that both $W(G)$ and $W(U)$ act naturally on T and that $W(G) \supset W(U)$. Then the cohomology algebra of the classifying space BG is known to be isomorphic to the subalgebra $H^*(T)^{W(G)} \subset H^*(T)$ of elements which are stable under the natural action of the Weil group $W(G)$, induced by its action on the maximal torus T. Similarly, $H^*(BU) \cong H^*(BT)^{W(U)}$. It can also be shown that under these identifications, the projection p^* in the universal fibration $G/U \longrightarrow BU \longrightarrow BG$ can be identified with the natural inclusion $H^*(BT)^{W(U)} \hookrightarrow H^*(BT)^{W(G)}$, therefore the theorem above gives rise to the isomorphism

$$H^*(G/U) \cong H^*(BT)^{W(U)}/(H^+(BT)^{W(G)}) \tag{1.1}$$

of graded algebras. To obtain a more transparent description of $H^*(G/U)$, use the isomorphism

$$H^*(BT) \cong \mathbf{k}[t_1, \dots, t_r], \deg(t_i) = 2, 1 \le i \le r.$$

Then

$$H^*(BT)^{W(G)} \cong \mathbf{k}[f_1, \dots, f_r],$$

where $f_1, \dots, f_r \in \mathbf{k}[t_1, \dots, t_r]$ are some elements of even degrees. Similarly

$$H^*(BT)^{W(U)} \cong \mathbf{k}[x_1, \dots, x_r]$$

where again $x_1, \dots, x_r$ are elements of $\mathbf{k}[t_1, \dots, t_r]$ of even degrees. Because of the clear inclusion

$$H^*(BT)^{W(G)} \subset H^*(BT)^{W(U)},$$

the elements $f_1, \dots, f_r$ can be considered as polynomials from $\mathbf{k}[x_1, \dots, x_r]$ and the formula (1.1) gives

$$H^*(G/U) \cong \mathbf{k}[x_1, \dots, x_r]/(f_1, \dots, f_r). \tag{1.2}$$

As the dimension of $H^*(G/U)$ is clearly finite, the sequence $f_1, \dots, f_r$ must be regular in the polynomial ring $\mathbf{k}[x_1, \dots, x_r]$.

Notice also that the space G/U is simply connected – this is an easy consequence of the fact that the inclusion of the maximal torus induces an epimorphism of fundamental groups.

The theorem above gives rise to the question whether the collapsing of Serre spectral sequence is the common property of fibrations having the fiber of the described type. This was really proved in some special cases by W. Meier [6] and then in full generality by Shiga and Tezuka [8]. What they proved is the following theorem.

THEOREM. *Every orientable fibration $F \overset{i}{\hookrightarrow} E \overset{p}{\rightarrow} B$ of connected spaces having the fiber F of the form G/U for an equal-rank pair $G \supset U$ of compact Lie groups is totally noncohomologous to zero.*

Let us sum up the most basic properties of spaces F of the form G/U as above:

1. *$H^*(F)$ is finite dimensional* - this easily follows from the fact that F is a compact CW-complex,
2. *the "rational homotopy" $\pi_*(F) \otimes \mathbf{Q}$ is finite dimensional, too* - this may be seen looking at the exact homotopy sequence of the universal fibration "modulo torsions" or using the most elementary tools of the rational homotopy theory,
3. *$H^*(F)$ is evenly graded (i.e., $H^*(F)$ is zero in odd dimensions)* - this follows immediately from (1.2).

Recall that, according to [3], a simply connected space satisfying both the conditions (1) and (2) above, is called to be of type (F).

The observations above led to the formulation of the following conjecture, which is due to S. Halperin [10].

CONJECTURE. *Let F be a simply connected space of type (F) having evenly graded rational cohomology. Then every orientable fibration $F \overset{\iota}{\hookrightarrow} E \overset{p}{\to} B$ of simply connected spaces is totally noncohomologous to zero.*

Besides the already quoted result of Shiga and Tezuka, J.-C. Thomas proved in [10] this conjecture for spaces, whose cohomology algebra is generated by no more than two generators.

The main tool of approaching this conjecture is the following surprising reformulation, which can be deduced using either the multiplicative properties of the Serre spectral sequence as in [6], or using the Sullivan classifying space as in [8].

