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SQUARE INTEGRABILITY OF GROUP REPRESENTATIONS ON HOMOGENEOUS SPACES AND GENERALIZED COHERENT STATES

S.Twareque Ali, J.-P. Antoine and J.-P. Gazeau

In previous papers [1,2], we have obtained coherent states for the Poincaré group in 1+1 dimensions, a case where the standard methods do not work. It turns out that the technique used there is a special case of a much more general construction, that we will outline here. A detailed analysis will be published elsewhere [3,4].

1. THE STANDARD CONSTRUCTION OF COHERENT STATES

Let G be a locally compact group (not necessarily unimodular), with (left invariant) Haar measure dg, and consider a strongly continuous, irreducible, unitary representation U of G into a Hilbert space $\mathcal{H}$. The representation U is said to be *square integrable* if there exists a vector $\eta \in \mathcal{H}$ such that :

$$\int_G |\langle U(g)\,\eta \,|\, \phi \rangle|^2 \, dg < \infty, \forall \phi \in \mathcal{H} \tag{1.1}$$

(equivalently, U belongs to the discrete series). Choose a fixed vector η that satisfies the admissibility condition (1.1). Then the orbit of η under U,

$$\mathfrak{S} = \{ \eta_g = U(g)\eta \mid g \in G \}, \tag{1.2}$$

is an overcomplete family of vectors, called *coherent states* associated to the representation U. Alternatively, the family $\mathfrak{S}$ determines a resolution of the identity

$$\int_G |\eta_g\rangle\langle\eta_g| \, dg = I. \tag{1.3}$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

This is the familiar construction of coherent states [10,11]. A particularly interesting example is the affine group of the line (the "ax+b" group), in which case one obtain *wavelet analysis* (η is then called the analyzing wavelet) [9].

2. THE PERELOMOV CONSTRUCTION

In many cases, the admissibility condition (1.1) is too strong, and one has to use the generalization due to Perelomov [11]. Given η, define H as the subgroup of G that leaves η invariant up to a phase (this is obviously motivated by Quantum Mechanics, where states are defined only up to the phase) :

$$U(h)\,\eta = e^{i\alpha(h)}\,\eta. \tag{2.1}$$

Let ν be a (left) invariant measure on the coset space $X = G/H$. Then we say that the representation U is *square integrable modH* if the following admissibility condition holds :

$$\int_{G/H} |\langle U(g)\,\eta\,|\,\phi\rangle|^2\, d\nu(x) < \infty, \ \forall\ \phi \in \mathcal{H} \tag{2.2}$$

(by (2.1) the integrand depends only on the coset x = gH). When this is the case, the construction proceeds as before. We denote again by $\mathfrak{S}$ the orbit of η under U, but now the elements of $\mathfrak{S}$ are indexed by points of the coset space X. Hence one obtains a set of generalized coherent states (GCS) :

$$\mathfrak{S} = \{\eta_x \,|\, x = gH \in X = G/H\}, \tag{2.3}$$

which yields again a resolution of the identity :

$$\int_X |\eta_x\rangle\langle\eta_x|\, d\nu(x) = I. \tag{2.4}$$

Examples are plentiful ; for instance :

(i) when G is the Weyl-Heisenberg group, one gets the canonical coherent states, familiar in Quantum Mechanics [10];

(ii) G = SU(2) or a compact simple Lie group ;

(iii) G = SU(1,1) or a noncompact simple Lie group, and U a representation of the discrete series.

The last two classes are treated in detail in the monograph of Perelomov [11]. It should be remarked that GCS may be defined also for a representation that is not square integrable mod H, e.g. a principal series representation, but those vectors lack many of the nice properties of the square integrable ones [11].

Now this method, however nice and powerful it may be, does not always work ; for instance, it is inapplicable in the case of the Galilei group or the Poincaré groups $\mathcal{P}_+^{\uparrow}(1,1)$ or $\mathcal{P}_+^{\uparrow}(1,3)$, more generally for a semidirect product $G = S \wedge V$, where V is a vector space and $S \subset GL(V)$ (this is the case treated by DeBièvre [6]).

