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GENERIC ORBITS OF THE DIFFEOMORPHISM GROUP OF A TWO-MANIFOLD IN THE SPACE $\mathfrak{g}^*_{reg}$ OF REGULAR MOMENTA.

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In this paper we analyze the structure of generic orbits of the diffeomorphisme group $\mathcal{D}iff\,(M^2)$ of a two-dimensional smooth compact orientable manifold M^2 in the space $\mathfrak{g}^*_{reg}$ of regular momenta. For the group $\mathcal{D}iff_+(S^1)$ a complete description of the orbits of finite codimension in $\mathfrak{g}^*_{reg}$ was given by Kirillov A.A. [4]. The study of the one-dimensional case has led to the conjecture that orbits of finite codimension always exist and, moreover, thuse orbits are generic, i.e. they constitute an open everywhere dense set in the space of all orbits. In this paper we show that the existence of orbits of finite codimension in $\mathfrak{g}^*_{reg}$ depends on the topoligy of M^2. In particular, we show that the group $\mathcal{D}iff\,(S^2)$ of diffeomorphisms of the two-sphere has no orbits of finite codimension in $\mathfrak{g}^*_{reg}$. Even in cases when orbits of finite codimension exist (such orbits are constructed here for the diffeomorphism group $\mathcal{D}iff\,(T^2)$ of the two-dimensional torus) they are very far from being typical and be in the complement to the open dense set of orbits of infinite codimension. Thus, with regard to the structure of orbits the one-dimensional case is exceptional.

In this paper we obtain global functional moduli for momenta under the coadjoint action of $\mathcal{D}iff\,(M^2)$. Recently some work has been done on the construction of functional moduli of smooth classification in related problems. For instance, functional moduli occur in the C^1-classification of mappings of the line or the circle with more then one hyperbolic point (see [3])

This paper is in final form and no version of it will be submitted for publication elsewhere.

I. Functional moduli of the momenta for the group $\mathcal{D}iff\,(M^2)$.

Recall that for the diffeomorphism group $\mathcal{D}iff\,(M^2)$ of a smooth compact orientable two-manifold, its Lie algebra $\mathfrak{g}$ is identified with the space of smooth vector fields on M^2, and the dual space $\mathfrak{g}^*_{reg}$ of regular momenta is identified with

$$\Omega^1(M^2) \otimes_{\Omega^0(M^2)} \Omega^2(M^2)$$

That is every momentum can be written as $p = \alpha \otimes \omega$, where α is a differential I-form and ω is an area form on M^2. Both form α and ω are determined by momentum p up to gauge transformation $\alpha \mapsto \rho\alpha$, $\omega \mapsto \rho^{-1}\omega$, where ρ smooth function on M^2.

The momentum $p = \alpha \otimes \omega$ gives rise to a one-dimensional foliation on M^2 tangent to the direction field $\alpha = 0$ (null-distribution of α). Singular points of this foliation are called the singular poins of p. They are given by $\alpha(x) = 0$ If p is a generic momentun its singular points are isolated and the 2-form $d\alpha$ has rank 2 at these poins.

In a neighborhood of a non-singular point for p, there exist so-called accommodation coordinates (u, v) such that p can be written as

$$du \otimes dv \wedge du$$

If (u_1, v_1) is another set of accommodation coordinates then

$$\begin{aligned} u_1 &= U(u) \\ v_1 &= [\,U'(u)\,]^{-2}\, v + b(u) \end{aligned} \qquad \text{(I)}$$

The geometric meaning of (I) is that in a neighborhood of a nonsingular point every momentun determines a one-dimensional foliation with an orientation and an affine structure on the fibres. More precisely, the foliation is given by the equation $u = const$, and the local coordinate v defines an affine structure and an orientation on each fibre.

The affine structure and orientation on the fibres $u = const$ can be defined by means of the vector field tangent to the fibre and related to the affine parameter v by $\xi = 2v\,\partial/\partial v$.

At a singular point the momentum $p = \alpha \otimes \omega$ has an invariant (modulus) λ which varies continuously with the momentum, and a discrete invariant $\varepsilon = \pm 1$. The invariant λ defined to be the eigenvalue of greatest absolute value of the operator of linear part of the vector field $\xi = \alpha/d\alpha$ at the singular point. This operator is independent of the choice of α, and its eigenvalues λ_1, λ_2 are related by $\lambda_1 + \lambda_2 = 1$. If the set $\{\lambda_1, \lambda_2\}$ is non-resonant the germ of the momemtum at the singular poin is smoothly I-determined. In this case, in a neighborhood of the singular point the momentum p can be expressed uniquely as $\alpha_0 \otimes \varepsilon\, d\alpha_0$. In what follows we shall consider momenta with only this type of singular points. Note that the singular point of the vector field $\xi = \alpha/d\alpha$ is always unstable.

<u>Lemma I</u>. In a neighborhood of a singular point $x \in M^2$ of the momentum $p = \alpha \otimes \omega$ ther exists a smooth function v, unique up to a constant factor, which is affine on each curve of the foliation $\alpha = 0$ and vanishes at the singular poin.