PROPOSITION. *A space F, satisfying all the assumptions of Halperin conjecture, satisfies also its conclusion if and only if*

$$Der_{<0}(H^*(F)) = 0$$

where $Der_{<0}(H^*(F))$ denotes the space of negative-degree derivations of the cohomology algebra,

$$Der_{<0}(H^*(F)) = \left\{ \begin{array}{c} \text{linear maps } \theta\colon H^*(F) \to H^*(F) \text{ of negative} \\ \text{degrees with } \theta(ab) = \theta(a)b + a\theta(b) \end{array} \right\}$$

2. RESULTS

Let us say that a (simply connected) space X satisfies the condition (†) if the Serre spectral sequence of every orientable fibration of the form $X \overset{\iota}{\hookrightarrow} Y \overset{p}{\to} Z$ collapses. The conjecture of S. Halperin is then equivalent to say that the condition (†) is satisfied by all spaces of type (F) having evenly graded cohomology. The first result of us shows that the category of simply connected spaces satisfying (†) is closed under taking fibrations.

THEOREM 1. *Let $X \overset{\iota}{\hookrightarrow} Y \overset{p}{\to} Z$ be a rational fibration (for example, a Serre fibration of simply connected spaces, see [10, I.5.(4)]) in which both X and Z satisfy (†). Then (†) is satisfied also by the total space E.*

Before going to the furter result, recall briefly the notion of the homotopy Lie algebra. For a simply connected space X denote by ΩX its loop space. It is well-known, that the Samelson product, comming from the Hopf structure on ΩX, converts $\pi_*(\Omega X) \cong \pi_{*-1}(X)$ into a graded Lie ring. Then $\pi_*(\Omega X) \otimes \mathbf{Q}$, endowed with the induced Lie algebra structure, is called the *homotopy Lie algebra* of the space X. Let us denote by $C^*(L_*)$ the usual cochain functor on a graded Lie algebra L_* (see [9;p.24]).

THEOREM 2. *Every simply connected space X of type (F) having evenly graded cohomology, whose homotopy Lie algebra is of finite cohomological dimension, i.e.*

$$\dim_{\mathbf{Q}}(H^*(C^*(\pi_*(\Omega X) \otimes \mathbf{Q}))) < \infty$$

has the property stated in Halperin conjecture. Especially, the conjecture is true for all coformal spaces of type (F) having evenly graded cohomology (see [9] for the definition of coformality).

Recall that, according to [4], the cohomology algebra of an arbitrary space of type (F) with evenly graded cohomology can be represented in the form

$$\mathbf{k}[x_1, \ldots, x_r]/(f_1, \ldots, f_r)$$

where $x_1, \ldots, x_r$ are indeterminates of even degrees and the sequence $(f_1, \ldots, f_r)$ is regular in $\mathbf{k}[x_1, \ldots, x_r]$ (to be compared with (1.2)). The following result says that the conjecture is valid in the most honest algebro-geometric situation, when all the polynomials $f_1, \ldots, f_r$ are homogeneous.

THEOREM 3. *Halperin conjecture is true for every space X, for which in the presentation*

$$H^*(X) \cong \mathbf{k}[x_1, \ldots, x_r]/(f_1, \ldots, f_r)$$

all the polynomials $f_1, \ldots, f_r$ are homogeneous in the grading given by the lenght of monomials.

3. PROOFS

PROOF OF THEOREM 1: As X is supposed to satisfy (†), the Serre spectral sequence of the fibration $X \overset{\iota}{\hookrightarrow} Y \overset{p}{\to} Z$ collapses. Thus there exists an isomorphism of $H^*(X)$-modules $H^*(Y)$ and $H^*(Z) \otimes H^*(X)$; in what follows we will identify these modules. Due to this observaiton the cohomology algebra $H^*(Y)$ can be written in the form

$$(H^*(Z) \otimes H^*(X), *),$$

where the product $*$ may possibly differ from the usual one. Newertheless, from the fact that our isomorphism is $H^*(Z)$-modular, we easily obtain that

$$(a \otimes 1) * (a' \otimes b') = aa' \otimes b', \text{ for } a, a' \in H^*(Z), b' \in H^*(X). \tag{3.1}$$