We will discuss in detail the case of the 1+1 dimensional Poincaré group $\mathcal{P}_+^{\uparrow}(1,1)$, in Section 5 below. The natural representation is the familiar Wigner representation U_W, but it is not square integrable. An obvious manifold X to consider here is the *phase space* $\Gamma = \mathcal{P}_+^{\uparrow}(1,1)/T$, where T is the time translation subgroup, but T is not the stability subgroup under U_W of any vector η. Thus Perelomov's construction fails to give relativistic coherent states. The formalism we are going to develop will enable us to do so, but it is much more general than this particular example.

3. REPRODUCING TRIPLES

In fact, it makes no reference at all to a group representation. The only ingredient we need is a generalization of the resolution of the identity (2.4). Let (X, ν) be a measure space and $\mathcal{H}$ a Hilbert space. Then we replace (2.4) by the following relation :

$$\int_X F(x)\, d\nu(x) = A\,, \tag{3.1}$$

where A is a bounded, positive, invertible operator on $\mathcal{H}$, $F : X \to \mathcal{L}(\mathcal{H})^+$ is a ν - measurable, positive operator valued function, and the operator integral in (3.1) is to be understood in the weak sense. Given such elements, we call $\{\mathcal{H},F,A\}$ a *reproducing triple* . This is the central notion in the sequel.

3.1. Coherent states

For each $x \in X$, we denote by $\mathbb{P}(x)$ the projection on the orthogonal complement of $\mathrm{Ker}\, F(x)$: $\mathbb{P}(x)\mathcal{H} = (\mathrm{Ker}\, F(x))^{\perp}$, and define on $\mathbb{P}(x)\mathcal{H}$ the new scalar product :

$$\langle\phi_x \mid \psi_x\rangle_x = \langle\phi_x \mid F(x)\, \psi_x\rangle_{\mathcal{H}}, \ \forall\, \phi_x, \psi_x \in \mathbb{P}(x)\mathcal{H} \tag{3.2}$$

Let $\mathcal{K}_x$ be the completion of $\mathbb{P}(x)\mathcal{H}$ with respect to the corresponding norm $\|\cdot\|_x$, and $\{v_i(x),\ i = 1,\dots, d(x) \equiv \dim \mathbb{P}(x)\mathcal{H}\}$ an orthonormal basis in $\mathbb{P}(x)\mathcal{H}$. Then one obtains an overcomplete family of *coherent states*

$$\mathfrak{S} = \{\, \eta_x^i \mid \eta_x^i = F(x)v_i(x)\,,\ i = 1,2,\dots, d(x)\,;\ x \in X\,\}, \tag{3.3}$$

such that :

$$F(x) = \sum_{i=1}^{d(x)} |\eta_x^i><\eta_x^i|, \tag{3.4}$$

$$\int_X \sum_{i=1}^{d(x)} |\eta_x^i><\eta_x^i| \, d\nu(x) = A \text{ (weak integral).} \tag{3.5}$$

Overcompleteness means that $\mathfrak{S}$ is a total set in $\mathcal{H}$: $\mathfrak{S}^{\perp} = 0$.

3.2. Reproducing kernel Hilbert space

A nice feature of the usual construction of coherent states is the presence of a reproducing kernel Hilbert space. The same is true here. To see that, let us introduce successively :

. the direct integral $\tilde{\mathcal{H}} = \int_X^{\oplus} \mathcal{K}_x \, d\nu(x)$,

. the map $W_K : \mathcal{H} \to \tilde{\mathcal{H}}$, defined by

$$(W_K \phi)(x) = \sum_{i=1}^{d(x)} <\eta_x^i | \phi> v_i(x) = \mathbb{P}(x)\phi \in \mathcal{K}_x, \tag{3.6}$$

. the operator $F_K(x) = W_K F(x) W_K^{-1}$,

. A_K, the self-adjoint extension of $W_K A W_K^{-1} | \mathrm{Ran}(W_K)$,

. A_K^{-1}, the self-adjoint extension of $W_K A^{-1} W_K^{-1} | W_K[\mathcal{D}(A^{-1})]$

(of course, the existence and uniqueness of these two self-adjoint extensions requires a proof, see [3,4]).

