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CELLS OF HARMONICITY

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The study of complexified partial differential equations was inspired by quantum field theory. The analytic continuation of fields from the Euclidean region into Minkowski space is a standard procedure in QFT nowadays. From the mathematical point of view, we are interested in partial differential equations on domains in $\mathbf{C}^n$. It is a typical feature of this study that eliptic and hyperbolic equations, formely considered separately, appear as restrictions of one common complex equation. In the complexified version of a partial differential equation, real derivatives $\dfrac{\partial}{\partial x_i}$ are replaced by complex $\dfrac{\partial}{\partial z_i}$ while we assume $\dfrac{\partial f}{\partial \bar{z}_i} = 0$. So all solutions are holomorphic maps which moreover satisfy the complex differential equation.

One of the most natural questions is that of analytic continuation of solutions and domains of holomorphy. In the classical case, there is a nice theory which gives full description of domains of holomorphy in terms of local behaviour of their boundaries. When we take instead of the whole class of holomorphic maps a subclass made up by the solutions of a complex differential equation, the situation is different. For example, there are no longer smooth domains of holomorphy.

Our aim in this paper is to describe the domains of holomorphy for solutions to the complex Laplace and Dirac equations. We call them cells of harmonicity. We deduce their properties mostly by examining geometrical properties of the characteristic surface (which is the same for both equations), namely the complex null cone. We consider only even dimensions. The methods we use are not instantly applicable in odd dimensions. After some examples we find a necessary condition for cells of harmonicity. For a certain class of domains we obtain full characterization of cells of harmonicity. Further we apply these results to the case of analytic continuation from the Euclidean region into Minkowski space. We get a simple proof of a result by Gindikin and Henkin in dimension 4 and its generalization to higher dimensions.

In $\mathbf{C}^n$ we use the following notation. Let $z = (z_1, z_2, \dots, z_n), z_i = x_i + iy_i$:

(euclidean norm) $$\|z\| = \left(\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2\right)^{\frac{1}{2}}$$

$$|z|^2 = \sum_{i=1}^{n} z_i^2$$

(real scalar product) $$\langle z, z' \rangle_{\mathbf{R}} = \sum_{i=1}^{n} x_i x_i' + y_i y_i'$$

(hermitian scalar product) $$\langle z, z' \rangle_{\mathbf{C}} = \sum_{i=1}^{n} z_i \bar{z}_i'$$

(Euclidean space) $$\mathbf{E}^n = \{z \in \mathbf{C}^n : y_i = 0 \text{ for all } i = 1, 2, ..., n\}$$

The set

$$CN(z') = \left\{z \in \mathbf{C}^n : |z' - z|^2 = 0\right\}$$

is the complex null cone of a point z'.

The real Laplace operator is defined by

$$\Delta_R = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

and its complex version

$$\Delta_C = \sum_{i=1}^{n} \frac{\partial^2}{\partial z_i^2}.$$

A solution to the complex Laplace equation $\Delta_C f = 0$ on a domain $\Omega \subseteq \mathbf{C}^n$ is a holomorphic function of n complex variables which satisfies $\sum_{i=1}^{n} \frac{\partial^2 f}{\partial z_i^2} = 0$ in Ω .

In order to introduce the complex Dirac operator, let us consider the complex Clifford algebra $C_n^{\mathbf{C}}$ over $\mathbf{C}^n$. We have an embedding of $\mathbf{C}^n$ into $C_n^{\mathbf{C}}$ and the vectors $\{e_i\}$ of the canonical basis are generators of $C_n^{\mathbf{C}}$ satisfying $e_i e_j + e_j e_i = 2\delta_{ij}$. Every element $z \in C_n^{\mathbf{C}}$ can be expressed in the form

$$z = z_0 1 + \dots + z_{i_1 \dots i_k} e_{i_1} \dots e_{i_k} + \dots + z_{1 \dots n} e_1 \dots e_n$$

We define its norm

$$\|z\| = \left(\|z_0\|^2 + \dots + \|z_{1\dots n}\|^2\right)^{\frac{1}{2}}$$

which for $\mathbf{C}^n \subseteq C_n^{\mathbf{C}}$ coincides with the previous definition. Let us denote

$$D_{\mathbf{R}} = \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}$$

$$D_{\mathbf{C}} = \sum_{i=1}^{n} e_i \frac{\partial}{\partial z_i}$$

Let S be any left ideal in $C_n^{\mathbf{C}}$. A solution to the complex Dirac equation $D_{\mathbf{C}} f = 0$ is a map from a domain in $\mathbf{C}^n$ to S which components are again supposed to be holomorphic functions.

