Jarolím Bureš; Vladimír Souček

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1 Introduction

The aim of the presented paper is to study the Penrose transform for solutions of the Laplace equation by means of Clifford analysis.

The description of the Penrose transform in a general situation can be found in the book by R. Baston and M. Eastwood (see [2,5]). We shall discuss here the special case corresponding to orthogonal groups in even dimensions. The Penrose transform maps certain cohomology groups to solutions of (complex) Laplace equation in this case.

The discussion of the Penrose transform in a general case ([5]) uses quite advanced and sophisticated tools, such as direct images of sheaves, sheaf cohomologies, the Bernstein-Gel'fand-Gel'fand resolution and spectral sequences. We are presenting here a more simple approach using the Dolbeault realization of the corresponding cohomology groups and a simple calculus with differential forms (as it was done in 4-dimensional case in [9]) The main new tools used are the Cauchy integral formula for solutions of the Laplace equation ([2,4]) and the Leray residue for closed differential forms (see [10,7]).

The presented paper contains a short description of the results, the full version with all proofs will be published later.

2 Isotropic Grassmannians

The Penrose transform is always based on a diagram of homogeneous spaces (see [2] and [5]). In our case (i.e. for the description of solutions of Laplace equation in higher even dimensions), we shall need the following homogeneous spaces of the group $SO(2n, \mathbf{C})$.

Let us consider the quadratic form

$$Q(Z) = \sum_{1}^{n+1} Z'_j Z''_j, Z = [Z', Z'']^t; Z', Z'' \in \mathbf{C}_{n+1}$$

on the vector space $\mathbf{C}_{2n+2}$. The corresponding bilinear form will be denoted by $\langle\ ,\ \rangle$. We shall need the following type of flag manifolds:

$$IG_{i_1,\ldots,i_j;2n+2} := \{[L_{i_1},\ldots,L_{i_j}] | L_{i_1} \subset \ldots \subset L_{i_j} \subset \mathbf{C}_{2n+2};\ Q|L_{i_j} = 0\}.$$

We shall drop the dimension of the ambient space if it is clear from the context. We shall use mainly the complex quadric IG_1 (which can be considered as the compactification of the complex Minkowski or Euclidean space) and the spaces IG_{n+1} and $IG_{1,n+1}$. The last two spaces are not connected, we shall work always with one of their connected components. The space IG_{n+1} can be interpreted either as the space of all maximal isotropic subspaces in the quadric IG_1 or as the space of all pure spinors. Together with the natural forgetting maps, they form the basic diagram

$$\begin{array}{ccc} & IG_{1,n+1} & \\ \mu \swarrow & & \searrow \nu \\ IG_{n+1} & & IG_1 \end{array}$$

3 Isotropic Stieffel manifolds

The standard Stieffel manifolds are principal fibre bundles over Grassmannians (formed by bases of corresponding spaces) for the general linear group. The same is true for isotropic Stieffel manifolds. Elements of cohomology classes with values in sections of some line bundles will be conveniently described by $\overline{\partial}$-closed differential forms on these isotropic Stieffel manifolds transforming properly with respect to the action of the general linear group.

In particular, we shall consider the space

$$\begin{aligned} ISt_{n+1} :=& \\ & \{Z = [Z^0,\ldots,Z^n] | Z^i \in \mathbf{C}_{2n+2}, \operatorname{rank} Z = n+1, \\ & \langle Z^i, Z^j \rangle = 0; i,j = 0,\ldots,n\} \end{aligned}$$

as a principal fibre bundle over IG_{n+1} with the group $G = \mathrm{GL}(n+1,\mathbf{C})$ acting from the right. The corresponding projection will be denoted by π. The same space will be considered as a principal fibre bundle over $IG_{1,n+1}$

$$\pi' : ISt_{n+1} \mapsto IG_{1,n+1}$$

with

$$\pi'(\mathcal{Z}) = [L_1, L_{n+1}],\ L_1 = \text{span}\{Z^0\}, L_{n+1} = \text{span}\{Z^0, \ldots, Z^n\}.$$

The group of the fibration consists of all regular matrices having the form

$$g = \begin{pmatrix} a & v \\ 0 & \gamma \end{pmatrix}, a \in \mathbf{C},\ v^t \in \mathbf{C}_n,\ \gamma \in \text{GL}(n, \mathbf{C}).$$

4 Minkowski space

We shall consider solutions of (complex) Laplace equation only on subsets of the (complexified) Minkowski or Euclidean space

$$\mathbf{C}M \subset IG_1, \mathbf{C}M := \{L = \text{span}\{Z\} | Z \in \mathbf{C}_{2n+2},\ Z_{n+1} \neq 0\}.$$

It is an open dense subset of IG_1. It is useful to consider nonhomogeneous coordinates (x, y) on $\mathbf{C}M$ by the identification

$$\mathbf{C}M = \{[x, 1, y, -x \cdot y]^t | x, y \in \mathbf{C}_n\},$$

where $x \cdot y := \sum_1^n x_j y_j$.

