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Geometric constructions and representations

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**Geometric Constructions of Representations**

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## SECTION 0. BACKGROUND.

I'll describe certain geometric constructions of representations for a class of groups that is important in differential geometry, harmonic analysis, and certain aspects of relativity and particle physics: the semisimple Lie groups. In order to do that, I'll first sketch some of the basic facts in representation theory, then in turn discuss discrete series representations, tempered series representations and standard admissible representations.

The constructions I'm going to describe are constructions connected with the idea of geometric quantization. I won't describe the Beilinson-Bernstein method **[3]** based on $\mathcal{D}$-modules and localization; but see **[23]** and **[56]** for the connection with the constructions I will describe. I won't describe ring-theoretic methods based on the enveloping algebra. I won't describe Howe's method of dual reductive pairs. And I won't describe a variety of methods adapted to particular circumstances in physics and in special function theory.

Let $G$ be a locally compact hausdorff topological group with a countable basis for open sets. A **unitary representation** of $G$ is a homomorphism $\pi : G \to \mathbf{U}(H_\pi)$ where $H_\pi$ is a separable Hilbert space and $x \mapsto \langle u, \pi(x)v\rangle$ is a continuous function on $G$ for every $u, v \in H_\pi$. Two unitary representations $\pi, \pi'$ and **equivalent** if there is a unitary map $\alpha : H_\pi \to H_{\pi'}$ such that $\alpha \cdot \pi(x) = \pi'(x)$ for all $g \in G$. A unitary representation $\pi$ of $G$ is called **irreducible** if $H_\pi$ has no proper closed subspace invariant under all the $\pi(x), x \in G$. The set $\widehat{G}$ of all equivalence classes of irreducible unitary representations of $G$ is called the **unitary dual** of $G$.

Fix a unitary representation $\pi$ of $G$ as above. If $H_\pi$ is finite dimensional, say with basis $\{v_1, \ldots, v_n\}$, then the functions $x \mapsto \langle \pi(x)u, v\rangle$ are the ordinary matrix coefficients of $\pi$. In general we will refer to the functions $f_{u,v}(x) = \langle u, \pi(x)v\rangle$ as **coefficients** of $\pi$. They are really coefficients of the dual, $\pi^*$, but this is compensated by the convenient property $f_{u,v}(g_1^{-1}xg_2) = f_{\pi(g_1)u,\pi(g_2)v}(x)$. In other words, the left and right regular representations of $G$ on $L^2(G)$,

$$[L(x)f](y) = f(x^{-1}y) \qquad \text{and} \qquad [R(x)f](y) = f(yx),$$

act on $f_{u,v}$, $L$ acting by $\pi$ on $u$ and $R$ acting by $\pi^*$ on $v$. So we can view the space of coefficients of $\pi$ as the irreducible $G \times G$-module $H_\pi \otimes H_\pi^*$. This is the fundamental connection between representation theory and harmonic analysis.

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EXAMPLE 1: Let $G$ be compact. Then every irreducible unitary representation of $G$ is finite dimensional. The Peter–Weyl Theorem expresses $L^2(G) = \sum_{\pi\in\widehat{G}} H_\pi \otimes H_\pi^*$, orthogonal direct sum of the spaces of coefficients of (the equivalence classes of) the irreducible unitary representations of $G$. In the case where $G$ is the circle group $\{z \in \mathbb{C} : \|z\| = 1\}$, $\widehat{G}$ consists of the characters $\chi_n(z) = z^n$, $n \in \mathbb{Z}$. $\chi_n$ is a 1-dimensional representation. Here the Peter–Weyl Theorem simply interprets the classical Fourier expansion of periodic functions as an orthogonal direct sum decomposition $L^2(G) = \sum_{n\in\mathbb{Z}} \mathbb{C}\chi_n$.

EXAMPLE 2: Let $G$ be commutative. Then $\widehat{G}$ has a natural structure of locally compact abelian group, and the Fourier transform carries a function $f \in L^1(G)$ to a continuous function $\widehat{f}$ vanishing at infinity on $\widehat{G}$ defined by $\widehat{f}(\chi) = \int_G f(x)\overline{\chi(x)}dx$. There is a unique choice of invariant measure on $\widehat{G}$ such that $f \mapsto \widehat{f}$ maps $L^1(G) \cap L^2(G)$ isometrically (for $L^2$ norm) into $L^2(\widehat{G})$. This extends by continuity to an isometry of $L^2(G)$ onto $L^2(\widehat{G})$. The inverse Fourier transform $f(x) = \int_{\widehat{G}} \widehat{f}(\chi)\chi(x)d\chi$ expresses $L^2(G)$ as a continuous direct sum $\int_{\widehat{G}} \mathbb{C}\chi\, d\chi$. In the case $G = \mathbb{R}^n$ this comes down to the classical Fourier transform formula $\widehat{f}(x) = (\frac{1}{2\pi})^{n/2}\int_{\mathbb{R}^n} f(x)e^{-i\xi\cdot x}dx$ and the classical Fourier inversion formula $f(x) = (\frac{1}{2\pi})^{n/2}\int_{\mathbb{R}^n}\widehat{f}(\xi)e^{i\xi\cdot x}d\xi$. They express $L^2(\mathbb{R}^n)$ as $\int_{\mathbb{R}^n} \mathbb{C}\chi_\xi\, d\xi$, $\chi_\xi(x) = e^{i\xi\cdot x}$. Representation theory and harmonic analysis on a semisimple Lie group $G$ combine aspects of both examples.

The set $Car(G)$ of conjugacy classes of Cartan subgroups $H \subset G$ is finite. Given a Cartan subgroup $H \subset G$ there is a distinguished decomposition $H = T \times A$ where $T$ is compact and $A$ is isomorphic to an $\mathbb{R}^n$. So $\widehat{H}$ consists of the $\chi \otimes \sigma$ with $\chi \in \widehat{T}$ and $\sigma \in \widehat{A}$. Furthermore, there is a series $\widehat{G}_H \subset \widehat{G}$ of unitary representations $\pi = \pi_{\chi,\sigma}$ of $G$ more or less parameterized by $\widehat{H}$. Roughly speaking $\pi_{\chi,\sigma}$ acts like Example 1 in the $\chi$ variable, like Example 2 in the $\sigma$ variable. This series depends (up to equivalence) only on the class of $H$ in $Car(G)$. These series are the **tempered series** of representations of $G$. The $\pi_{\chi,\sigma}$ are the **standard tempered series** representations of $G$.

The various tempered series exhaust enough of $\widehat{G}$ for a decomposition of $L^2(G)$ essentially as $\sum_{H\in Car(G)} \sum_{\chi\in\widehat{T}} \int_{\widehat{A}} H_{\pi_{\chi,\sigma}} \otimes H^*_{\pi_{\chi,\sigma}} m(H : \chi : \sigma)d\sigma$. Here $m(H : \chi : \sigma)d\sigma$ is the **Plancherel measure** on $\widehat{G}$. This was worked out by Harish–Chandra ([20], [21], [22]) for groups of what is now called "Harish–Chandra class", and somewhat more generally by Herb and myself ([61]; [25], [26]; [29], [30]). Harish–Chandra's approach is based on an analysis of the structure of the Schwartz space, while Herb and I use explicit character formulae (compare [11], [49], [25], [27]). Analysis of the Schwartz space is not yet complete in the general case, essentially because of interference between the various series (compare [37]), but at the moment it seems pretty sure that a certain extension of [54] will control this.

In any case, our objective in these notes is much more modest: to understand the tempered series representations, and more generally the standard admissible representations, for semisimple Lie groups, in a concrete geometric manner.

If $H$ is compact the corresponding tempered series is the "discrete series". In general, $H = T \times A$ defines $M \times A$, the centralizer of $A$ in $G$, and $\widehat{M_T}$ is the discrete series for $M$. A certain construction, starting with $\widehat{M_T}$, gives $\widehat{G}_H$. We also consider non-tempered representations, but there the results depend very strongly on the ideas for tempered representations. For these reasons we start with the discrete series.

## Section 1. Discrete Series: Definition, Parameterization and Geometric Construction.

### The General Notion of Discrete Series

The **discrete series** of a unimodular locally compact group $G$ is the subset $\widehat{G}_{\text{disc}} \subset \widehat{G}$ consisting of those irreducible unitary representation classes $[\pi]$ such that $\pi$ is equivalent to a subrepresentation of the left regular representation $L$ of $G$. One can check that the following are equivalent: (i) $\pi$ is a discrete series representation of $G$, (ii) every coefficient $f_{u,v}(x) = \langle u, \pi(x)v\rangle$ belongs to $L^2(G)$, (iii) for some nonzero $u, v \in H_\pi$, the coefficient $f_{u,v} \in L^2(G)$. Then one has orthogonality relations much as in the case of finite groups: there is a real number $deg(\pi) > 0$ such that the $L^2(G)$-inner product of coefficients of $\pi$ is given by

$$\langle f_{u,v}, f_{s,t}\rangle = \frac{1}{deg(\pi)}\langle u, s\rangle\overline{\langle v, t\rangle} \text{ for } s, t, u, v \in H_\pi.$$

Furthermore, if $\pi'$ is another discrete series representation of $G$, and is not equivalent to $\pi$, then

$$\langle f_{u,v}, f_{u',v'}\rangle = 0 \text{ for } u, v \in H_\pi \text{ and } u', v' \in H_{\pi'}.$$

In fact these orthogonality relations come out of convolution formulae. With the usual

$$f * h(x) = [L(f)h](x) = \int_G f(y)h(y^{-1}x)\, dy$$

we have

$$f_{u,v} * f_{s,t} = \frac{1}{deg(\pi)}\langle u, t\rangle f_{s,v} \quad \text{for} \quad s, t, u, v \in H_\pi$$

and

$$f_{u,v} * f_{u',v'} = 0 \text{ for } u, v \in H_\pi \text{ and } u', v' \in H_{\pi'}$$

whenever $\pi$ and $\pi'$ are inequivalent discrete series representations of $G$. These results were proved by independently—under somewhat different conditions—by Godement **[7]** and Harish–Chandra **[14]**. One can derive them directly as in Dixmier (**[6]**, Sec. 14) or by applying Rieffel's results **[48]** to the convolution algebra $L^1(G) \cap L^2(G)$.