We will often think of Laplace and Dirac equation at once. That is why we denote the two operators $D_{\mathbf{C}}$ and $\Delta_{\mathbf{C}}$ by a common symbol $d_{\mathbf{C}}$. So $d_{\mathbf{C}}$ denotes either the complex Laplace operator $\Delta_{\mathbf{C}}$ or the complex Dirac operator $D_{\mathbf{C}}$. We denote by $H_d(\Omega)$ the space of all solutions to the equation $d_{\mathbf{C}} f = 0$ in Ω.

DEFINITION 1. *A domain $\Omega \subseteq \mathbf{C}^n$ is called the domain of holomorphy of a holomorphic map f if for any point $z \in \partial\Omega$ and for any neighbourhood $U(z)$ there is no holomorphic map $\tilde{f}$ such that $\tilde{f} = f$ on a connected component of $U(z) \cap \Omega$.*

We say that Ω is a cell of harmonicity if it is the domain of holomorphy of a map $f \in H_d(\Omega)$.

One way to prove that a domain is a cell of harmonicity is to find a solution unbounded on the boundary of Ω, or it is enough to find for each boundary point a solution unbounded in that point. It suffices to do so for a dense subset of the boundary.

THEOREM 2. *Let E be a dense subset of $\partial\Omega$ and suppose that for any $x \in E$ there is a map from $H_d(\Omega)$ which is unbounded in x. Then there is a map $f \in H_d(\Omega)$ unbounded on $\partial\Omega$, so Ω is a cell of harmonicity.*

PROOF: Let E' be a countable subset of E. Let $\{x_n\}$ be such a sequence of points from E' that each point appears infinitely many times in the sequence. Denote by h_n a map from $H_d(\Omega)$ unbounded in x_n. Consider an increasing sequence of compact subsets $K_n \subseteq \Omega$ such that $\bigcup_{n=1}^{\infty} K_n = \Omega$. By induction we construct a sequence $\{z_n\} \subseteq \Omega$ and a sequence of maps $f_n \in H_d(\Omega)$:

(1) $z_1 \in \Omega$ arbitrary, $f_1 = h_1$

(2) We take $z_n \in \Omega$ such that $\|x_n - z_n\| < \frac{1}{n}$ and

$$\|h_n(z_n)\| \geq 2^n \|h_n\|_{K_n} \left(n + \| \sum_{i=1}^{n-1} f_i(z_n)\| \right)$$

where $\|h_n\|_{K_n} = \sup_{x \in K_n} \|h_n(x)\|$. Let us denote

$$f_n = \frac{1}{2^n} \frac{h_n}{\|h_n\|_{K_n}}$$

The map $f = \sum_{n=1}^{\infty} f_n$ converges uniformely on every K_n , because

$$\|f_m\|_{K_n} \leq \frac{1}{2^m} \text{ for every } m \geq n$$

. So we have $f \in H_d(\Omega)$. But

$$\|f(z_n)\| \geq n$$

and any point of $\partial\Omega$ is an accumulating point of the sequence $\{z_n\}$, so f is unbounded in $\partial\Omega$.

EXAMPLE 3. *(a) Let $p \in \mathbf{C}^n$. The map*

$$f(z) = \frac{1}{|z-p|^{n-2}}$$

is an elementary solution to the complex Laplace equation. Similarly

$$f(z) = \frac{z-p}{|z-p|^n}$$

is an elementary solution to the complex Dirac equation. Both elementary solutions are unbounded on the complex null cone $CN(p)$.