Lemma I shows that each singular point imposes certain "rigidity" on the choice accommodation coordinates. By itself this rigidity carries no information. But if the family of integral curves defined by the momentum $p = \alpha \otimes \omega$ begins at one singular point and ends at another singular point then the two rigidities interact. This interaction results in the appearance of functional moduli of the momentum.

<u>Theorem I</u>. Let $p = \alpha \otimes \omega$ be a momentum on M^2 and let Γ be the set of integral curves of the field $\xi = \alpha/\omega$ which begin at one singular point and end at another singular point. Then there are two functional moduli for p on the set Γ.

<u>Proof</u>. In a neighborhood containing the integral curves of Γ there exist two functions v_1 and v_2 uniquely determined up to constant factors, which are affine on the integral curves of Γ, extend smoothly respectively to the initial and final singular points, and vanish there. Each of thuse functions determines uniquely a coordinate u_i, $i = 1, 2$ on Γ in such a way that the momentum in question is $du_i \otimes dv_i \wedge du_i$, $i = 1, 2$. We can identify Γ with some curve γ which is

transversal to all integral curves of Γ. Then each coordinate u_i, $i=1,2$ defines a parametrization of γ. The function $a(u_1)=du_2/du_1$ is the first functional modulus. The function $b(u_1)=v_2-a^{-2}v_1$ is the second functional modulus of p. Since the functions v_1 and v_2 are determined by p up to constant factors, there is an equivalence relation for pairs of functional moduli: $(a_1;\ b_1)\sim(a_2,\ b_2)$, if

$$(2)\quad a_1(u)=\lambda\mu^{-1}a_2(\lambda u),\quad b_1(u)=\mu^2 b_2(\lambda u),\quad \lambda,\mu\in\mathbb{R}.$$

The proof is complete.

Theorem I shows the existence of functional moduli for a momentum. The next natural step is to show that these moduli are nontrivial. For the proof it suffices to show that there is a nontrivial dependence between small variations of the momentum and small variations of the moduli obtained.

Theorem I also extends to the case where the one-dimensional foliation associated with the momentum has limit cycles.

Theorem 2. Let $p=\alpha\otimes\omega$ be a momentum on M^2 and let Γ be the set of integral curves of the field $\xi=\alpha/\omega$ which have the same α-limit set and the same ω-limit set. Suppose that each of these limit sets is either a singular point of node or fucus type or a limit cycle with multiplier different from I. Then there are two functional moduli for p on Γ.

Theorems I and 2 describe the appearance of functional moduli for the orbits of the diffeomorphism group $\mathcal{D}iff(M^2)$ of a two-dimensional smooth compact oriented manifold in the space $\mathfrak{g}^*_{ug}$ and assert that these moduli are non-trivial. The natural question which arises next is whether these moduli are sufficient. We shell give an answer to this question in some particular cases.

2. Orbits of $\mathcal{D}iff(M^2)$ in $\mathfrak{g}^*_{ug}$.

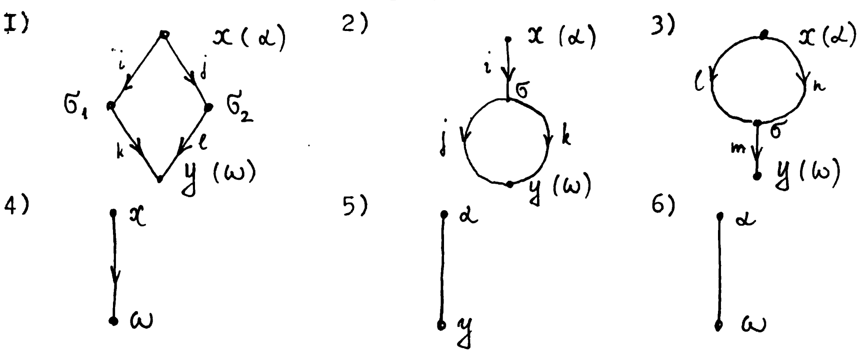
Associated to the momentum $p=\alpha\otimes\omega$ there is a vector field $\xi=\alpha/\omega$ determined by p up to multiplication by a smooth positive function. The field ξ defines on M^2 a dynamical system whose phase portrait does not depend on the ambiguity in the definition of ξ . We shell consider the subspace of those momenta in M^2 which give rise to Morse-Smale systems on M^2 (see [5]). Such momenta form a dense open subset of $\mathfrak{g}^*_{reg}$. This follows from the fact that the Morse-Smale systems on a compact two-dimensional manifold are a dense open set in the space of all smooth systems. The distinguishing graph of the flows is also an orbital invariant. Let $G^*(p)$ denote the distinguishing graph of the system associated with the momentum p .