If $G$ is compact, then every class in $\widehat{G}$ is discrete series, and if Haar measure is normalized as usual to total volume 1 then $deg(\pi)$ has the usual meaning, the dimension of $H_\pi$. So the orthogonality relations for irreducible unitary representations of compact groups are more or less equivalent to the Peter–Weyl Theorem. More generally, if $G$ is a unimodular locally compact group then $L^2(G) = {}^0L^2(G) \oplus {}'L^2(G)$, orthogonal direct sum, where ${}^0L^2(G) = \sum_{[\pi]\in\widehat{G}_{\rm disc}} H_\pi \otimes H_\pi^*$, the "discrete" part, and ${}'L^2(G) = {}^0L^2(G)^\perp$, the "continuous" part. If, further, $G$ is a group of type I then ${}'L^2(G)$ is a continuous direct sum (direct integral) over $\widehat{G} \setminus \widehat{G}_{\rm disc}$ of the Hilbert spaces $H_\pi \otimes H_\pi^*$.

## Harish-Chandra Class

As hinted at the end of the previous section, we will be constructing representations of semisimple Lie groups from certain representations of certain subgroups. One problem is that these subgroups generally will not be semisimple. So we want to work with a class of groups that is hereditary in the sense that it includes all the connected semisimple Lie groups and also includes the above-mentioned subgroups of groups in the class. This forces us to work with a somewhat technical class [**61**] of reductive Lie groups. (A Lie group is called **reductive** if its Lie algebra is the direct sum of a semisimple Lie algebra and a commutative Lie algebra.) For ease of exposition, I'll work here with a smaller hereditary class. Instead of all connected semisimple Lie groups it contains the connected semisimple Lie groups with finite center. That is the **Harish–Chandra class**, or **class** $\mathcal{H}$, from [**20**], [**21**] and [**22**], defined as follows.

Let $G$ be a reductive Lie group, $G^0$ its identity component, $\mathfrak{g}_0$ its Lie algebra, and $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$. Suppose that $[G^0, G^0]$ has finite center, that $G/G^0$ is finite, and that if $x \in G$ then $Ad(x)$ is an inner automorphism of $\mathfrak{g}$. Then we say that $G$ belongs to class $\mathcal{H}$. From now on we will assume that $G$ belongs to class $\mathcal{H}$. A good example to keep in mind is the indefinite-metric unitary group $U(p, q)$.

## Characters of Representations

If $\pi$ is a unitary representation of $G$, and if $f \in L^1(G)$, we have the bounded operator $\pi(f) = \int_G f(x)\pi(x)dx$ on $H_\pi$. Now suppose that $\pi$ has finite composition series, i.e., is a finite sum of irreducible representations. If $f \in C_c^\infty(G)$ then $\pi(f)$ is of trace class. Furthermore, the map

$$\Theta_\pi : C_c^\infty(G) \to \mathbb{C} \text{ defined by } \Theta_\pi(f) = \text{ trace } \pi(f)$$

is a distribution on $G$. $\Theta_\pi$ is called the **character**, the **distribution character** or the **global character** of $\pi$. Character theory for representations of semisimple Lie groups was developed in Harish–Chandra's papers [8], [9], [**10**], [**12**], [**13**], [**14**], [**15**] and [**16**].

Let $\mathcal{Z}(\mathfrak{g})$ denote the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. If we interpret $\mathcal{U}(\mathfrak{g})$ as the algebra of all left-invariant differential operators on $G$ then $\mathcal{Z}(\mathfrak{g})$ is the subalgebra of those which are also invariant under right translations. If $\pi$ is irreducible then $d\pi|_{\mathcal{Z}(\mathfrak{g})}$ is an associative algebra homomorphism $\chi_\pi : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ called the **infinitesimal character** of $\pi$. We say that $\pi$ is **quasi-simple** if it has an infinitesimal character, i.e. if it is a direct sum of irreducible representations that have the same infinitesimal character.

Let $\pi$ be quasi-simple. Then the distribution character $\Theta_\pi$ satisfies a system of differential equations

$$z \cdot \Theta_\pi = \chi_\pi(z)\Theta_\pi \text{ for all } z \in \mathcal{Z}(\mathfrak{g})$$

The regular set

$$G' = \{x \in G : \mathfrak{g}^{Ad(x)} \text{ is a Cartan subalgebra of } \mathfrak{g}\}$$

is a dense open subset whose complement has codimension $\geqq 2$. Every $x \in G'$ has a neighborhood on which at least one of the operators $z \in \mathcal{Z}(\mathfrak{g})$ is elliptic. It follows that $\Theta_\pi|_{G'}$ is integration against a real analytic function $T_\pi$ on $G'$. A much deeper fact **[15]** is that $\Theta_\pi$ has only finite jump singularities across the singular set $G \setminus G'$, so $T_\pi$ is locally $L^1$ and $\Theta_\pi$ is integration against it,

$$\Theta_\pi(f) = \int_G f(x)T_\pi(x)dx \text{ for all } f \in C_c^\infty(G).$$

So we may (and do) identify $\Theta_\pi$ with the function $T_\pi$. This key element of Harish–Chandra's theory allows the possibility of *a priori* estimates on characters and coefficients as well as explicit character formulae.

## Discrete Series for Groups of Class $\mathcal{H}$

Fix a Cartan involution $\theta$ of $G$. In other words, $\theta$ is an automorphism of $G$, $\theta^2$ is the identity, and the fixed point set $K = G^\theta$ is a maximal compact subgroup of $G$. The choice is essentially unique, because the Cartan involutions of $G$ are just the $Ad(x) \cdot \theta \cdot Ad(x)^{-1}$, $x \in G^0$. If $G = U(p,q)$ then $\theta(x) = {}^t x^{-1}$ and $K = U(p) \times U(q)$.

Every Cartan subgroup of $G$ is $Ad(G^0)$-conjugate to a $\theta$-stable Cartan subgroup. In particular, $G$ has compact Cartan subgroups if and only if $K$ contains a Cartan subgroup of $G$.

Harish–Chandra proved (**[14]**, **[19]**) that $G$ has discrete series representations if and only if it has a compact Cartan subgroup. Suppose that this is the case and fix a compact Cartan subgroup $T \subset K$ of $G$. Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{t})$ be the root system, $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{t})$ a choice of positive root system, and let $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$, half the trace of $ad(\mathfrak{t})$ on $\sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$.

If $\pi$ is a discrete series representation of $G$ and $\Theta_\pi$ is its distribution character, then the equivalence class of $\pi$ is determined by the restriction of $\Theta_\pi$ to $T \cap G'$. So we can parameterize the discrete series of $G$ by parameterizing those restrictions. Here we follow **[19]**, **[61]** and **[20]**.

Let $G^\dagger$ denote the finite index subgroup $TG^0 = Z_G(G^0)G^0$ of $G$. The Weyl group $W^\dagger = W(G^\dagger, T)$ coincides with $W^0 = W(G^0, T^0)$ and is a normal subgroup of $W = W(G,T)$.

Let $\chi \in \widehat{T}$. Since $T^0$ is commutative, $\chi$ has differential $d\chi(\xi) = \lambda(\xi)I$ where $\lambda \in i\mathfrak{t}_0^*$ and where $I$ is the identity on the representation space of $\chi$. Suppose that $\lambda + \rho$ is regular, i.e., that $\langle \lambda + \rho, \alpha \rangle \neq 0$ for all $\alpha \in \Phi$. Then there are unique discrete series representations $\pi_\chi^0$

of $G^0$ and $\pi_\chi^\dagger$ of $G^\dagger$ whose distribution characters satisfy

$$\Theta_{\pi_\chi^0}(x) = \pm\frac{\sum_{w\in W^0} sign(w)e^{w(\lambda+\rho)}}{\prod_{\alpha\in\Phi^+}(e^{\alpha/2}-e^{-\alpha/2})} \text{ and } \Theta_{\pi_\chi^\dagger}(zx) = \chi(z)\Theta_{\pi_\chi^0}(x)$$

for $z \in Z_G(G^0)$ and $x \in T^0 \cap G'$. Here note that $\pi_\chi^\dagger = \chi|_{Z_G(G^0)} \otimes \pi_\chi^0$. The same datum $\chi$ specifies a discrete series representation $\pi_\chi$ of $G$, $\pi_\chi = \mathrm{Ind}_{G^\dagger}^G(\pi_\chi^\dagger)$. $\pi_\chi$ is characterized by the fact that its distribution character is supported in $G^\dagger$, where

$$\Theta_{\pi_\chi} = \sum_{1\leqq i\leqq r} \Theta_{\pi_\chi^\dagger}\cdot\gamma_i^{-1}$$

with $\gamma_i = Ad(g_i)|_{G^\dagger}$ where $\{g_1,\dots,g_r\}$ is any system of coset representatives of $G$ modulo $G^\dagger$. To combine these into a single formula one chooses the $g_i$ so that they normalize $T$, i.e. chooses the $\gamma_i$ to be a system of coset representatives of $W$ modulo $W^\dagger$.

Every discrete series representation of $G$ is equivalent to a representation $\pi_\chi$ as just described. Discrete series representations $\pi_\chi$ and $\pi_{\chi'}$ are equivalent if and only if $\chi' = \chi\cdot w^{-1}$ for some $w \in W$.

## Geometric Realizations of Discrete Series Representations on Spaces of Square Integrable Harmonic Differential Forms

If $G$ is a compact connected Lie group and $T$ is a maximal torus, then a choice $\Phi^+ = \Phi^+(\mathfrak{g},\mathfrak{t})$ of positive root system defines a $G$-invariant complex manifold structure on $G/T$ by: $\sum_{\alpha\in\Phi^+}\mathfrak{g}_\alpha$ represents the holomorphic tangent space. Now fix that structure and let $\lambda \in i\mathfrak{t}_0^*$ be integral, that is, $e^\lambda$ is a well defined character of $T$. View $e^\lambda$ as a representation of $T$ on a 1-dimensional vector space $L_\lambda$ and let $\mathbb{L}_\lambda \to G/T$ denote the associated homogeneous holomorphic hermitian line bundle. We write $\mathcal{O}(\mathbb{L}_\lambda) \to G/T$ for the sheaf of germs of holomorphic sections of $\mathbb{L}_\lambda \to G/T$. The group $G$ acts on everything here, including the cohomologies $H^q(G/T;\mathcal{O}(\mathbb{L}_\lambda))$. As before let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. The Bott–Borel–Weil Theorem ([**5**], [**36**]) is

Theorem. *If $\lambda+\rho$ is singular then every $H^q(G/T;\mathcal{O}(\mathbb{L}_\lambda)) = 0$. Now suppose that $\lambda+\rho$ is regular, let $w$ denote the unique Weyl group element such that $\langle w(\lambda+\rho),\alpha\rangle > 0$ for all $\alpha\in\Phi^+$, and let $\ell(w)$ denote its length as a word in the simple root reflections. Then* (i) *$H^q(G/T;\mathcal{O}(\mathbb{L}_\lambda)) = 0$ for $q \neq \ell(w)$, and* (ii) *$G$ acts irreducibly on $H^{\ell(w)}(G/T;\mathcal{O}(\mathbb{L}_\lambda))$ by the representation with highest weight $w(\lambda+\rho)-\rho$.*

In the Bott–Borel–Weil Theorem, $\ell(w)$ can be described as the number of positive roots that $w$ carries to negative roots, the representation of $G$ with highest weight $w(\lambda+\rho)-\rho$ can be described as the discrete series representation with Harish–Chandra parameter $w(\lambda+\rho)$, and, by Kodaira–Hodge Theory, $H^q(G/T;\mathcal{O}(\mathbb{L}_\lambda))$ is naturally $G$-isomorphic to the space of harmonic differential forms of bidegree $(0,q)$ on $G/T$ with values in $\mathbb{L}_\lambda$.