(b) The second basic type of singularity is a complex hyperplane. Let us consider a map

$$f(z) = \frac{1}{\sum_{i=1}^n z_i v_i} \quad (v \in \mathbf{C}^n)$$

We have

$$\frac{\partial^2 f}{\partial z_i^2} = v_i^2 \frac{2}{(\sum_{i=1}^n z_i v_i)^3} \text{ and } \Delta_{\mathbf{C}} f = |v|^2 \frac{2}{(\sum_{i=1}^n z_i v_i)^3}$$

So f is a solution to the complex Laplace equation if and only if $v \in CN$. The singularity for f is then the complex hyperplane perpendicular to $\bar{v}$. In the case of Dirac operator, the same is true for

$$f(z) = \frac{v}{\sum_{i=1}^n z_i v_i}$$

Example 3(a) leads to the following theorem which contains as a special case cells of harmonicity defined over domains in $\mathbf{E}^n$ and over some more general manifolds which were studied in [4] and [5].

THEOREM 4. *Let γ be a closed subset of* $\mathbf{C}^n$. *Then each connected component of the set* $\mathbf{C}^n \setminus \bigcup_{x\in\gamma} CN(x)$ *is a cell of harmonicity.*

PROOF: Let Ω be one of the components and $x \in \partial\Omega$. There is a point $x' \in \gamma$ for which $|x-x'|^2 = 0$. The elementary solution for the point x' is defined in Ω and it is unbounded in x . By Theorem 2 Ω is a cell of harmonicity.

As in the classical case we can define the notion of envelope of holomorphy of a domain Ω to be the minimal cell of harmonicity containing Ω. There are several examples of constructing such envelopes in [4]. We mention one of them here in a form that will be usefull in the proof of Theorem 7.

THEOREM 5. *Let P be a real n-dimensional affine subspace of* $\mathbf{C}^n$ *which contains no non-zero null vectors. Let M be an open connected subset of P . Let f be a solution to the equation $d_{\mathbf{C}}f = 0$ defined on a neighbourhood of M in* $\mathbf{C}^n$. *Then we can always continue f analytically to the connected component of the set* $\mathbf{C}^n \setminus \bigcup_{x\in\partial M} CN(x)$ *determined by M .*

Theorem 5 gives an idea what the boundary of a cell of harmonicity looks like. If we can find a real n-dimensional subspace without null vectors near the boundary of a domain we are able to continue all solutions beyond the boundary and such domain is not a cell of harmonicity.

LEMMA 6. *Let T be a (2n-1)-dimensional linear subspace of* $\mathbf{C}^n$ *and let v be its normal vector . If $|v|^2 \neq 0$, then T contains a real n-dimensional subspace which does not contain any non-zero null vectors.*

PROOF: Let $T_C = \langle v\rangle^{\perp_C}$ be the complex orthogonal complement to v . Let us consider a complex basis $\{z^1,...,z^{n-1}\}$ of T_C such that $|z^i|^2 \neq 0$ and $\sum_{k=1}^{n} z^i_k z^j_k = 0$ for $i \neq j$. There is a complex number c such that

$$\{z^1, iz^1, ..., z^{n-1}, iz^{n-1}, c\bar{v}\}$$

is a real basis of T . Let us choose n vectors $\{x^1,...,x^n\}$ from that basis in such a way that all the numbers $|x^1|^2, |x^2|^2, ..., |x^n|^2$ lie in one halfplane in $\mathbf{C}$ determined by a straightline passing through the origin. For all non-zero real combinations of these vectors we have

$$|\sum_{i=1}^{n} a_i x^i|^2 = \sum_{i=1}^{n} a_i^2 |x^i|^2 \neq 0$$

THEOREM 7. *Let Ω be a cell of harmonicity with Lipschitz boundary. Let $x_0 \in \partial\Omega$ and suppose that $T_{x_0}\partial\Omega$ exists. Let n be the unit normal vector. Then $|n|^2 = 0$.*

Proof: Suppose $|n|^2 \neq 0$. By Lemma 6 there is an n-dimensional subspace $P \subseteq T_{x_0}\partial\Omega$ without null vectors. The following number expresses the angle between P and $CN(x_0)$:

$$\alpha = \arccos\left[\inf_{\substack{p\in P\setminus\{x_0\}\\ q\in CN(x_0)\setminus\{x_0\}}} \left|\frac{(p-x_0,q-x_0)_R}{\|p-x_0\|\|q-x_0\|}\right|\right]$$