Let p be a momentum on M^2 which generates a Morse-Smale system. Denote by $\Omega(p)$ the set of singular trajectories of this system. Let

$$\Omega(p) = \{\underbrace{x_1,\dots,x_p}_{\substack{\text{singular points}\\ \text{sources}}};\ \underbrace{\alpha_1,\dots,\alpha_m}_{\substack{\text{cycles}\\ \text{sources}}};\ \underbrace{\sigma_1,\dots,\sigma_s}_{\text{saddles}}$$

$$\underbrace{y_1,\dots,y_q}_{\text{singular points sinks}};\ \underbrace{\omega_1,\dots,\omega_k}_{\text{cycles sinks}}\}$$

The distinguishing graph $G^*(p)$ is a collectio of distinguishing sets of the following types I) - 6)



This collection must be subject to certain conditions which ensure that the graph can be realized as the graph of a flow. These conditions are given by Peixoto [5]. In addition, one must have the relation $p+q-s=\chi(M^2)$ where $\chi(M^2)$ is the Euler characteristic of M^2.

There is a one-to-one correspondence between the distinguishing sets and the canonical domains into which M^2 breaks up aftes the removal of the singular curves. In each of these domains the momentum p gives rise to a set of functional moduli (a, b) with respect to the condjoint action of $\mathcal{D}iff(M^2)$. These functional moduli are determined up to the equivalence given by (2). In addition, to each vetice of the graph one can assing a certain number λ. If the vertice corresponds to a singular point then λ is the local modulus at this point; if the vertice corresponds to a limit cycle then $e^{2\pi\lambda}$ is the multiplier of this cycle.

In this way we get a set of invariants determined by the orbit of $\mathcal{D}iff(M^2)$ in $\mathfrak{g}^*_{reg}$: if p and $\tilde{p}$ lie on the same orbit then the above invariants of p coincide with those of $\tilde{p}$.

<u>Theorem 3</u>. If for two momenta p and $\tilde{p}$ the invariants defined above coincide (up to equivalence). Then p and $\tilde{p}$ belong to the same orbit of $\mathcal{D}iff(M^2)$ in $\mathfrak{g}^*_{reg}$.

Theorem 2 and the fact that $\chi(S^2)=2$ yield the following theorem

<u>Theoreme 4</u>. The diffeomorphism group $\mathcal{D}iff(S^2)$ of the two-sphere has no orbits of finite codimension in $\mathfrak{g}^*_{reg}$.

As we have mentioned, the orbits of $\mathcal{D}iff(M^2)$ in $\mathfrak{g}^*_{reg}$ consisting of those momenta which give rise to Morse-Smale systems on M^2 form a dense open set in the space of all orbits. Although these orbits have infinite codimention in $\mathfrak{g}^*_{reg}$, they are determined by a finite number of functional moduli. Therefor in the definition of generic orbits one should apparently speak of finite functional codimension in $\mathfrak{g}^*_{reg}$ rather than just finite codimension.

<u>Definition</u>. A generic orbit of $\mathcal{D}iff(M^2)$ in $\mathfrak{g}^*_{reg}$ is an orbit consisting of momenta which give rise to Morse-Smale systems on M^2.

3. <u>Orbits of finite codimension for $\mathcal{D}iff(T^2)$</u>.

For the diffeomorphism group of a one-dimensional manifold the most interesting orbits are thore of finite codimension in $\mathfrak{g}^*_{reg}$. This is no longer so in the two-dimensional case. For instance, in the case of $\mathcal{D}iff\,(T^2)$ there exist orbits of finite codimension in $\mathfrak{g}^*_{reg}$ which are not generic. It means that the small deformation of these orbits reduces to orbits of infinite codimension. On the contrary, generic orbits have infinite codimension in $\mathfrak{g}^*_{reg}$. In this section we give an example of nongeneric orbits of codimention two for the diffeomorphism group $\mathcal{D}iff\,(T^2)$ of the two-dimensional torus. The choice of the torus as a base manifold is not accidental. In fact, by the theorem on the sum of idices of a vector field, the torus is the only compact orientable surface which allows vector fields without singular points.

Lemma 2. Let $p = \alpha \otimes \omega$, $\alpha \in \Omega^1(T^2)$, $\omega \in \Omega^2(T^2)$ be a momentum on $T^2 = \{(x, y) \bmod 2\pi\}$ which gives rise to a flow with an irrational rotation number μ (see [1]). Suppose that μ has the type (K, ε), $K>0$, $\varepsilon>0$, i.e. for any irreducible fraction p/q we have

$$|\mu - p/q| > K\,|q|^{-(2+\varepsilon)}$$

Then p can be written as

$$p = (dy - \mu\, dx) \otimes \rho\; dx \wedge dy \tag{3}$$

where $\rho \in C^\infty(T^2)$.

The proof follows from a theorem of Hermani [1].

Let p_ρ denot a momentum which has the form (3) in a given coordinate systen.

Theorem 5. Suppose that the momentum p satisfied the hypotheses of lemma 2 and let C be the mean value of ρ on T^2. Then p can reduced to the form

$$p = (dy - \mu\, dx) \otimes c\; dx \wedge dy$$

This theorem can be proofed by homotopical method (see [2]).

Corollary. If the momentum $p \in \mathfrak{g}^*_{reg}$ satisfies the hypotheses of lemma 2, then the orbit Ω_p of $\mathcal{D}iff\,(T^2)$ through p has codimension two in $\mathfrak{g}^*_{reg}$, and $\Omega_p \simeq \mathcal{D}iff\,(T^2)/T^2$.

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