Kostant and Langlands independently conjectured an analog of the Bott–Borel–Weil Theorem for connected noncompact semisimple Lie groups with finite center. The conjecture was proved in two stages by Schmid ([**51**], [**53**]) and the result was extended to general semisimple Lie groups in [**61**].

Let $G$ again be a reductive Lie group of class $\mathcal{H}$ that has discrete series representations, i.e. that has a compact Cartan subgroup $T$. The root system $\Phi = \Phi(\mathfrak{g}, \mathfrak{t})$ decomposes as the disjoint union of the **compact roots** $\Phi_K = \Phi(\mathfrak{k}, \mathfrak{t}) = \{\alpha \in \Phi : \mathfrak{g}_\alpha \subset \mathfrak{k}\}$ and the **noncompact roots** $\Phi_{G/K} = \Phi \setminus \Phi_K$.

As in the compact case, a choice $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{t})$ of positive root system defines a $G$-invariant complex manifold structure on $G/T$ such that $\sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$ represents the holomorphic tangent space. Fix a choice of $\Phi^+$. Write $\Phi_K^+$ for $\Phi^+ \cap \Phi_K$ and $\Phi_{G/K}^+$ for $\Phi^+ \cap \Phi_{G/K}$. Let $\chi \in \widehat{T}$, let $E_\chi$ be the representation space, and let $\mathbb{E}_\chi \to G/T$ denote the associated hermitian homogeneous holomorphic vector bundle. Let $\Box$ denote the Kodaira-Hodge-Laplace operator $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ on $\mathbb{E}_\chi$. Then we have spaces

$$\mathcal{H}^q(G/T; \mathbb{E}_\chi): \text{ harmonic } L^2\ \mathbb{E}_\chi\text{-valued } (0,q)\text{-forms on } G/T$$

on which $G$ acts naturally and the natural action of $G$ is a unitary representation.

As noted above, if $G$ is compact then the space $\mathcal{H}^q(G/T; \mathbb{E}_\chi)$ of $L^2$ harmonic forms is naturally identified with the sheaf cohomology $H^q(G/T; \mathcal{O}(\mathbb{E}_\chi))$.

In general, for reductive Lie groups $G$ of class $\mathcal{H}$, let $\lambda \in i\mathfrak{t}_0^*$ such that $d\chi = \lambda I$ where $I$ is the identity transformation of $E_\chi$.

THEOREM. *If $\lambda + \rho$ is singular then every $\mathcal{H}^q(G/T; \mathbb{E}_\chi) = 0$. Now suppose that $\lambda + \rho$ is regular and let*

$$q(\lambda + \rho) = |\{\alpha \in \Phi_K^+ : \langle \lambda + \rho, \alpha \rangle < 0\}| + |\{\beta \in \Phi_{G/K}^+ : \langle \lambda + \rho, \beta \rangle > 0\}|.$$

*Then $\mathcal{H}^q(G/T; \mathbb{E}_\chi) = 0$ for $q \neq q(\lambda + \rho)$, and $G$ acts irreducibly on $\mathcal{H}^{q(\lambda+\rho)}(G/T; \mathbb{E}_\chi)$ by the discrete series representation $\pi_\chi$.*

An interesting variation on this result realizes the discrete series on spaces of $L^2$ bundle-valued harmonic spinors. See [**47**], [**52**] and [**62**].

### GEOMETRIC REALIZATIONS OF DISCRETE SERIES REPRESENTATIONS ON DOLBEAULT COHOMOLOGY SPACES

Another important variation on the Kostant–Langlands Conjecture result—which in fact preceded its solution—is Schmid's Dolbeault cohomology realization [**50**]. Note that $K/T$ is a maximal compact complex submanifold of $G/T$ and denote $s = \dim_{\mathbb{C}} K/T$. Whenever

$$\lambda + \rho \text{ is antidominant: } \langle \lambda + \rho, \gamma \rangle < 0 \text{ for all } \gamma \in \Phi^+$$

we have $s = q(\lambda+\rho)$. This is the case where the bundle $\mathbb{E}_\chi \to G/T$ is negative.

If $\pi$ is a discrete series representation of $G$, we can choose the positive root system $\Phi^+$ so that $\pi = \pi_\chi$ where $\lambda = d\chi$ is such that $\lambda+\rho$ is antidominant. Thus there is no restriction on $\pi_\chi$ in

THEOREM. *Suppose that $\lambda+\rho$ is antidominant, so $s = q(\lambda+\rho)$. Then $H^s(G/T;\mathcal{O}(\mathbb{E}_\chi))$ has a natural Fréchet space structure, $G$ acts naturally on $H^s(G/T;\mathcal{O}(\mathbb{E}_\chi))$ by a continuous representation, and this representation is infinitesimally equivalent to $\pi_\chi$.*

Here we use the following definition of infinitesimal equivalence. Let $\pi$ and $\psi$ be continuous representations of $G$ on complete locally convex topological vector spaces $V_\pi$ and $V_\psi$. Let $(V_\pi)_{(K)}$ and $(V_\psi)_{(K)}$ denote the respective subspaces of $K$-finite vectors. They are modules for the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. An *infinitesimal equivalence* of $V_\pi$ with $V_\psi$ means a $\mathcal{U}(\mathfrak{g})$-isomorphism of $(V_\pi)_{(K)}$ onto $(V_\psi)_{(K)}$. In the case of the Theorem just stated, the infinitesimal equivalence,

$$\mathcal{H}^s(G/T;\mathbb{E}_\chi)_{(K)} \to H^s(G/T;\mathcal{O}(\mathbb{E}_\chi))_{(K)} ,$$

as a map on spaces of $K$-finite vectors, is the map that sends an $L^2$ harmonic form to (the sheaf cohomology class that corresponds to) its Dolbeault class.

## FORMULATION BY MEANS OF BASIC DATA

We rephrase the theorem of Dolbeault cohomology realizations of the discrete series in anticipation of our realization of the tempered series.

By **basic datum** for $G$ we mean a triple $(H,\mathfrak{b},\chi)$ such that

(i) $H$ is a Cartan subgroup of $G$

(ii) $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ with $\mathfrak{h} \subset \mathfrak{b}$

(iii) $\chi$ is a finite dimensional representation of $(\mathfrak{b}, H)$

Here (i) means that $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ where $\mathfrak{n} = [\mathfrak{b},\mathfrak{b}] = \sum_{\alpha\in\Phi^+} \mathfrak{g}_{-\alpha}$ for some positive root system $\Phi^+ = \Phi^+(\mathfrak{g},\mathfrak{h})$. And (ii) means that $\chi$ consists of a Lie group representation of $H$ on a finite dimensional vector space $E_\chi$ and a Lie algebra representation of $\mathfrak{b}$ on $E_\chi$, that are consistent in the sense that they satisfy both (a) the restriction to $\mathfrak{h}$ of the representation of $\mathfrak{b}$ is the differential of the representation of $H$ and (b) if $h \in H$ and $\xi \in \mathfrak{b}$ then $\chi(Ad(h)\xi) = \chi(h)\chi(\xi)\chi(h)^{-1}$. Of course (a) implies (b) in case $H$ is connected.

Now let $X$ denote the **flag variety** of all Borel subalgebras $\mathfrak{b} \subset \mathfrak{g}$. If $B$ is a Borel subgroup of $G_\mathbb{C}$ (normalizer of a Borel subalgebra of $\mathfrak{g}$) then we can identify $G_\mathbb{C}/B$ with $X$ by the usual $gB \leftrightarrow Ad(g)\mathfrak{b}$. The $G$-orbit structure of $X$ is pretty well understood [60].

Fix a basic datum $(H,\mathfrak{b},\chi)$ and the associated homogeneous vector bundle $\mathbb{E}_\chi \to G/H$. Let $\mathcal{O}_\mathfrak{n}(\mathbb{E}_\chi) \to G/H$ be the sheaf of germs of smooth sections annihilated by the right action of $\mathfrak{n} = [\mathfrak{b},\mathfrak{b}]$. Then $G$ acts naturally on the cohomologies $H^q(G/H;\mathcal{O}_\mathfrak{n}(\mathbb{E}_\chi))$.

The theorem of Dolbeault cohomology realizations of the discrete series identifies discrete series representations and the representations infinitesimally equivalent to cohomology representations for basic data $(H, \mathfrak{b}, \chi)$ with $H$ compact. We see this as follows. When $H$ is compact, $gH \mapsto Ad(g)\mathfrak{b}$ gives a diffeomorphism of $G/H$ onto the orbit $G\cdot\mathfrak{b}$. The orbit $G\cdot\mathfrak{b}$ is open in $X$, thus inherits the structure of complex manifold from $X$. This is the same complex structure that we considered before—$\mathfrak{n}$ represents the antiholomorphic tangent space—and $\mathcal{O}_{\mathfrak{n}}(\mathbb{E}_\chi)$ is the sheaf of germs of holomorphic sections of $\mathbb{E}_\chi \to G/H$. Thus, for $H$ compact and $\mathbb{E}_\chi \to G/H$ sufficiently negative, the $H^q(G/H; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}_\chi))$ yield the discrete series representations of $G$ up to infinitesimal equivalence.

We will take this viewpoint in describing the tempered series of representations and, more generally, the standard admissible representations of $G$.

## SECTION 2. TEMPERED SERIES: DEFINITION, PARAMETERIZATION AND GEOMETRIC CONSTRUCTION.

### THE GENERAL NOTION OF INDUCED REPRESENTATION

Let $G$ be a separable locally compact group, $dx = d\mu_G(x)$ its left Haar measure, and $\Delta_G$ its modular function. Thus

$$\int_G f(xy)dy = \int_G f(y)dy = \int_G f(y^{-1})\Delta_G(y^{-1})dy$$

and

$$\int_G f(xyx^{-1})dy = \Delta_G(x)\int_G f(y)dy = \int_G f(yx^{-1})dy$$

for $y \in G$ and $f \in C_c(G)$.