On $W_K[\mathcal{D}(A^{-1})]$, define a new scalar product

$$<\Phi | \Psi>_K = <\Phi | A_K^{-1} \Psi>_{\tilde{\mathcal{H}}}$$

and denote by $\mathcal{H}_K$ the completion of $W_K[\mathcal{D}(A^{-1})]$ with respect to the norm $\| \cdot \|_K$. Then it turns out that $\mathcal{H}_K = \mathrm{Ran}(W_K) \subset \tilde{\mathcal{H}}$ and $W_K : \mathcal{H} \to \mathcal{H}_K$ is a unitary map. As a consequence, we get a new reproducing triple $\{ \mathcal{H}_K, F_K, A_K \}$, where $\mathcal{H}_K$ is a space of vector-valued *functions* , with a *reproducing kernel* $K(x,y) : \mathcal{K}_y \to \mathcal{K}_x$, enjoying the following properties :

. $K(x,x) > 0, \forall x \in X$;
. $K(x,y)^* = K(y,x)$;
. $\Phi(x) = \int_X K(x,y)\, \Phi(y)\, d\nu(y), \forall\, \Phi \in \mathcal{H}_K$ (reproducing property).

3.3. Continuous frames

All the results above simplify if Rank $F(x) = n < \infty$, for every $x \in X$, since then $\tilde{\mathcal{H}}$ may be mapped unitarily onto $L^2(X,\nu;\mathbb{C}^n)$. If, in addition, A^{-1} is a bounded operator, then $\{ \mathcal{H}, F, A\}$ is called a (continuous) *frame* . In that case, indeed, (3.5) leads to the "frame" condition :

$$m(A)\, \| \phi \|^2 \leq \sum_{i=1}^{n} \int_X | < \eta_x^i \mid \phi > |^2 \, d\nu(x) \leq M(A)\, \| \phi \|^2, \qquad (3.7)$$

where $m(A) = \inf \sigma(A)$, $M(A) = \sup \sigma(A)$ (the so-called "frame bounds"). The frame is called *tight* if $m(A) = M(A)$, or equivalently if $A = \lambda$ I. The terminology used here is borrowed from the theory of nonorthogonal expansions [5,7], since the family $\{\eta_x^i\}$ is a frame in the usual sense when X is discrete and ν the counting measure.

4. SQUARE INTEGRABILITY OF A GROUP REPRESENTATION ON A COSET SPACE

Now we apply the general formalism just developed to the situation described at the beginning. Let G be a l.c. group and H a closed subgroup of G such that the coset space $X = G/H$ has an invariant measure ν. Choose a (global) Borel section $\beta : X \to G$. Given a representation $U : G \to \mathcal{H}$ and an operator $F \in \mathcal{L}(\mathcal{H})^+$, define the function $F_\beta : X \to \mathcal{L}(\mathcal{H})^+$ by covariance

$$F_\beta(x) = U(\beta(x))\, F\, U(\beta(x))^*. \qquad (4.1)$$

In this notation, the representation U is said to be *square integrable mod(H,β)* if there exist $F, A_\beta \in \mathcal{L}(\mathcal{H})^+$, with rank $F = n$, such that $\{\mathcal{H}, F_\beta, A_\beta\}$ is a reproducing triple, i.e. A_β is invertible and the following operator integral converges weakly :

$$\int_X F_\beta(x)\, d\nu(x) = A_\beta . \qquad (4.2)$$

This class of representations enjoys many properties analogous to those of the square integrable ones, in the standard sense.

4.1. Covariance

If U is square integrable mod(H,β), then it is also square integrable mod(H,β_g), $\forall g \in G$, where $\beta_g(x) = g\beta(g^{-1}.x)$ $(x \in X)$ is the invariant measure obtained from β by the action of G on X.