The corresponding spaces in the double fibration are then defined as

$$\mathbf{F} := \nu^{-1}(\mathbf{C}M) = \{[L_1, L_{n+1}] | L_1 \in \mathbf{C}M\},$$
$$\mathbf{T} := \mu(\mathbf{F}) = \{L_{n+1} | L_{n+1} \cap \mathbf{C}M \neq \emptyset\}.$$

In this situation, it is very useful to restrict further allowed bases in ISt_{n+1}. Any basis $\mathcal{Z} \in \mathbf{F}$ can be transformed into a basis of the form

$$\begin{bmatrix} x & x^1 & \ldots & x^n \\ 1 & 0 & \ldots & 0 \\ y & y^1 & \ldots & y^n \\ -x \cdot y, & -x \cdot y^1 - x^1 \cdot y & \ldots & -x \cdot y^n - x^n \cdot y \end{bmatrix},$$

where $x^j, y^j \in C_n$; $x^i \cdot y^j + x^j \cdot y^i = 0$; $i, j = 1, \ldots, n$.

So we shall consider the fibration of $\tilde{\mathbf{F}} = \mathbf{C}M \times ISt_n$ over $\mathbf{F}$, where $\tilde{\mathbf{F}}$ is formed by all $(n+1) \times (2n+2)$-matrices of the form shown above. The fibration $\pi : \tilde{\mathbf{F}} \mapsto \mathbf{F}$ is a principal fibration with the group formed by all matrices of the form $g = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}$, where $\gamma \in \text{GL}(n, \mathbf{C})$.

The fibration over the twistor space looks then like $\pi' : \tilde{\mathbf{F}} \mapsto \mathbf{T}$, which is a principal fibration with the group of all matrices of the form

$$g = \begin{pmatrix} 1 & 0 \\ v & \gamma \end{pmatrix}, v \in C_n,\ \gamma \in \text{GL}(n, \mathbf{C}). \tag{1}$$

5 Invariant forms

We shall need some nice forms on the twistor space IG_n.

We shall define first the form ω on the ordinary Stieffel manifold St_n over the Grassmannian $G_{n;2n}$ by

$$\omega_n(\mathcal{Z}) := \bigwedge_{i=1}^{n} \det(Z^1, \ldots, Z^n, \underbrace{dZ^i \ldots, dZ^i}_{n}),$$

where $\mathcal{Z}$ is $n \times 2n$-matrix $\mathcal{Z} = (Z^1, \ldots, Z^n) \in St_n$ (compare [8]). Under the substitution

$$\mathcal{Z} \mapsto \mathcal{Z} \cdot g,\ d\mathcal{Z} \mapsto d\mathcal{Z} \cdot g + \mathcal{Z} \cdot dg,\ g \in \mathrm{GL}(n, \mathbf{C}), \tag{2}$$

the form ω_n transforms as $\omega_n \mapsto (\det g)^{2n} \omega_n$.

The isotropic Stieffel manifold ISt_n is defined by the set of equations $\varphi_{ij}(\mathcal{Z}) := \langle Z^i, Z^j \rangle = 0, i, j, = 1, \ldots, n; i \geq j$. The form α_n is then defined by

$$\alpha_n := d\varphi_{11} \rfloor (d\varphi_{12} \rfloor \ldots (d\varphi_{nn} \rfloor \omega_n) \ldots)|_{ISt_n},$$

where $\rfloor$ denotes the division of differential forms. The $(n-1)n/2$-form α_n on ISt_n transforms as $\alpha_n \mapsto (\det g)^{n-1} \alpha_n$ under the substitution (2).

We shall define a form

$$\kappa_n(\mathcal{Z}) := \frac{\alpha_n(\mathcal{Z}) \wedge \overline{\alpha_n(\mathcal{Z})}}{(\sum_I |M_I|^2)^{n-1}},$$

where the sum is taken over all subsets $I \subset \{1, \ldots, 2n\}$ with n elements and M_I denotes the determinant of the corresponding $n \times n$ minor of the matrix $\mathcal{Z}$. The form κ_n is invariant under the substitution (2), so it is a well-defined continuous form of the top degree on the isotropic Grassmannian IG_n. We shall normalize it in such a way that its integral over IG_n is equal to one.