Let $K$ be a closed subgroup, $dk = d\mu_K(k)$ its left Haar measure, and $\Delta_K$ its modular function. If $\eta$ is a weakly continuous homomorphism from $K$ to the bounded linear operators on a Hilbert space $V_\eta$ we denote

$$C_c(G/K; V_\eta) = C_c(G/K;\eta) = \left\{ f : G \to V_\eta : \begin{array}{c} f \text{ is continuous,} \\ f \text{ is compactly supported mod } K, \text{ and} \\ f(xk) = \eta(k)^{-1}f(x) \text{ for } x \in G,\ k \in K \end{array} \right\}$$

Consider $C_c(G/K; \Delta_{G/K})$, where $\Delta_{G/K} : K \to \mathbb{R}^+$ by $\Delta_{G/K}(k) = \Delta_G(k)/\Delta_K(k)$. The linear map $\tau : C_c(G) \to C_c(G/K; \Delta_{G/K})$, defined by $(\tau f)(x) = \int_K f(xk)dk$, is surjective, and $\tau f = 0$ implies $\int_G f(x)dx = 0$. Now $\int_G (\tau F)(x)dx$ is a positive continuous $G$-invariant linear functional on $C_c(G/K; \Delta_{G/K})$. We write it as integration against a measure $d(xK) = d\mu_{G/K}(xK)$:

$$\int_{G/K} F(xK)d(xK) = \int_{G/K} F(xK)d\mu_{G/K}(xK) = \int_G (\tau F)(x)dx$$

for $F \in C_c(G/K; \Delta_{G/K})$. One can in fact express this as a genuine integral.

Now let $\eta$ be a (weakly continuous) unitary representation of $K$, let $V_\eta$ denote the representation space of $\eta$, and notice that $\eta \otimes \Delta_{G/K}^{1/2}$ is a weakly continuous homomorphism from $K$ to the bounded linear operators on the Hilbert space $V_\eta$. So we have the space $C_c(G/K; \eta \otimes \Delta_{G/K}^{1/2})$. If $F_1$ , $F_2 \in C_c(G/K; \eta \otimes \Delta_{G/K}^{1/2})$ then, since $\eta$ is unitary, the pointwise inner product satisfies

$$\langle F_1(xk), F_2(xk)\rangle_{V_\eta} = \Delta_{G/K}(k)^{-1}\langle F_1(x), F_2(x)\rangle_{V_\eta}$$

so our "integral" over $G/K$ defines a global inner product

$$\langle F_1, F_2\rangle = \int_{G/K} \langle F_1(x), F_2(x)\rangle_{V_\eta}\, d\mu_{G/K}(xK).$$

$G$ acts on the Hilbert space completion of $C_c(G/K; \eta \otimes \Delta_{G/K}^{1/2})$ with respect to the global inner product defined just above: $[\pi(x)F](y) = F(x^{-1}y)$. This natural action is a unitary representation of $G$, denoted $\mathrm{Ind}_{K\uparrow G}(\eta)$ and called the *representation of $G$* **induced** *or* **unitarily induced** *by $\eta$*.

It is often convenient to view $\mathrm{Ind}_{K\uparrow G}(\eta)$ as the action of $G$ on $L^2$ sections of the Hilbert space bundle over $G/K$ associated to $G \to G/K$ by the action $\eta \otimes \Delta_{G/K}^{1/2}$ of $K$ on $V_\eta$. And of course the notion of induced representation is simplified when $\Delta_{G/K} \equiv 1$.

A useful example: Suppose that $\Delta_{G/K} \equiv 1$ and let $1_K$ denote the trivial 1-dimensional representation of $K$. Then $\mathrm{Ind}_{K\uparrow G}(1_K)$ is the left regular representation of $G$ on $L^2(G/K)$. In particular, $\mathrm{Ind}_{\{1\}\uparrow G}(\{1\})$ is the left regular representation of $G$.

An important tool in working with induced representations is **induction by stages**: given closed subgroups $K \subset M \subset G$ one has

$$\mathrm{Ind}_{K\uparrow G}(\eta) \text{ is equivalent to } \mathrm{Ind}_{M\uparrow G}(\mathrm{Ind}_{K\uparrow M}(\eta)).$$

This shows, for example, that if $K$ is a closed subgroup of $G$ then the left regular representations satisfy $L_G \simeq \mathrm{Ind}_{K\uparrow G}(L_K)$.

For more details on induced representations, including projective representations, the Mackey little-group method, and complete proofs, see papers of Mackey ([**39**], [**40**], [**41**], [**42**], [**43**], [**44**]), Moore ([**2**], [**45**], [**46**]) and Duflo ([**4**], Ch. V), and the summary ([**63**], Appendix).

## Cuspidal Parabolic Subgroups and Tempered Series

Let $H$ be a Cartan subgroup in the reductive Lie group $G$ of class $\mathcal{H}$. Fix a Cartan involution $\theta$ of $G$ such that $\theta(H) = H$. Its fixed point set $K = G^\theta$ is a maximal compact subgroup of $G$. We decompose

$$\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0 \text{ and } H = T \times A \quad \text{where} \quad T = H \cap K \text{ and } A = \exp_G(\mathfrak{a}_0).$$

Then the centralizer $Z_G(A)$ of $A$ in $G$ has form $M \times A$ where $\theta(M) = H$. Now **[61]** $M$ is a reductive Lie group of class $\mathcal{H}$, and $T$ is a compact Cartan subgroup of $M$, so $M$ has discrete series representations.

Suppose that the positive root system $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{h})$ is defined by positive root systems $\Phi^+(\mathfrak{m}, \mathfrak{t})$ and $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$. This means that

$$\Phi^+(\mathfrak{m}, \mathfrak{t}) = \{\alpha|_{\mathfrak{t}} : \alpha \in \Phi^+(\mathfrak{g}, \mathfrak{h}),\ \alpha|_{\mathfrak{a}} = 0\}$$

and

$$\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0) = \{\beta|_{\mathfrak{a}_0} : \beta \in \Phi^+(\mathfrak{g}, \mathfrak{h}),\ \beta|_{\mathfrak{a}} \neq 0\}.$$

In other words, given $\mathfrak{h}$, the Borel subalgebra

$$\mathfrak{b} = \mathfrak{h} + \mathfrak{n} \text{ where } \mathfrak{n} = [\mathfrak{b}, \mathfrak{b}] = \sum\nolimits_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{h})} \mathfrak{g}_{-\alpha}$$

is chosen to maximize $\mathfrak{u} = \mathfrak{n} \cap \overline{\mathfrak{n}}$. Note that $\mathfrak{u}$ has real form $\mathfrak{u}_0 = \mathfrak{u} \cap \mathfrak{g}_0$, which is the sum $\sum_{\gamma \in \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)} (\mathfrak{g}_0)_\gamma$ of the positive restricted root spaces.

A subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is called **parabolic** if it contains a Borel subalgebra. A subgroup $P \subset G$ is called **parabolic** if (i) its complexified Lie algebra $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ and (ii) $P$ is the normalizer of $\mathfrak{p}$ in $G$. A parabolic subgroup $P \subset G$ is called **cuspidal** if the Levi component $L \subset P$ has a Cartan subgroup that is compact modulo the center of $L$.

We now have the cuspidal parabolic subgroup $P = MAU$ of $G$, where $M$ and $A$ are as before, where $MA = M \times A$ is the Levi component of $P$, and where $U = exp_G(\mathfrak{u}_0)$.

Let $\chi \in \widehat{H}$ and consider the basic datum $(H, \mathfrak{b}, \chi)$. The representation of $\mathfrak{b}$ is determined because $\chi$ represents $H$ irreducibly: $\chi(\mathfrak{n}) = 0$ and $\chi|_{\mathfrak{h}}$ is the differential of the representation of $H$. Decompose $\chi = \psi \otimes e^{i\sigma}$, $\psi \in \widehat{T}$, $\sigma \in \mathfrak{a}_0^*$. Suppose that $\nu + \rho_M$ is $\Phi^+(\mathfrak{m}, \mathfrak{t})$-nonsingular where $d\psi = \nu I$ with $\nu \in\subset \mathfrak{t}_0^*$. Then $\psi$ specifies a discrete series representation $\eta_\psi$ of $M$. The Levi component $M \times A$ of $P$ acts irreducibly and unitarily on $H_{\eta_\psi}$ by $\eta_\psi \otimes e^{i\sigma}$. That extends uniquely to a representation (which we still denote $\eta_\psi \otimes e^{i\sigma}$) of $P$ on $H_{\eta_\psi}$ whose kernel contains $U$. Now we have the **standard tempered representation**

$$\pi_\chi = \pi_{\psi,\sigma} = \mathrm{Ind}_{P\uparrow G}(\eta_\psi \otimes e^{i\sigma})$$

of $G$. One can compute the character of $\pi_\chi$ and see that it is independent of the choice of positive root system $\Phi^+(\mathfrak{g}, \mathfrak{h})$ that is defined by choices of $\Phi^+(\mathfrak{m}, \mathfrak{t})$ and $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$. With $H$ fixed up to conjugacy, and as $\psi$ and $\sigma$ vary, we have the **$H$-series** of tempered representations of $G$.