Similarly, we easily deduce that

$$(1 \otimes b) * (1 \otimes b') = 1 \otimes bb' + \chi(b, b'), \text{ for } b, b' \in H^*(X) \tag{3.2}$$

where the "twisting function" χ is a linear map

$$\chi: H^*(X) \otimes H^*(X) \to H^+(Z) \otimes H^*(X).$$

According to the proposion of the previous paragraph, the condition (†) is equivalent to the triviality of the Lie algebra of all negative-degree derivations of the cohomology algebra $H^*(Y) \cong (H^*(Z) \otimes H^*(X), *)$. So suppose that $\Theta \in Der_{<0}((H^*(Z) \otimes H^*(X), *))$. We shall then show that the derivation Θ is trivial. At first, $\Theta|H^*(Z)$ can be clearly expressed in the form

$$\Theta|H^*(Z) = \sum \Xi_i \otimes b_i,$$

where $\Xi_i \colon H^*(Z) \to H^*(Z)$ are linear maps, $b_i \in H^*(X)$ and $\deg(\Xi_i) = \deg(\Theta) - \deg(b_i) < 0$. From (3.1) and from the fact that Θ is a derivation we easily infer that, for $a, a' \in H^*(Z)$,

$$\begin{aligned}\Theta(a * a') &= a * \Theta(a') + \Theta(a) * a' = \sum \{a * [\Xi_i(a') \otimes b_i] + [\Xi_i(a) \otimes b_i] * a'\}\\ &= \sum (a\Xi_i(a') + \Xi_i(a)a') \otimes b_i.\end{aligned}$$

On the other hand

$$\Theta(a * a') = \sum \Xi_i(a * a') \otimes b_i = \sum \Xi_i(aa') \otimes b_i.$$

As b_i's can be choosen to be linearly independent, Ξ_i must be a negative-degree derivation of $H^*(Z)$, hence $\Xi_i = 0$ since Z satisfies the condition (†) by our assumption, and

$$\Theta|H^*(Z) = 0. \tag{3.3}$$

Similarly as above, $\Theta|H^*(X)$ can be written in the form

$$\Theta|H^*(X) = \sum a_j \otimes \Omega_j + \Omega_0,$$

where $\Omega_j, \Omega_0 \colon H^*(X) \to H^*(X)$ are linear maps, $\{a_i\}_{i\in I}$ is an additive basis of $H^+(Z)$, $\deg(\Omega_0) = \deg(\Theta) < 0$, $\deg(\Omega_j) = \deg(\Theta) - \deg(a_j) < 0$. Using the basis $\{a_i\}_{i\in I}$, we can expand the twisting function χ as

$$\chi = \sum a_i \otimes \chi_i$$

for some bilinear functions $\chi_i \colon H^*(X) \otimes H^*(X) \to H^*(X)$. For $b, b' \in H^*(X)$ we have the following terrifying formula:

(3.4)

$$\begin{aligned}\Theta(b * b') =& \{\sum_j a_j \otimes \Omega_j(b) + 1 \otimes \Omega_0(b)\} * b' + b * \{\sum_j a_j \otimes \Omega_j(b') + 1 \otimes \Omega_0(b')\}\\ =& \sum_{i,j} a_i a_j \otimes \chi_i(\Omega_j(b), b') + \sum_j a_j \otimes \Omega_j(b)b' + \sum_i a_i \otimes \chi_i(\Omega_0(b), b') + 1 \otimes \Omega_0(b)b'\\ &+ \sum_{i,j} a_i a_j \otimes \chi_i(b_i, \Omega_j(b')) + \sum_j a_j \otimes b\Omega_j(b') + \sum_i a_i \otimes \chi_i(b, \Omega_0(b')) + 1 \otimes b\Omega_0(b')\\ =& \sum_{i,j} a_i a_j \otimes [\chi_i(\Omega_j(b), b') + \chi_i(b, \Omega_j(b'))] + \sum_i a_i \otimes [\chi_i(\Omega_0(b), b') + \chi_i(b, \Omega_0(b'))]\\ &+ \sum_i a_i \otimes [\Omega_j(b)b' + b\Omega_j(b')] + 1 \otimes [\Omega_0(b)b' + b\Omega_0(b')].\end{aligned}$$