4.2. Coherent states

Using the spectral decomposition

$$F = \sum_{i=1}^{n} \lambda_i \, | u_i > < u_i |, \tag{4.3}$$

where $\lambda_i > 0$ and $\{u_i\}_{i=1}^{n}$ is an orthonormal set, define the vectors :

$$\eta^{i}_{\beta(x)} = \lambda_i^{1/2} \, U(\beta(x)) u_i \, . \tag{4.4}$$

Then, as in Section 3.1, the vectors (4.4) constitute an overcomplete family of coherent states

$$\mathfrak{S}_\beta = \{ \eta^{i}_{\beta(x)} \mid i = 1,2,\ldots,n, x \in X \} \tag{4.5}$$

and

$$A_\beta = \int_X \sum_{i=1}^{n} | \eta^{i}_{\beta(x)} > < \eta^{i}_{\beta(x)} | \, d\nu(x), \tag{4.6}$$

the integral converging weakly as usual.

4.3. Equivalent families of coherent states

Let β, β' two sections $X \to G$, such that U is square-integrable *both* mod(H,β) and mod(H,β'). Then the corresponding families of coherent states $\mathfrak{S}_\beta$ and $\mathfrak{S}_{\beta'}$ are *equivalent* , in the following sense : for every $x \in X$, there exists an operator $T(x) \in \mathcal{L}(\mathcal{H})$, with $T(x)^{-1} \in \mathcal{L}(\mathcal{H})$, such that

$$\eta^{i}_{\beta'(x)} = T(x) \, \eta^{i}_{\beta(x)} \quad \text{and} \quad \eta^{i}_{\beta(x)} = T(x)^{-1} \eta^{i}_{\beta'(x)}. \tag{4.7}$$

Thus, in this precise sense, we may say that the set of coherent states associated to the representation U does not depend on the choice of the section.

<u>Example</u> : $\mathfrak{S}_{\beta_g}$ is equivalent to $\mathfrak{S}_\beta$, $\forall g \in G$.

5. EXAMPLE : THE POINCARE GROUP $\mathcal{P}_+^{\uparrow}(1,1) = SO_0(1,1) \wedge \mathbb{R}^2$

As an illustration of the general theory developed in Section 4, we shall now construct systems of coherent states for the Poincaré group in one space and one time dimensions, $\mathcal{P}_+^{\uparrow}(1,1) = SO_0(1,1) \wedge \mathbb{R}^2$ (see also [1,2]).

The group is parameterized in the standard way : $(a, \Lambda_p) \in \mathcal{P}_+^{\uparrow}(1,1)$, where

$$a = (a_o, \mathbf{a}) \equiv (a_o, \underline{a}) \in \mathbb{R}^2 \qquad (5.1a)$$

$$\Lambda_p = \begin{pmatrix} \frac{p_o}{m} & \frac{p}{m} \\ \frac{p}{m} & \frac{p_o}{m} \end{pmatrix} \in SO_0(1,1), \text{ with } p_o = (p^2 + m^2)^{1/2}, \qquad (5.1b)$$

We consider the familiar Wigner representation :

$$(U_w(a,\Lambda)\phi)(k) = e^{ik.a}\, \phi(\Lambda_p{}^{-1}k), \quad \phi \in \mathcal{H}_w = L^2(\mathcal{V}_m^+, dk/k_o) \qquad (5.2)$$

where $\mathcal{V}_m^+ = \{ k = (k_o,\mathbf{k}),\ k_o{}^2 - \mathbf{k}^2 = m^2, k_o > 0 \}$ and $k.a = k_o\, a_o - \mathbf{k} \cdot \mathbf{a}$. A straightforward calculation shows that this representation U_W is *not* square integrable over $\mathcal{P}_+^{\uparrow}(1,1)$!

Instead we introduce the *phase space* $\Gamma = \mathcal{P}_+^{\uparrow}(1,1)/T$, where T is the time translation subgroup. Notice that T is not the stability subgroup of *any* vector in $\mathcal{H}_w$, so that the method of Perelomov is indeed inapplicable here.