## Geometric Realizations of Tempered Series Representations on Hilbert Spaces of Partially Harmonic Differential Forms

Fix a $\theta$-stable Cartan subgroup $H \subset G$ and a positive root system $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{h})$ defined by positive root systems $\Phi^+(\mathfrak{m}, \mathfrak{t})$ and $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$ as above. The associated cuspidal parabolic subgroup $P = MAU$ contains $TAU = HU$ as a closed subgroup, and we have a $G$-equivariant fibration

$$G/TAU \to G/P \text{ with structure group } M \text{ and typical fibre } M/T.$$

Notice that the fibre $M/T$ has an $M$-invariant complex structure for which $\sum_{\alpha\in\Phi^+(\mathfrak{m},\mathfrak{t})} \mathfrak{m}_\alpha$ is the holomorphic tangent space. So an irreducible unitary representation

$$\chi = \psi \otimes e^{i\sigma} \in \widehat{H} = \widehat{TA}$$

defines a $G$-homogeneous hermitian vector bundle $\mathbb{E}_\chi \to G/TAU$ that is holomorphic over every fibre of $G/TAU \to G/P$. Define Hilbert spaces

$$\mathcal{H}^q(G/TAU;\mathbb{E}_\chi): \; L^2 \text{ sections of } \mathbb{H}^q(M/T;\mathbb{E}_\chi|_{M/T}) \to G/P$$

where the Hilbert space bundle

$$\mathbb{H}^q(M/T;\mathbb{E}_\chi|_{M/T}) \to G/P \text{ has structure group } MAU, \text{ typical fibre } \mathcal{H}^q(M/T;\mathbb{E}_\chi|_{M/T}).$$

Here, as in the discussion of realization of the discrete series, $\mathcal{H}^q(M/T;\mathbb{E}_\chi|_{M/T})$ is the Hilbert space of harmonic, $L^2(M/T)$, $\mathbb{E}_\chi|_{M/T}$-valued $(0.q)$-forms on $M/T$. Let $\eta$ denote the (necessarily unitary) representation of $M$ on $\mathcal{H}^q(M/T;\mathbb{E}_\chi|_{M/T})$. Then $MAU$ acts on $\mathcal{H}^q(M/T;\mathbb{E}_\chi|_{M/T})$ by $\eta \otimes e^{i\sigma}$.

As above define $\nu \in \mathfrak{t}_0^*$ by $d\psi = \nu I$. If $\nu + \rho_M$ is $\Phi^+(\mathfrak{m},\mathfrak{t})$-singular then

$$\mathcal{H}^q(M/T;\mathbb{E}_\chi|_{M/T}) = 0 \text{ for every } q \geqq 0.$$

If $\nu + \rho_M$ is $\Phi^+(\mathfrak{m},\mathfrak{t})$-nonsingular and

$$q(\nu + \rho_M) = |\{\alpha \in \Phi^+_{K\cap M} : \langle \nu + \rho_M, \alpha\rangle < 0\}| + |\{\beta \in \Phi^+_{M/K\cap M} : \langle \lambda + \rho_M, \beta\rangle > 0\}|$$

then as before $\mathcal{H}^q(M/T;\mathbb{E}_\chi|_{M/T}) = 0$ for $q \neq q(\nu + \rho_M)$, and $M$ acts irreducibly on $\mathcal{H}^{q(\nu+\rho_M)}(M/T;\mathbb{E}_\chi|_{M/T})$ by the discrete series representation $\eta_\psi$. Thus we have [**61**]

THEOREM. *If $\nu + \rho_M$ is $\Phi^+(\mathfrak{m},\mathfrak{t})$-singular then every $\mathcal{H}^q(G/TAU;\mathbb{E}_\chi) = 0$. Now suppose that $\nu+\rho_M$ is $\Phi^+(\mathfrak{m},\mathfrak{t})$-regular and let $q(\nu+\rho_M)$ be as just above. Then $\mathcal{H}^q(G/TAU;\mathbb{E}_\chi) = 0$ for $q \neq q(\nu + \rho_M)$, and $G$ acts on $\mathcal{H}^{q(\nu+\rho_M)}(G/TAU;\mathbb{E}_\chi)$ by the standard $H$-series representation $\pi_\chi = \pi_{\psi,\sigma} = Ind_{P\uparrow G}(\eta_\psi \otimes e^{i\sigma})$ of $G$.*

A variation on this theorem realizes the tempered series on spaces of $L^2$ bundle-valued partially harmonic spinors. See [**62**].

The standard $H$-representation $\pi_\chi = \pi_{\psi,\sigma}$ is irreducible whenever $\sigma \in \mathfrak{a}_0^*$ is $\Phi^+(\mathfrak{g}_0,\mathfrak{a}_0)$-nonsingular. Plancherel measure for $G$ thus is carried by the irreducible representations among the $H$-series representations realized above, as $H$ varies over the conjugacy classes of Cartan subgroups of $G$.

## Geometric Realizations of Tempered Series Representations on Partial Dolbeault Cohomology Spaces

We now replace $\mathcal{H}^q(M/T; \mathbb{E}_\chi|_{M/T})$ by the Dolbeault cohomology space that realizes the representations $\eta_\psi$ of $M$ and $\eta_\psi \otimes e^{i\sigma}$ of $MA$ and $P = MAU$. The space $(K \cap M)/T$ is a maximal compact complex submanifold of $G/TAU$. Let $s = \dim_{\mathbb{C}} (K \cap M)/T$. Whenever

$$\nu + \rho_M \text{ is } \Phi^+(\mathfrak{m}, \mathfrak{t})\text{-antidominant: } \langle \nu + \rho_M, \gamma \rangle < 0 \text{ for all } \gamma \in \Phi^+(\mathfrak{m}, \mathfrak{t})$$

we have $s = q(\nu + \rho_M)$. Much as before, this is the case where the bundle $\mathbb{E}_\chi|_{M/T} \to M/T$ is negative.

Let $\mathcal{O}_{\mathfrak{n}}(\mathbb{E}_\chi) \to G/TAU$ denote the sheaf of germs of $C^\infty$ sections of $\mathbb{E}_\chi \to G/TAU$ that are holomorphic along the fibres of $G/TAU \to G/P$, i.e. are annihilated by the right action of $\mathfrak{n}$.

If $\eta$ is a discrete series representation of $M$, we can choose the positive root system $\Phi^+$ to be defined by $\Phi^+(\mathfrak{m}, \mathfrak{t})$ and $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$ (any choice of the latter will do) so that $\nu + \rho_M$ is $\Phi^+(\mathfrak{m}, \mathfrak{t})$-antidominant. Thus there is no restriction on $\pi_\chi$ in

THEOREM. *Suppose that $\nu + \rho_M$ is $\Phi^+(\mathfrak{m}, \mathfrak{t})$-antidominant, so $s = q(\nu + \rho_M)$. Then $H^s(G/TAU; \mathcal{O}(\mathbb{E}_\chi))$ has a natural Fréchet space structure, the natural action of $G$ on $H^s(G/TAU; \mathcal{O}(\mathbb{E}_\chi))$ is a continuous representation, and this representation is infinitesimally equivalent to $\pi_\chi$.*

## Formulation Using Basic Data

Fix a basic datum $(H, \mathfrak{b}, \chi)$: $H$ is a Cartan subgroup of $G$, $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{b}$, and $\chi$ is a finite dimensional representation of $(\mathfrak{b}, H)$. Then we have the associated hermitian homogeneous vector bundle $\mathbb{E}_\chi \to G/H$ and the sheaf of germs of $C^\infty$ sections annihilated by the right action of $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$, $\mathcal{O}_{\mathfrak{n}}(\mathbb{E}_\chi) \to G/H$. $G$ acts naturally on the cohomologies $H^q(G/H; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}_\chi))$.

Since $\chi$ is a representation of $(\mathfrak{b}, H)$, not just $H$, the bundle $\mathbb{E}_\chi \to G/H$ pushes down to a bundle $\mathbb{E}_\chi \to G/TAU$. The germs in $\mathcal{O}_{\mathfrak{n}}(\mathbb{E}_\chi) \to G/H$ are constant along the fibres of $G/H \to G/TAU$ and one verifies without difficulty that the spectral sequence collapses. Thus there are natural $G$-equivariant isomorphisms $H^q(G/H; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}_\chi)) \cong H^q(G/TAU; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}_\chi))$.

We identify $G/TAU$ with the $G$-orbit of $\mathfrak{b}$ in the flag variety $X$, as in **[61]**. The point here is that the isotropy subgroup $G \cap B = TAU$ and that the partial complex structure (CR structure) induced on $G \cdot \mathfrak{b}$ by $X$ is one for which the holomorphic tangent space of the typical fibre $M/T$ of $G/TAU \to G/P$ is $\sum_{\alpha \in \Phi^+(\mathfrak{m},\mathfrak{t})} \mathfrak{m}_\alpha$.

THEOREM. *Every standard tempered series representation $\pi_\chi$ of $G$, $\chi \in \widehat{H}$ and $H \in Car(G)$, is realized up to infinitesimal equivalence as the natural action of $G$ on a partial Dolbeault cohomology space $H^s(G \cdot \mathfrak{b}; \mathcal{O}_\mathfrak{n}(\mathbb{E}_\chi))$ for a basic datum $(H, \mathfrak{b}, \chi)$ as follows. $\mathfrak{b}$ is maximally real subject to the condition $\mathfrak{h} \in \mathfrak{b}$, $\chi \in \widehat{H}$, and $s = \dim_\mathbb{C} (K \cap M)/T$.*

This theorem is the starting point for the construction of standard admissible representations of $G$.

## SECTION 3. ADMISSIBLE REPRESENTATIONS AND THEIR GEOMETRIC CONSTRUCTIONS.

### CLASSICAL GEOMETRIC QUANTIZATION

In [**56**], Schmid and I formulate "classical" geometric quantization for semisimple Lie groups in a manner consistent with the constructions just described for the tempered series. Fix a basic datum $(H, \mathfrak{b}, \chi)$, $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ with $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}] = \sum_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}$, as before. Then we have the associated homogeneous vector bundle $\mathbb{E}_\chi \to G/H$ and the sheaf

$$\begin{aligned} \mathcal{O}_\mathfrak{n}(\mathbb{E}_\chi) \to G/H : \text{germs of } C^\infty \text{ sections } f \text{ of } \mathbb{E}_\chi \to G/H \text{ such that} \\ f(x;\xi) + \chi(\xi) \cdot f(x) = 0 \text{ for all } x \in G \text{ and } \xi \in \mathfrak{b}. \end{aligned} \tag{3.1}$$

We view $\mathfrak{b}$ as a choice of $G$-invariant polarization on the homogeneous space $G/H$, and we associate the representations of $G$ on the cohomologies $H^q(G/H; \mathcal{O}_\mathfrak{n}(\mathbb{E}_\chi))$.

In the earlier discussion of the tempered series, $\chi$ was irreducible as a representation of $H$ on $E_\chi$, so necessarily $\chi(\mathfrak{n}) = 0$. Thus, there the differential equation in (3.1) reduced to

$$f(x;\xi) = 0 \text{ for all } x \in G \text{ and all } \xi \in \mathfrak{b}.$$

In general $f(x;\xi) + \chi(\xi) \cdot f(x) = 0$ is the appropriate equation; see [**57**].