The twistor space IG_n is, in fact, a nontrivial fibration over the sphere $S_{2n-2} \subset IG_{1;2n}$ with fibers diffeomorphic to $IG_{n-1;2n-2}$. The fibration can be covered by two trivial products $E_{2n-2} \times IG_{n-1;2n-2}$ (where E_{2n-2} denotes a Euclidean slice of $\mathbf{C}_{2n-2} \subset IG_{1;2n}$) and the forms κ_{n-1} (constant on E_{2n-2}) coincide on the intersection of both maps, so they give us a well-defined form, say τ_n, on IG_n. We shall consider also the form τ_n lifted to ISt_n.

6 The Penrose transform

The fibration $ISt_{n+1} \mapsto IG_{n+1}$ together with the representation

$$g \in \mathrm{GL}(n+1, \mathbf{C}) \mapsto (\det g) \in \mathbf{C}$$

defines the corresponding associated line bundle over IG_{n+1}, we shall denote it by $\mathcal{L}$.

Take now a cohomology class $\beta \in H^{(0,\frac{n(n-1)}{2})}(\mathbf{T}, \mathcal{L}^{1-n})$. We can represent it by a $\overline{\partial}$-closed form β of the degree $(0, \frac{n(n-1)}{2})$ over $\tilde{\mathbf{F}}$ transforming as $\beta \mapsto (\det \gamma)^{1-k}\beta$ under the substitution (2) with $\mathcal{Z} \in \tilde{\mathbf{F}}$ and g of the form (1).

Then clearly the restriction of the form $\beta \wedge \alpha_n$ is well-defined on every fiber $\nu^{-1}(Z) \subset \mathbf{F}$ and we define the function

$$\phi(Z) := \int_{\nu^{-1}(Z)} \beta \wedge \alpha_n$$

on the Minkowski space. This function is called the Penrose transform of the element β. It can be shown that it is a holomorphic function satisfying the (complex) Laplace equation on $\mathbf{C}M$ and that the function depends only on the cohomology class of β.

The Penrose transform is a bijective map of the corresponding cohomology group onto the set of all holomorphic solutions on $\mathbf{C}M$. The injectivity is proved by complex analytic methods in [6]. The most difficult part of the construction is to prove that the Penrose transform is surjective. We shall describe a new method of the proof, based on Clifford analysis, bellow.

7 Integral formulae

Let us describe now how to reconstruct the twistor form from the knowledge of a solution of the Dirac equation. Let us suppose that a function $f = f(Z)$ on $\mathbf{C}M$ is a holomorphic solution of the Laplace equation $\Delta f = 0$ on $\mathbf{C}M$, where $\Delta = \sum_1^{2n} \frac{\partial^2}{\partial Z_i^2}$.

Let us denote the complex null cone of the point Z by $\mathbf{C}N_Z$, i.e.

$$\mathbf{C}N_Z := \{W \in \mathbf{C}M \,|\, |W - Z|^2 = 0\}.$$

For any $Z \in \mathbf{C}M$, the value of the function f in the point Z can be expressed using values of the function (and its derivatives) on a $(2n-1)$-dimensional cycle Σ in $\mathbf{C}M \setminus CN_Z$ such that $\mathrm{Ind}_\Sigma Z = 1$ (see [2,7]).

The corresponding integral formula looks like:

$$f(Z) = \frac{1}{K}\int_\Sigma \frac{\sum_1^n(-1)^i(Z_i - W_i)D\widehat{W_i}}{|Z-W|^{2n}} + \frac{\sum_1^n(-1)^i\frac{\partial f}{\partial W_i}D\widehat{W_i}}{(2n-2)|Z-W|^{2n-2}},$$

where K is the area of the unit sphere in E_{2n} and $D\widehat{W_i} = dW_1\wedge\ldots\wedge\widehat{dW_i}\wedge\ldots\wedge dW_n$ (the hat over dW_i means that this one-form is omitted). The form under the integral sign is the holomorphic continuation of the traditional form used for integral formulae for harmonic functions on Euclidean space.

Consider now the $(4n-1)$-form

$$\omega = \frac{1}{K}dZ\wedge\{\frac{\sum_1^n(-1)^i(Z_i - W_i)D\widehat{W_i}}{|Z-W|^{2n}} + \frac{\sum_1^n(-1)^i\frac{\partial f}{\partial W_i}D\widehat{W_i}}{(2n-2)|Z-W|^{2n-2}}\},$$

where $dZ = dZ_1\wedge\ldots\wedge dZ_{2n}$. The form ω is a closed form on

$$(\mathbf{C}M\times\mathbf{C}M)\setminus\mathcal{M},\ \mathcal{M} := \{[Z,W]|Z,W\in\mathbf{C}M;\ |Z-W|^2 = 0\}.$$

Hence its Leray residue ([7,10]) is a well-defined cohomology class represented by a closed form $[\text{Res}\ \omega]\in H^{4n-2}(\mathcal{M}\setminus\Delta), \Delta := \{[Z,W]|Z = W\}$. The value $f(Z)$ of the field can be reconstructed not only by the integration of ω over the cycle Σ, but by the integration of Res ω over a $(4n-2)$-dimensional cycle $\Gamma\subset\mathbf{C}N_Z$ such that its Leray cobord has the same properties as Σ does (for details see [7]).