Using standard coordinates (q, p), one gets the $\mathcal{P}_+^{\uparrow}(1,1)$-invariant measure dq dp on Γ. Define first the basic section $\beta_o : \Gamma \to \mathcal{P}_+^{\uparrow}(1,1)$ by $\beta_o(q,p) = ((0, q), \Lambda_p)$. Then a general section may be written as $\beta(q,p) = \beta_o(q,p)\ ((f(q,p), 0), I)$. We will consider, in particular, the class $\mathcal{S}_A$ of *affine* sections defined by the two conditions :

$$f(q,p) = \varphi(p) + q \,.\, \theta(p), \qquad (5.3)$$

$$-\frac{p_o - p}{m} < \theta(p) < \frac{p_o + p}{m} \qquad (5.4)$$

(condition (5.4) is equivalent to $\hat{q}_o^2 - \hat{q}^2 < 0$, where $\beta(q,p) = (\hat{q}, \Lambda_{\hat{p}})$ and $\varphi = 0$).

According to the general theory, a vector $\eta \in \mathcal{H}_w$ is admissible mod(T,β) for $\beta \in \mathcal{S}_A$ if

$$A_\beta = \int_{\mathbb{R}^2} U_w(\beta(q,p))\,|\,\eta > < \eta\,|\,U_w(\beta(q,p))^*\ dq\, dp \qquad (5.5)$$

is a bounded, invertible operator on $\mathcal{H}_w$. Then an explicit calculation yields the following results.

(1) The vector $\eta \in \mathcal{H}_w$ is admissible mod(T,β), for any section $\beta \in \mathcal{S}_A$ iff $\eta \in \mathcal{D}(P_o^{1/2})$, where P_o is the energy operator :

$$P_o \eta(k) = k_o \eta(k). \tag{5.6}$$

(2) When η is admissible, $A_\beta \equiv A_\beta^\eta$ is a multiplication operator given by

$$(A_\beta^\eta \psi)(k) = A_\beta^\eta(k)\, \psi(k) \tag{5.7}$$

$$A_\beta^\eta(k) = \int_{\mathcal{V}_m^+} \mathcal{A}_\beta(k, p)\, |\eta(p)|^2\, \frac{dp}{p_o} \tag{5.8}$$

where
$$\mathcal{A}_\beta(k, p) = 2\pi\, \frac{(\Lambda_k^{-1} p)_o}{k_o - \theta(\underline{\Lambda}_p^{-1} \underline{k}) \cdot \mathbf{p}}. \tag{5.9}$$

The crucial fact is that the kernel $\mathcal{A}_\beta$ satisfies the following inequalities, which result from (5.4) :

$$\frac{2\pi}{m}(p_o - |\mathbf{p}|) < \mathcal{A}_\beta(k, p) < \frac{2\pi}{m}(p_o + |\mathbf{p}|), \tag{5.10}$$

for all $k, p \in \mathcal{V}_m^+$ and any θ obeying (5.4), in particular, for any section $\beta \in \mathcal{S}_A$.

(3) As a consequence, every vector η which is admissible mod(T,β) generates a reproducing triple $\{\mathcal{H}_w, F_\beta, A_\beta^\eta\}$, of constant rank $n = 1$, and in fact a frame. The frame bounds obey the estimates (independent of the choice of the section $\beta \in \mathcal{S}_A$) :

$$m(A_\beta^\eta) \geq m(\eta) \equiv 2\pi < \eta \,|\, \frac{P_o - |\mathbf{P}|}{m}\, \eta >, \tag{5.11}$$

$$M(A_\beta^\eta) \leq M(\eta) \equiv 2\pi < \eta \,|\, \frac{P_o + |\mathbf{P}|}{m}\, \eta >, \tag{5.12}$$

where $\mathbf{P}$ is the momentum operator :

$$\mathbf{P}\, \phi(k) = \mathbf{k}\, \phi(k). \tag{5.13}$$

Furthermore, there exist sections β for which the frame is tight (sometimes, only for a suitable choice of η, as in the case of the basic section β_o) and other ones (e.g. the section β_s used in [2]) for which the frame is never tight, that is, the spectrum of A_β^η is always purely continuous, whatever state η is used.

(4) It follows from the general theory developed in Section 4 that, for every affine section $\beta \in \mathcal{S}_A$, any vector η which is admissible mod(T,β) generates a family of coherent states indexed by points in phase space :

$$\mathfrak{S}_\beta = \{ \eta_{\beta(q,p)} \mid (q,p) \in \Gamma \}. \tag{5.14}$$

Furthermore, all these systems of coherent states are equivalent.

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