We try to calculate the $G$-modules $H^q(G/H; \mathcal{O}_\mathfrak{n}(\mathbb{E}_\chi))$ from the complex

$$C^\infty(G/H; \mathbb{E}_\chi \otimes \Lambda^\bullet \mathbb{N}^*),\ d_\mathfrak{n} \tag{3.2}$$

where $d_\mathfrak{n} : C^\infty(G/H; \mathbb{E}_\chi \otimes \Lambda^p \mathbb{N}^*) \to C^\infty(G/H; \mathbb{E}_\chi \otimes \Lambda^{p+1} \mathbb{N}^*)$ is the unique first order $G$-invariant differential operator with symbol $(\mathfrak{g}/\mathfrak{h})^* \otimes E_\chi \otimes \Lambda^p \mathfrak{n}^* \to E_\chi \otimes \Lambda^{p+1} \mathfrak{n}^*$ given by $(\phi, e, \omega) \mapsto e \otimes (q(\phi) \wedge \omega)$ where $q : (\mathfrak{g}/\mathfrak{h})^* \to \mathfrak{n}^*$ is dual to $\mathfrak{n} \cong \mathfrak{b}/\mathfrak{h} \hookrightarrow \mathfrak{g}/\mathfrak{h}$.

It is convenient to pull the bundles $\mathbb{E}_\chi \otimes \Lambda^\bullet \mathbb{N}^* \to G/H$ back to $G$. Then the complex (3.2) becomes

$$\{C^\infty(G) \otimes E_\chi \otimes \Lambda^\bullet \mathfrak{n}^*\}^H,\ d_\mathfrak{n} \tag{3.3}$$

where $H$ and $\mathfrak{n}$ act from the right on $C^\infty(G)$, where $\{\dots\}^H$ denotes the space of $H$-invariants, and where $d_\mathfrak{n}$ is the coboundary operator for Lie algebra cohomology of $\mathfrak{n}$. If

$\mathfrak{n} \cap \bar{\mathfrak{n}} = 0$, i.e. if the polarization $\mathfrak{b}$ is totally complex, then (i) $H$ is as compact as possible among the Cartan subgroups of $G$, (ii) $G/H$ has an invariant complex structure as the open orbit $G \cdot \mathfrak{b}$ in the flag variety $X$, (iii) $\mathsf{E}_\chi \to G \cdot \mathfrak{b}$ is a $G$-homogeneous holomorphic vector bundle, and (iv) $\mathcal{O}_\mathfrak{n}(\mathsf{E}_\chi) \to G \cdot \mathfrak{b}$ is the sheaf of germs of holomorphic sections of $\mathsf{E}_\chi$. This is always the case when $H$ is compact, where it leads to the $C^\infty$ discrete series of $G$. If $H$ is not necessarily compact, but $\mathfrak{b}$ is maximally real in the sense that $\mathfrak{u} = \mathfrak{n} \cap \bar{\mathfrak{n}}$ is maximized for the given choice of $H$, then we have seen that the complex (3.3) leads to the $C^\infty$ realization of the $H$-series of tempered representations of $G$. There are serious problems with the complex (3.3) for general $(H, \mathfrak{b})$, and one sees them already when the polarization $\mathfrak{b}$ is totally complex but $H$ is noncompact, *e.g.* in $SL(3;\mathbb{C})$. First, (3.3) may fail to be acyclic, so it will compute the hypercohomology of a complex of sheaves rather than the cohomology of a single sheaf. Second, there is no reason to expect $d_\mathfrak{n}$ to have closed range. In fact, even when things go well, say for the discrete series, the proof of closed range is delicate. Zuckerman's derived functor construction [**58**], an algebraic version of (3.3), avoids these problems. But it leads to $(\mathfrak{g}, K)$-modules rather than $G$-modules. Now I am going to describe some geometric complexes, variations on (3.3), which also circumvent the problems with (3.3), and which effectively yield all standard representations. Then I'll indicate how these geometric complex constructions of $G$-modules relate to Zuckerman's construction of $(\mathfrak{g}, K)$-modules. This is joint work with W. Schmid [**56**].

## Harish-Chandra Modules and Globalizations

By **representation** of $G$ we mean a continuous representation $(\pi, \tilde{V})$ of $G$ with finite composition series, on a complete locally convex Hausdorff topological vector space $\tilde{V}$. By **Harish–Chandra module** for $G$ we mean a $\mathcal{U}(\mathfrak{g})$-finite $K$-semisimple $(\mathfrak{g}, K)$-module in which every vector is $K$-finite and the $K$-multiplicities are finite.

If $(\pi, \tilde{V})$ is a representation of $G$, then $V = \{v \in \tilde{V} : v \text{ is } K\text{-finite}\}$ is a Harish–Chandra module for $G$, is dense in $\tilde{V}$, and consists of smooth (in fact analytic) vectors.

If $V$ is a Harish–Chandra module for $G$, and if $(\pi, \tilde{V})$ is a representation of $G$ such that $V$ is $(\mathfrak{g}, K)$-isomorphic to the space of $K$-finite vectors in $\tilde{V}$, then $(\pi, \tilde{V})$ is a **globalization** of $V$.

For the derived functor construction let $\mathcal{M}(\mathfrak{g}, K)_{(K)}$ denote the category of $K$-finite $(\mathfrak{g}, K)$-modules, let $\mathcal{M}(\mathfrak{g}, H \cap K)_{(H\cap K)}$ be the category of $H \cap K$-finite $(\mathfrak{g}, H \cap K)$-modules, and let

$$\Gamma : \mathcal{M}(\mathfrak{g}, H \cap K)_{(H\cap K)} \to \mathcal{M}(\mathfrak{g}, K)_{(K)}$$

denote the functor that sends a module to its maximal $K$-finite $K$-semisimple submodule. $\Gamma$ is left exact. Its right derived functors $R^q(\Gamma)$ are the **Zuckerman functors**. Now the basic datum $(H, \mathfrak{b}, \chi)$ specifies $(\mathfrak{g}, K)$-modules

$$A^q(G, H, \mathfrak{b}, \chi) = (R^p)\{\mathrm{Hom}_\mathfrak{b}(\mathcal{U}(\mathfrak{g}), E_\chi)_{(H\cap K)}\}. \tag{3.4}$$

These Zuckerman derived functor modules are Harish–Chandra modules for $G$. The modules (3.4) at first glance seem rather far from the cohomologies of the complex (3.3), but in fact there is a tight connection. Write $C^{\mathrm{for}}$ for formal power series sections at $1\cdot H \in G/H$. Evaluation of formal power series sections at $1 \cdot H$ maps $C^{\mathrm{for}}(G/H; \mathbb{E}_\chi \otimes \wedge^p \mathbb{N}^*)$ isomorphically (as $(\mathfrak{g}, K)$-module) onto $\mathrm{Hom}_{\mathfrak{h}}(\mathcal{U}(\mathfrak{g}), E_\chi \otimes \wedge^p \mathfrak{n}^*)_{(H\cap K)}$. The complex

$$C^{\mathrm{for}}(G/H; \mathbb{E}_\chi \otimes \wedge^{\bullet} \mathbb{N}^*),\ d_{\mathfrak{n}}$$

gives a well defined injective resolution of $\mathrm{Hom}_{\mathfrak{b}}(\mathcal{U}(\mathfrak{g}), E_\chi)_{(H\cap K)}$, its 0-cohomology, so

$$A^q(G, H, \mathfrak{b}, \chi) \cong H^q(C^{\mathrm{for}}(G/H; \mathbb{E}_\chi \otimes \wedge^{\bullet} \mathbb{N}^*)_{(K)},\ d_{\mathfrak{n}}). \tag{3.5}$$

This isomorphism defines a map from the $K$-finite version of (3.3) to the derived functor module,

$$H^q(C^\infty(G/H; \mathbb{E}_\chi \otimes \wedge^{\bullet} \mathbb{N}^*)_{(K)},\ d_{\mathfrak{n}}) \to H^q(C^{\mathrm{for}}(G/H; \mathbb{E}_\chi \otimes \wedge^{\bullet} \mathbb{N}^*)_{(K)},\ d_{\mathfrak{n}}), \tag{3.6}$$

which is the coefficient morphism defined by the Taylor series expansion at $1 \cdot H$. This will be a key point in showing that various geometrically defined representations of $G$ are globalizations of the modules (3.4).

There are four functorial globalizations, the $C^\infty$ and $C^{-\infty}$ (distribution) globalizations of Casselman and Wallach [**59**] and the minimal and maximal globalizations of Schmid [**55**]. For reasons to be explained shortly, the maximal globalization is the one that is appropriate here.

Here are the basic facts on the maximal globalization [**55**]. Let $(\pi, \tilde{V})$ be any globalization of a Harish–Chandra module $V$. Every element of the dual Harish–Chandra module

$$V' : K\text{-finite vectors in the algebraic dual of } V$$

extends to a continuous linear functional on $\tilde{V}$. If $v \in V$ and $v' \in V'$, the coefficient $f_{v,v'}(x) = (v', \pi(x)v)$ is a $C^\infty$ function on $G$. The Taylor series of $f_{v,v'}$ at 1 depends only on the action of $\mathcal{U}(\mathfrak{g})$ on $V$. Any finite $\mathcal{U}(\mathfrak{g})$ generating set $\{v'_1, \ldots, v'_m\} \subset V'$ defines an injection $V \hookrightarrow C^\infty(G)^m$ by $v \mapsto (f_{v,v'_1}, \ldots, f_{v,v'_m})$. The induced topology is independent of $\{v'_i\}$ because we pass between any two such generating sets by a $\mathcal{U}(\mathfrak{g})$-valued matrix. So we have

$$V_{\max} : \text{completion of } V \text{ in the topology induced by } C^\infty(G)^m. \tag{3.7}$$

$V_{\max}$ is a globalization of $V$. It is called the **maximal globalization** because, if $\tilde{U}$ is any globalization then the identity map $\tilde{U}_{(K)} \to V$ extends to a $G$-equivariant continuous injection $\tilde{U} \hookrightarrow V_{\max}$.

In a Banach globalization $(\pi, \tilde{V})$ of $V$ the subspace $\tilde{V}^\omega$ of analytic ($C^\omega$) vectors has a natural structure of complete locally convex Hausdorff topological vector space. Suppose that the Banach space $\tilde{V}$ is reflexive, $\tilde{V}'$ is its Banach space dual, and $\pi'$ is the dual of $\pi$. Then

$$\tilde{V}^{-\omega} : \text{strong topological dual of } (\tilde{V}')^\omega$$

is the space of **hyperfunction vectors** ($C^{-\omega}$ vectors) of $\tilde{V}'$. It is another globalization and ([**55**], p. 317) the inclusion $\tilde{V}^{-\omega} \hookrightarrow V_{\max}$ is a topological isomorphism. This has an extremely important consequence: $V \mapsto V_{\max}$ is an exact functor.

See [**34**] and [**35**] for an introduction to hyperfunctions.