. Since P contains no null vectors, we have $\alpha > 0$. Denote

$$Q_\epsilon = \left\{y \in U(x_0,\epsilon) : \arccos\frac{(y-x_0,n)_R}{\|y-x_0\|} \leq \frac{\Pi-\alpha}{2}\right\}$$

. By the Lipschitz property, there is $\epsilon_0 > 0$ for which $Q_{\epsilon_0} \subseteq \Omega$. We shift the space P in the direction of the inner normal vector and define

$$P' = P + \frac{\epsilon_0}{2}n$$

. Then $M = P' \cap Q_{\epsilon_0}$ is a convex open set in P' and we have $CN(x) \cap P' \subseteq \Omega$ for every x on the segment $(x_0, x_0 + \frac{\epsilon_0}{2}n)$. So x_0 lies in the component of $\mathbf{C}^n \setminus \bigcup_{x\in\partial M} CN(x)$ determined by M and by Theorem 5 we can continue all solutions to a neighbourhood of x_0 , so Ω is not a cell of harmonicity.

Corollary 8. *There are no bounded cells of harmonicity with smooth boundary.*

Proof: For a smooth compact manifold of codimension 1 each unit vector occurs as a normal vector in some point.

It is not difficult to find an example which shows that the condition from Theorem 7 is not sufficient. It is sufficient when we confine to convex domains.

Theorem 9. *Let Ω be a convex domain in $\mathbf{C}^n$. Then the following assertions are equivalent:*

(i) *Ω is a cell of harmonicity.*
(ii) *Whenever the normal vector n_x exists in a point $x \in \partial\Omega$, it is a null vector , i. e. $|n_x|^2 = 0$.*

Proof: Denote

$$V = \{x \in \partial\Omega \text{ such that } n_x \text{ exists }\}$$

By Rademacher theorem, V is dense in $\partial\Omega$. So it suffices to prove that for each point $x \in V$ there is a map $f_x \in H_d(\Omega)$ which is unbounded in x . In order to do so, take

$$f_x(z) = \frac{1}{\sum_{i=1}^n z_i(\bar{n}_x)_i}$$

and

$$f_x(z) = \frac{\bar{n}_x}{\sum_{i=1}^{n} z_i(\bar{n}_x)_i}$$

for Laplace and Dirac operators respectively.

We apply Theorem 9 to give further examples of cells of harmonicity.

EXAMPLE 10. *Let us take an orthogonal basis of* $\mathbf{R}^{2n}$ *containing only null vectors, e. g.*

$$\{z^1, ..., z^n, iz^1, ..., iz^n\}$$

where

$$z_k^k = 1, z_{k+1}^k = i \text{ for } k \text{ odd}$$
$$z_{k-1}^k = i, z_k^k = 1 \text{ for } k \text{ even}$$

and $z_i^j = 0$ *otherwise. The cube spanned by these vectors has normal vectors from* CN *on all its faces and so by Theorem 9 it is a cell of harmonicity.*

EXAMPLE 11. *Let us denote*

$$M_+ = \{z = x + iy : x_n^2 - \sum_{k=1}^{n-1} y_k^2 > 0\}$$

For $z \in \partial M_+$ *we have* $x_n^2 - \sum_{k=1}^{n-1} y_k^2 = 0$ *and the null vector*

$$(-iy_1, -iy_2, ..., -iy_{n-1}, x_n)$$

is a normal vector to ∂M_+ *in the point* z . *Since* M_+ *is convex, we conclude that it is a cell of harmonicity*

In dimension 4, the domain M_+ was studied by Gindikin and Henkin in [3].

They proved that any solution to the Laplace and Dirac equations can be holomorphically continued from the Euclidean halfspace $x_n > 0$ to M_+ and that M_+ is the maximal domain with this property. The result concerning maximality they proved by applying Penrose twistor transform, which allows to reduce the problem of holomorphic continuation of solutions to the classical theory of functions of several complex variables (results by Andreotti-Norguet about cohomologies of 1-pseudoconvex domains with coefficients in holomorphic sheaves). So as a consequence of Theorem 9 we got a straight and simple proof of this result and its generalization to arbitrary even dimension.

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