8 The choice of the map ϕ.

To get back the twistor form representing the field f and to prove that the Penrose transform is surjective, we have to choose a map

$$\phi : \mathbf{C}M\times ISt_n \mapsto (\mathcal{M}\setminus\Delta)\times ISt_n.$$

It has the following geometrical meaning: for any point

$$\mathcal{Z} = [Z^0; Z^1,\ldots,Z^n]\in\mathbf{C}M\times ISt_n$$

we want to choose a point $\tilde{Z}^0\neq Z^0$ in the α-plane span$\{Z^0,\ldots,Z^n\}$. So we are looking for a map ϕ satisfying the following conditions:

$$\phi(Z) = [Z^0,\tilde{Z}^0; Z^1,\ldots,Z^n]$$

such that $\tilde{Z}^0\in\text{span}\{Z^0,\ldots,Z^n\}, \tilde{Z}^0\neq Z^0$.

Example 1.

We shall describe now an example of such a map ϕ. Let us consider a point $\mathcal{Z} = [Z^0; Z^1, \dots, Z^n] \in \mathbf{C}M \times ISt_n$. Let us suppose that $Z^0 = [x, 1; y, -x \cdot y]^t \in \mathbf{C}M$ and let us consider the Euclidean subspace $E_{2n} \subset \mathbf{C}M$ defined by

$$E_{2n} := \{W \in \mathbf{C}M | W = = [w + x - 1, 1; \bar{w} + y + 1, -(w + x - 1) \cdot (\bar{w} + y + 1)]^t, w \in C_n\}.$$

It is the standard Euclidean subspace shifted to the point $Z^0 + U$, where $U = [-1_n, 1; 1_n, n]$ is a point on the imaginary axis. Now we define the map

$$\bar{\phi}([Z^0; Z^1, \dots, Z^n]) = [Z^0, \tilde{Z}^0; Z^1, \dots, Z^n],$$

where

$$\tilde{Z}^0 = E_{2n} \cap \mathrm{span}\{Z^0, Z^1, \dots, Z^n\}.$$

The intersection of E_{2n} with the null cone $\mathbf{CN}_{Z_0}$ is a $(2n-2)$-dimensional sphere and the described construction gives us a fibration of $\{Z_0\} \times ISt_n$ over this sphere.

9 The inverse Penrose transform

Given a solution f of the complex Laplace equation on $\mathbf{C}M$, we can reconstruct the corresponding form on the twistor space $\mathbf{T}$ by the following procedure.

For a given field f, we shall consider first the form ω (given by the Cauchy integral formula, see Sect. 7) and its Leray residuum Res ω, which is a $(4n-2)$-form on $\mathcal{M} \setminus \Delta$.

Let us choose any map $\phi : \mathbf{C}M \times ISt_n \mapsto (\mathcal{M} \setminus \Delta) \times ISt_n$ which is homotopic to the map $\bar{\phi}$ constructed in the example above. If π denotes the natural projection from $(\mathcal{M} \setminus \Delta) \times ISt_n$ onto $(\mathcal{M} \setminus \Delta)$, then the form $\phi^* \pi^*(\mathrm{Res}\ \omega) \wedge \tau_n$ represents a well-defined cohomology class in the (de Rham) cohomology group $H_{DR}^{n(n+1)}(\tilde{\mathbf{F}}, \mathbf{C})$ (the form τ_n was defined at the last paragraph of Sect. 5). The cohomology class does not depend on the choice of the map ϕ with the properties described above.

The map of $H^{(0, \frac{n(n-1)}{2})}(\mathbf{T}, \mathcal{L}^{(1-n)})$ into $H_{DR}^{n(n+1)}(\tilde{\mathbf{F}}, \mathbf{C})$ given by the map

$$\beta \mapsto dZ \wedge \mu^* \beta \wedge \alpha_n$$

is a well-defined injective homomorphism. It is possible to show by explicit computations (using the map $\bar{\phi}$) that the cohomology class of the form $\phi^* \pi^*(\mathrm{Res}\ \omega) \wedge \tau_n$ belongs to the image of the homomorpism. It gives us a well-defined cohomology class on the twistor space and it is not difficult to verify (using the Cauchy integral formula and the described construction) that this class is mapped to the field f by the Penrose transform.

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Jarolim BUREŠ Vladimir SOUČEK
Math. Institute
Charles University
Sokolovská 83
186 00 Praha
Czechoslovakia