### Hyperfunction Quantization

The Dolbeault lemma holds with $C^{\infty}$, $C^{-\infty}$, or $C^{-\omega}$ coefficients. So for compact $H$ the complex (3.3) computes Dolbeault cohomology $H^*(G/H;\mathcal{O}_{\mathfrak{n}}(\mathbb{E}_\chi))$—even when $C^{\infty}$ is replaced by $C^{-\infty}$ or $C^{-\omega}$. Thus each Dolbeault cohomology realization of a discrete series representation of $M$ already is the maximal globalization of its underlying Harish–Chandra module. Thus, in the induction step for constructing a tempered series representation of $G$, if we use any coefficients other than $C^{-\omega}$, we will not get a canonical globalization of the underlying Harish–Chandra module. This is one reason to use hyperfunction coefficients instead of $C^{\infty}$ coefficients in (3.3). Also, the general theory of hyperfunctions suggests that $C^{-\omega}$ coefficients might take care of the problem of closed range for $d_{\mathfrak{n}}$. (In fact that will turn out to be the case.) Thus, for several reasons, we study the hyperfunction analog

$$\{C^{-\omega}(G) \otimes E_\chi \otimes \Lambda^\bullet \mathfrak{n}^*\}^H,\ d_{\mathfrak{n}} \tag{3.8}$$

of the classical complex (3.3). This is the pull back to $G$ of the hyperfunction analog of (3.2), so of course $H^q(\{C^{-\omega}(G)\otimes E_\chi \otimes \Lambda^\bullet \mathfrak{n}^*\}^H, d_{\mathfrak{n}}) \cong H^q(C^{-\omega}(G/H;\mathbb{E}_\chi\otimes\Lambda^\bullet\mathbb{N}^*), d_{\mathfrak{n}})$.

As before, note that $\chi$ is a representation not just of $H$ but of $(\mathfrak{b}, H)$, so the bundle $\mathbb{E}_\chi \to G/H$ pushes down to the orbit of $\mathfrak{b}$ in the flag variety. Now we have

$$\mathbb{E}_\chi \to S:\ G\text{-homogeneous vector bundle where } S = G\cdot\mathfrak{b} \subset X.$$

By homogeneity, the $G$-orbit $S \subset X$ is a partially complex manifold whose holomorphic tangent spaces (intersection of the complexified tangent space of $S$ with the holomorphic tangent space of $X$) have constant dimension. So we also have

$$\mathbb{N}_S \to S:\ \text{antiholomorphic tangent bundle of } S\subset X.$$

The Dolbeault operator $\overline{\partial} = \overline{\partial}_X$ of $X$ involves differentiation in all directions given by an antiholomorphic frame. If we start that frame with an antiholomorphic frame for $S$, i.e. with a basis of the fibre of $\mathbb{N}_S$, then we have a well defined Cauchy–Riemann operator (partial Dolbeault operator)

$$\overline{\partial}_S:\ \text{the part of } \overline{\partial} = \overline{\partial}_X \text{ defined on } S.$$

That gives us the **Cauchy–Riemann complex**

$$C^{-\omega}(S;\mathbb{E}_\chi\otimes\Lambda^\bullet\mathbb{N}_S^*),\ \overline{\partial}_S \tag{3.9}$$

with hyperfunction coefficients. In the case of a maximally real polarization $\mathfrak{b}$ we had a fibration $S = G/TAU \to G/MAU = G/P$, where $P$ is a cuspidal parabolic subgroup of

$G$. The fibres $M/T$ were the maximal complex submanifolds and the base $G/P$ gave the totally real directions. The situation is a bit more complicated when $\mathfrak{b}$ is not maximally real, but in all cases the Cauchy–Riemann complex (3.9) is closely related to the geometry underlying the structure of representations constructed from $(H, \mathfrak{b}, \chi)$.

Let $\tilde{S}$ denote the germ of an open neighborhood of $S$ in $X$. Then $\mathbb{E}_\chi \to S$ has a unique holomorphic $\mathfrak{g}$-equivariant extension $\tilde{\mathbb{E}}_\chi \to \tilde{S}$ to $S$. Write $\mathbb{N}_X \to X$ for the antiholomorphic tangent bundle. Now we have the Dolbeault complex

$$C_S^{-\omega}(\tilde{S}; \tilde{\mathbb{E}}_\chi \otimes \Lambda^\bullet \mathbb{N}_X^*),\ \overline{\partial} \tag{3.10}$$

with hyperfunction coefficients supported in $S$. A basic fact here [**35**] is that

$$H^q(C_S^{-\omega}(\tilde{S}; \tilde{\mathbb{E}}_\chi \otimes \Lambda^\bullet \mathbb{N}_X^*), \overline{\partial}) \cong H_S^q(\tilde{S}; \mathcal{O}(\tilde{\mathbb{E}}_\chi))$$

where the right hand side is local cohomology along $S$.

## The Comparison Theorem

One of the main results of [**56**] compares the cohomologies of the complexes (3.8), (3.9) and (3.10) with the Zuckerman modules (3.4):

THEOREM. *Fix a basic datum $(H, \mathfrak{b}, \chi)$, let $S = G \cdot \mathfrak{b} \subset X$, and let $u$ =codim$_{\mathbb{R}}(S \subset X)$. Then there are canonical isomorphisms*

$$\begin{aligned} H^q(C^{-\omega}(G/H; \mathbb{E}_\chi \otimes \Lambda^\bullet \mathbb{N}^*), d_\mathfrak{n}) &\cong H^q(C^{-\omega}(S; \mathbb{E}_\chi \otimes \Lambda^\bullet \mathbb{N}_S^*), \overline{\partial}_S) \\ &\cong H_S^{q+c}(\tilde{S}; \mathcal{O}(\tilde{\mathbb{E}}_\chi)) \end{aligned}$$

*These cohomologies carry natural Fréchet topologies. In those topologies the isomorphisms are topological and the action of $G$ is continuous. The resulting representations of $G$ are canonically and topologically isomorphic to the action of $G$ on the maximal globalization of $A^q(G, H, \mathfrak{b}, \chi)$.*

Cohomologies of the complexes (3.8), (3.9) and (3.10) play different roles here. The one closest to the classical construction, $H^q(C^{-\omega}(G/H; \mathbb{E}_\chi \otimes \Lambda^\bullet \mathbb{N}^*), d_\mathfrak{n})$, fits the orbit picture best, essentially as prequantization of a regular semisimple orbit, and it is tied rather directly to the Zuckerman construction. The cohomology $H^q(C^{-\omega}(S; \mathbb{E}_\chi \otimes \Lambda^\bullet \mathbb{N}_S^*), \overline{\partial}_S)$ is a natural geometric object, and as noted above the geometry of $S$ is tied to the structure of the representation of $G$ on that cohomology. Also, the Fréchet topologies will come out of the geometry of $S$. $H_S^{q+c}(\tilde{S}; \mathcal{O}(\tilde{\mathbb{E}}_\chi))$ is more of an analytic object, and we will need it to identify the infinitesimal characters of our representations.

The complexes (3.8), (3.9) and (3.10) do not have obvious Fréchet topologies because there is no reasonable topology for hyperfunctions on a noncompact manifold. So we have to be cautious about the meaning of the topological part of the theorem. We will see that the cohomology of the Cauchy–Riemann complex (3.9) can be calculated from a certain subcomplex that does have a natural Fréchet topology. In fact, from ([38], Theorem 3.13), that subcomplex will be a complex of strongly nuclear Fréchet spaces. A closed range theorem carries the topology down to the cohomology of (3.9) and then, through the isomorphisms, to the cohomologies of (3.8) and (3.10).

Now I'll try to indicate the idea of proof of the Comparison Theorem.

## The Algebraic Isomorphisms

The first isomorphism of the Theorem, as $G$-modules without topology, is pretty straightforward. The Cauchy–Riemann complex (3.9) pulls back to the complex

$$\{C^{-\omega}(G) \otimes E_\chi \otimes \wedge^\bullet(\mathfrak{n}/\mathfrak{n} \cap \overline{\mathfrak{n}})^*\}^{\mathfrak{n}\cap\overline{\mathfrak{n}},H},\ d_{\mathfrak{n},\mathfrak{n}\cap\overline{\mathfrak{n}}} \tag{3.11}$$

for relative Lie algebra cohomology of $(\mathfrak{n}, \mathfrak{n} \cap \overline{\mathfrak{n}})$ and hyperfunction coefficients. $G/H \to S$ has euclidean space fibres $HU/H \simeq U$, where $U$ is the unipotent Lie group corresponding to the real form $\mathfrak{u}_0 = \mathfrak{u} \cap \mathfrak{g}_0$ of $\mathfrak{u} = \mathfrak{n} \cap \overline{\mathfrak{n}}$. We apply the Poincaré Lemma to those fibres to see that inclusion of the complex (3.11) in the complex (3.8) induces cohomology isomorphisms.

The isomorphisms $H^q(C^{-\omega}(G/H;\mathsf{E}_\chi \otimes \wedge^\bullet \mathsf{N}^*), d_{\mathfrak{n}}) \cong H_S^{q+c}(\tilde{S};\mathcal{O}(\tilde{\mathsf{E}}_\chi))$ are not as obvious. They depend on some local cohomology arguments.

## Construction of the Topology

If $\Phi^+$ is a positive root system for $(\mathfrak{g}, \mathfrak{h})$ that is not maximally real, then there is a complex simple root $\alpha$ such that $\overline{\alpha} \in -\Phi^+$. We use this to construct a maximally real positive root system $\Phi^+_{\max}$, the corresponding Borel subalgebra $\mathfrak{b}_{\max} \in X$ and the orbit $S_{\max} = G \cdot \mathfrak{b}_{\max}$, and the associated cuspidal parabolic subgroup $P = MAU \subset G$, in such a way that

$$\begin{array}{lll} g \cdot \mathfrak{b} \mapsto g \cdot \mathfrak{b}_{\max} & \text{defines a fibration} & S \to S_{\max}, \\ g \cdot \mathfrak{b}_{\max} \mapsto gP & \text{defines a fibration} & S_{\max} \to G/P, \text{ and} \\ g \cdot \mathfrak{b} \mapsto gP & \text{defines a fibration} & S \to G/P. \end{array} \tag{3.12}$$

Here all fibres are complex submanifolds of $X$.