On the other hand, we have

$$(3.5)\quad \Theta(b*b') = \Theta(\textstyle\sum_i a_i \otimes \chi_i(b,b') + 1 \otimes bb') = \displaystyle\sum_i a_i\Theta(1\otimes \chi_i(b,b')) + \Theta(1\otimes bb')$$
$$= \sum_{i,j} a_i a_j \otimes \Omega_j(\chi_i(b,b')) + \sum_i a_i \otimes \Omega_0(\chi_i(b,b')) + \sum_j a_j \otimes \Omega_j(bb') + 1 \otimes \Omega_0(bb').$$

Comparing (3.4) and (3.5) we get

$$(3.6)\qquad \begin{aligned} 0 = &\sum_{i,j} a_i a_j \otimes [\chi_i(\Omega_j(b),b') + \chi_i(b,\Omega_j(b')) - \Omega_j(\chi_i(b,b'))] \\ &+ \sum_j a_j \otimes [\Omega_j(b)b' + b\Omega_j(b') - \Omega_j(bb')] \\ &+ \sum_i a_i \otimes [\chi_i(\Omega_0(b),b') + \chi_i(b,\Omega_0(b') - \Omega_0(\chi_i(b,b'))] \\ &+ 1 \otimes [\Omega_0(b)b' + b\Omega_0(b') - \Omega_0(bb')]. \end{aligned}$$

Looking at the last term of (3.6), which is the only one belonging to $H^0(Z)\otimes H^*(X)$, we see that $\Omega_0 \in Der_{<0}(H^*(X))$, hence $\Omega_0 = 0$ and only the first two sums in (3.6) are nontrivial. Suppose that the additive basis $\{a_i\}_{i\in I}$ has been choosen so that $I = I_1 \cup I_2 \cup \ldots$, $\deg(a_i) = k$ for $i \in I_k$. The component of (3.6), which belongs to $H^1(Z)\otimes H^*(X)$ is plainly (as $a_i a_j \in H^{\geq 2}(Z)$) equal to

$$\sum_{j\in I_1} a_j \otimes [\Omega_j(b)b' + b\Omega_j(b') - \Omega_j(bb')],$$

hence $\Omega_j \in Der_{<0}(H^*(X))$ and $\Omega_j = 0$, for $j \in I_1$. We have reduced (3.6) to

$$(3.7)\quad 0 = \sum_{\substack{j\in I\setminus I_1\\ i\in I}} a_j a_i \otimes [\chi_i(\Omega_j(b),b') + \chi_i(b,\Omega_j(b')) - \Omega_j(\chi_i(b,b'))]$$
$$+ \sum_{j\in I\setminus I_1} a_j \otimes [\Omega_j(b)b' + b\Omega_j(b') - \Omega_j(bb')].$$

Again, as $\deg(a_i a_j) \geq 3$ for $i \in I$, $j \in I\setminus I_1$, the component of (3.7) in $H^2(Z)\otimes H^*(X)$ is

$$\sum_{j\in I_2} a_j \otimes [\Omega_j(b)b' + b\Omega_j(b') - \Omega_j(bb')],$$

hence $\Omega_j \in Der_{<0}(H^*(X))$ for $j \in I_2$, and $\Omega_j = 0$ for $j \in I_2$. Using this argument as many as necessary, we see that $\Omega_j = 0$ for $j \in I$, hence $\Theta(1\otimes b) = 0$ for each $b \in H^*(X)$. As $a\otimes b = (a\otimes 1)*(1\otimes b)$, the derivation Θ is zero identically on $H^*(Y)$, and Y satisfies (†).

PROOF OF THEOREM 3: Suppose that we have proved our theorem for all $r < q$. Prove it for $r = q$. We can suppose that $\deg(x_1) = \cdots = \deg(x_p)$, $\deg(x_i) > \deg(x_1)$ for $i > p$, with some $1 \leq p \leq q$. If $p = q$, i.e., if all indeterminates are of the same degree, it is nothing to prove, as all negative-degree derivations of our algebra clearly vanish (in the opposite case, $1 \in \mathbf{k}[x_1, \ldots, x_q]/(f_1, \ldots, f_q)$ must be an element of $\mathrm{Im}(\theta)$ for some negative-degree derivation θ, which is impossible by [10; Lemme V.3.(2)]). So, suppose $p < q$. For $i = 1, \ldots, q$ denote by Φ_i the set of all monomials occuring in f_i. Because our space is of type (F), $\Phi_1, \ldots, \Phi_q$ must satisfy the "polynomial condition" P.C. [3; p.119]. We can easily infer from this that, under an appropriate choice of indices, each $\Phi_1, \ldots, \Phi_p$ contains a monomial from $\mathbf{k}[x_1, \ldots, x_p]$. Using the homogenity and a simple degree argument, we see that $f_1, \ldots, f_p \in \mathbf{k}[x_1, \ldots, x_p]$. Notice also that

$$\begin{aligned}\dim(\mathbf{k}[x_1, \ldots, x_p]/(f_1, \ldots, f_p)) &= \dim(\mathbf{k}[x_1, \ldots, x_q]/(f_1, \ldots, f_q, x_{p+1}, \ldots, x_p)) \\ &\leq \dim(\mathbf{k}[x_1, \ldots, x_q]/(f_1, \ldots, f_q)) < \infty,\end{aligned}$$

therefore the rational fibration, reprezented by the model (see [5])

$$\begin{array}{ccc}(\Lambda(x_1, \ldots, x_p, y_1, \ldots, y_p), d) & \longrightarrow & (\Lambda(x_1, \ldots, x_q, y_1, \ldots, y_q), d) \\ & & \downarrow \\ & & (\Lambda(x_{p+1}, \ldots, x_q, y_{p+1}, \ldots, y_q), \overline{d}),\end{array}$$

where $d(y_i) = f_i$, $1 \leq i \leq q$, $\overline{d}(y_j) = \overline{f}_j$, $p+1 \leq j \leq q$, $\overline{f}_j$ is the image of f_j under the natural map $\mathbf{k}[x_1, \ldots, x_q] \to \mathbf{k}[x_1, \ldots, x_q]/(x_1, \ldots, x_p) \cong \mathbf{k}[x_{p+1}, \ldots, x_q]$ has both the base and fiber of type (F) having evenly graded cohomology, therefore they both satisfy (†) by induction, and the induction can go on due to Theorem 1.

PROOF OF THEOREM 2: Again suppose that our theorem has been proved for all spaces having the cohomology of the form $\mathbf{k}[x_1, \ldots, x_r]/(f_1, \ldots, f_r)$ (see the remark before Theorem 3) for all $r < q$ and prove it for $r = q$. Suppose again $\deg(x_1) = \cdots = \deg(x_p)$, $\deg(x_i) > \deg(x_1)$ for $q \geq i > p$, with some $p < q$. Now, for $1 \leq i \leq q$, let Ψ_i denote the set of all quadratic monomials occuring in f_i. As the coformal space, associated to X, is of type (F), $\Psi_1, \ldots, \Psi_q$ must satisfy the polynomial condition of [3]. Similarly as in the proof of Theorem 3 we show that, under some reindexing if necessary, the quadratic parts $(f_1)_2, \ldots, (f_p)_2$ belong to $\mathbf{k}[x_1, \ldots, x_p]$ and a simple degree checking shows that all f_i's are, for $1 \leq i \leq p$, quadratic, i.e., $f_1, \ldots, f_p \in \mathbf{k}[x_1, \ldots, x_p]$. Thus the same induction argument as in the proof of the previous theorem works as well.

REFERENCES

1. A. BOREL, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. 57 (1953), 115-207.

2. ________, "Topics in the homology theory of fiber bundles," Lecture Notes in Mathematics, Springer-Verlag.

3. J. FRIEDLANDER, S. HALPERIN, *An arithmetic characterization of the rational homotopy type of certain spaces*, Inv. Math. **53** (1979), 117-138.
4. S. HALPERIN, *Finiteness in the minimal models of Sullivan*, Trans. Amer. Math. Soc. **230** (1977), 173-199.
5. ————, *Lectures on minimal models*, Mémoires de SMF **9/10** (1983).
6. W. MEIER, *Rational universal fibrations and flag manifolds*, Math. Ann. **258** (1982), 329-340.
7. J.P. SERRE, *Homologie singulière des espaces fibrés*, Ann. of Math. **53** (1951), 425-505.
8. H. SHIGA, M. TEZUKA, *Rational fibrations, homogeneous spaces with positive Euler characteristic and Jacobians*, to appear.
9. D. TANRÉ, "Homotopie rationnelle: Modéles de Chen, Quillen, Sullivan," Lecture Notes in Mathematics 1025, Springer-Verlag 1983.
10. J.-C. THOMAS, "Homotopie rationnelle des fibrés de Serre," Thesis, University of Lille I, 1980.

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