Generally, when $W \to U$ is a $C^\omega$ fibration, one has a well defined sheaf $\mathcal{C}_U^{-\omega}(W)$ of germs of hyperfunctions on $W$ that are $C^\infty$ along the fibres. Apply this to (3.12) to get $\mathcal{C}_{G/P}^{-\omega}(S)$. Now we have a complex consisting of the differentials $\overline{\partial}_S$ and the sheaves

$$\mathcal{C}_{G/P}^{-\omega}(S;\mathsf{E}_\chi \otimes \wedge^p \mathsf{N}_S^*):\ \text{sections of } \mathsf{E}_\chi \otimes \wedge^p \mathsf{N}_S^* \to S \text{ with coefficients in } \mathcal{C}_{G/P}^{-\omega}(S)$$

Taking global sections we arrive at the **partial Cauchy–Riemann complex**

$$C_{G/P}^{-\omega}(S;\mathsf{E}_\chi \otimes \wedge^\bullet \mathsf{N}_S^*),\ \overline{\partial}_S \tag{3.13}$$

Here the basic facts are

1. The inclusion of (3.13) in the Cauchy–Riemann complex (3.9) induces isomorphisms in cohomology.

2. The $C^{-\omega}_{G/P}(S; \mathsf{E}_\chi \otimes \wedge^p \mathsf{N}^*_S)$ have natural Fréchet topologies. In those topologies, $\overline{\partial}_S$ is continuous and the actions of $G$ are continuous representations.

The topologies are constructed as follows. First note that standard hyperfunction theory can be developed just as well for functions with values in a reflexive Banach space. Now, suppose that $\mathbf{V} \to M$ is a $C^\omega$ vector bundle over a compact $C^\omega$ manifold $M$ whose typical fibre is a reflexive Banach space $V$. Let $\mathbf{V}^* \to M$ be the dual bundle. Then the space $C^{-\omega}(M; \mathbf{V})$ of hyperfunction sections has a natural Fréchet topology as the strong dual to $C^\omega(M; \mathbf{V}^*)$. We use this with $V = C^\infty(F)$ where $F$ is the typical fibre of $S \to G/P$. In fact $C^\infty(F)$ is the topological limit of Sobolev spaces $V_n$ of functions on $F$, so $\mathbf{V} = \mathbb{C}^\infty(F) \to G/P$ is a limit of Hilbert bundles $\mathbf{V}_n \to G/P$. Now

$$C^{-\omega}_{G/P}(S) = C^{-\omega}(G/P; \mathbb{C}^\infty(F)) = \lim C^{-\omega}(G/P; \mathbf{V}_n)$$

has a natural topology, $G$-invariant and $\overline{\partial}_S$-invariant by construction, and Fréchet because it is given by an increasing family of seminorms.

## Tensoring and the Underlying Harish–Chandra Module

Tensoring arguments reduce the proof of the Theorem to a fairly special case. We first indicate the properties that behave well with respect to tensoring.

We say that an admissible Fréchet $G$-**module has property (MG)** if it is the maximal globalization of its underlying Harish–Chandra module. We say that a **complex** $(C^\bullet, d)$ of Fréchet $G$-modules **has property (MG)** if (i) $d$ has closed range, (ii) the cohomologies $H^p(C^\bullet, d)$ are admissible and of finite length, and (iii) each $H^p(C^\bullet, d)$ has property (MG). Given a basic datum $(H, \mathfrak{b}, \chi)$ we say that the corresponding homogeneous vector **bundle** $\mathsf{E}_\chi \to S$ **has property (MG)** if the partially smooth Cauchy–Riemann complex (3.13) has property (MG).

Now of course we want to prove that $\mathsf{E}_\chi \to S$ has property (MG) for every choice of $(H, \mathfrak{b}, \chi)$. But the Comparison Theorem asserts a bit more. $H^p(S; \mathsf{E}_\chi) = H^p(C^{-\omega}(S; \mathsf{E}_\chi \otimes \wedge^\bullet \mathsf{N}^*_S), \overline{\partial}_S)$ is calculated by the Fréchet complex (3.13). As $\overline{\partial}_S$ commutes with projection to $K$-isotypic subspaces we can compute $H^p(S; \mathsf{E}_\chi)_{(K)}$ from the subcomplex of $K$-finite forms in (3.13). Those forms are smooth because they are smooth along the fibres of $S \to G/P$ by definition, and because $K$ is transitive on the transverse directions. In particular they have a formal power series expansion at $1 \cdot H$. Now one can use (3.6) to define morphisms

$$H^q(S; \mathsf{E}_\chi)_{(K)} \to A^p(G, H, \mathfrak{b}, \chi). \tag{3.14}$$

The other assertion of the Comparison Theorem is that the maps (3.14) are isomorphisms of Harish–Chandra modules. So we say that the homogeneous vector bundle $\mathsf{E}_\chi \to S$ **has property (Z)** if (3.14) are isomorphisms.

Let $F$ be a finite dimensional $G$-module, $\mathsf{F} \to S$ the associated homogeneous vector bundle. Using exactness of the maximal globalization functor one can prove: If $\mathsf{E}_\chi \to S$ has property (MG) so does $\mathsf{E}_\chi \otimes \mathsf{F} \to S$; if $\mathsf{E}_\chi \to S$ has property (Z) so does $\mathsf{E}_\chi \otimes \mathsf{F} \to S$. However, tensoring has to start somewhere. Consider the condition on $(H, \mathfrak{b})$

(3.15) There exist a positive root system $\Phi^+$ and a number $C > 0$ such that: if $\mathsf{E}_\chi \to S$ is irreducible, $\lambda = d\chi \in \mathfrak{h}^*$ and the $\langle \Re\lambda, \alpha \rangle > C$ for all $\alpha \in \Phi^+$ then $\mathsf{E}_\chi \to S$ has both properties $(Z)$ and $(MG)$.

The tensoring result for properties (MG) and (Z) is the analytic component of

Proposition. *Fix* $(H, \mathfrak{b})$ *and assume* (3.15). *Then, for all basic data of the form* $(H, \mathfrak{b}, \chi)$, $\mathsf{E}_\chi \to S$ *has both properties* (MG) *and* (Z).

## Maximally Real Polarizations

Fix $(H, \mathfrak{b})$ with $\mathfrak{b}$ maximally real. The point is to prove (3.15) deep in some Weyl chamber of $(\mathfrak{g}, \mathfrak{h})$. In doing this, $\Phi^+$ is as in (3.15); it does not denote the positive root that defines $\mathfrak{b}$.

Suppose first that $H$ is compact and let $\Phi^+$ correspond to the negative chamber. If $G$ is connected an idea of Aguilar–Rodríguez [1] extracts the (Z) assertion of (3.15) from Schmid's thesis [50] by using the Bott–Borel–Weil Theorem on an extremal $K$-type. The (Z) assertion of (3.15), for $G$ not necessarily connected, follows by considering the structure of $S$ and of $\widehat{G}_{\text{disc}}$. And for compact $H$ the (MG) assertion of (3.15) is essentially proved in [50].

Now suppose that $H$ is not necessarily compact. Let $\Phi^+$ be the maximally real positive root system defined from $\Phi^+(\mathfrak{g}, \mathfrak{a})$ and $-\Phi^+(\mathfrak{m}, \mathfrak{t})$. Compute the $A^q(G, H, \mathfrak{b}, \chi)$ from the complex of $K$-finite $\mathsf{E}_\chi$-valued forms on $G/H$ with formal power series coefficients. Watch the fibration $G/H \to S$ and $S \to G/P$ as in construction of the standard tempered series representations and their geometric realizations.. This gives the (Z) and (MG) assertions of (3.15).

At this point of the argument, the Comparison Theorem is proved for maximally real polarizations. In the geometric realizations of the tempered series, we used certain results that were only published for "sufficiently nonsingular" infinitesimal character. Now we have justified the statements of those results without any nonsingularity condition.

## Change of Polarization

Now consider $(H, \mathfrak{b})$ where the Borel subalgebra $\mathfrak{b} = \sum_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}$ is not maximally real. Then there is a complex simple root $\alpha$ such that $\overline{\alpha}$ is negative. We thus have $\Phi_0^+ = s_\alpha \Phi^+$, $\mathfrak{b}_0 = s_\alpha \mathfrak{b}$ and $S_0 = G \cdot \mathfrak{b}_0$, and $S$ fibres over $S_0$ with fibre $exp(\mathfrak{g}_\alpha) \cdot \mathfrak{b} \cong \mathbb{C}$. The point is to compare results for $(H, \mathfrak{b})$ with results for $(H, \mathfrak{b}_0)$.

Let $X_\alpha$ denote the flag manifold of parabolic subalgebras of $\mathfrak{g}$ that are Int($\mathfrak{g}$)-conjugate to $\mathfrak{b}_\alpha = \mathfrak{g} + \mathfrak{g}_\alpha$. The natural projection $p_\alpha : X \to X_\alpha$ maps $S$ onto $S_\alpha = G \cdot \mathfrak{b}_\alpha$ with fibre $\mathbb{C}$, maps $S_0$ onto $S_\alpha$ bijectively. We use the fibrations $S \to S_\alpha$ and $S_0 \to S_\alpha$ to compare cohomologies over $S$ and $S_0$. In effect, view a nonzero element $\omega^\alpha \in (\mathfrak{g}_{-\alpha})^*$ as an element of $\mathfrak{h}$-weight $\alpha$ in $\mathfrak{g}^*$. Exterior product with $\omega^\alpha$ with restriction from $\mathfrak{n}_0 + \mathfrak{g}_{-\alpha}$ to $\mathfrak{n}$ defines $e(\omega^\alpha) : \Lambda^q \mathfrak{n}_0^* \to \Lambda^{q+1}\mathfrak{n}^*$. One proves that $(-1)^q e(\omega^\alpha)$ induces a morphism of complexes

$$C^{-\omega}(S_0; \mathbb{E}_\chi \otimes \Lambda^q \mathbb{N}^*_{S_0}) \to C^{-\omega}(S; \mathbb{E}_\chi \otimes \mathbb{L}_{-\alpha} \otimes \Lambda^{q+1}\mathbb{N}^* S) \tag{3.16}$$

whose image consists of forms that are holomorphic along the fibres of $S \to S_0$. With that, one can reduce the proof of the Theorem to

(3.17) Let $\chi$ be irreducible, $d\chi = \lambda \in \mathfrak{h}^*$, with $\dfrac{2\langle \lambda + \rho - \alpha, \alpha\rangle}{\langle \alpha, \alpha\rangle}$ not a positive integer. Then (3.16) induces isomorphisms, in particular $H^0(C^{-\omega}(S; \mathbb{E}_\chi \otimes \mathbb{L}_{-\alpha}), \overline{\partial}_S) = 0$.

The proof of (3.17) is quite technical.

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