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ON CERTAIN CLASS OF UNITARIZABLE REPRESENTATIONS OF THE LIE ALGEBRA $u(p,q)$

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INTRODUCTION. T. Enright, R. Howe and N. Wallach [2] have given a complete classification of the unitary highest weight modules for Hermitian symmetric pairs. The highest weight modules are a special case of the Enright-Varadarajan modules [3]. In the present paper we formulate the theorem, which describes certain subclass of the unitarizable Enright -Varadarajan modules for the Lie algebra u (p,q). For these modules we construct the ortonormal bases of Gelfand-Tsetlin type. It turns out the unitarizable representations of u (p,q) which had been received by I.M. Gelfand and M.I.Graev in [4] belong to this subclass (see Theorem 2 below).

We recall now the definition of the Enright-Varadarajan modules for u (p,q) (see [1,3]). Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ be the Lie algebra of all complex n by n matrices, where n = p + q; $\mathfrak{k} = \mathfrak{gl}(p, \mathbb{C}) \oplus \mathfrak{gl}(q, \mathbb{C})$ be the obvious subalgebra of $\mathfrak{g}$.

The standard matrix units e^i_j , i,j = 1,..., n, form a basis of $\mathfrak{g}$ with the following relations:

$$[e^i_j \ e^k_m] = \delta^k_j e^i_m - \delta^i_m e^k_j .$$

Consider the usual triangular decomposition of $\mathfrak{k}$:

$$\mathfrak{k} = \mathfrak{k}^- \oplus \mathfrak{f} \oplus \mathfrak{k}^+ .$$

Let Δ and Δ_c denote the roots of $(\mathfrak{g}, \mathfrak{f})$ and $(\mathfrak{k}, \mathfrak{f})$ respectively. Put $\Delta_n = \Delta \setminus \Delta_c$. The elements of Δ_c and Δ_n are called compact and noncompact roots respectively. We choose the positive com-

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This paper is in final form and no version of it will be submitted for publication elsewhere.

pact roots $\Delta_c^+ \subset \Delta_c$ determined by the triangular decomposition of k. Let $\Delta^+ \subset \Delta$ be an arbitrary system of positive roots, such that $\Delta_c^+ \subset \Delta^+$. Put $\Delta_n^+ = \Delta^+ \setminus \Delta_c^+$,

$$\rho_c = \frac{1}{2} \sum_{\alpha \in \Delta_c^+} \alpha \, , \quad \rho_n = \frac{1}{2} \sum_{\alpha \in \Delta_n^+} \alpha \, ,$$

$$\rho = \rho_c - \rho_n .$$

Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be the basis of $\mathfrak{f}^*$ which is dual to $\{e_1^1, \ldots, e_n^n\}$. We shall consider a real span of $\{\varepsilon_1, \ldots, \varepsilon_n\}$, it will be identified with $\mathbb{R}^n$. The brackets (,) denote the standard inner product in $\mathbb{R}^n$.

For $\lambda \in \mathbb{R}^n$ let $V_0(\lambda)$ denote the Verma module for k relative to Δ_c^+ and $V(\lambda)$ denote the Verma module for $\mathfrak{g}$ relative to $-\omega_0 \Delta^+$. Here ω_0 is the element of the Weyl group W_c for Δ_c, which has the maximal length.

If $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ is Δ_c^+ - dominant integral (that is $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+$ for all $i \neq p$) we set

$$\omega_0 \cdot \lambda = \omega_0 (\lambda + \rho_c) - \rho_c .$$

PROPOSITION (see [1]). There exists the unique (up to isomorphism) $\mathfrak{g}$ - module $M(\lambda)$, which contains $V_0(\lambda)$, is generated by $V_0(\lambda)$ and has the following two properties:

1) If $x \in M(\lambda)$, $u \in U(k^-)$ and $ux = 0$ then either $u = 0$ or $x = 0$.

2) The submodule of $M(\lambda)$ which is generated by $V_0(\omega_0 \cdot \lambda)$ is equivalent to $V(\omega_0 \cdot \lambda)$.

DEFINITION. The Enright-Varadarajan module $D(\lambda)$ is the simple factor of $M(\lambda)$ (In [1,3] $D(\lambda)$ is denoted by $D_{P,\lambda}$, where $P = \Delta^+$).

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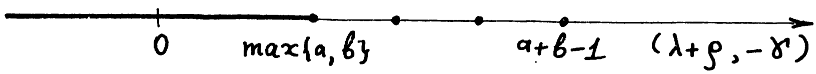
1. STATEMENT OF RESULTS

A $\mathfrak{g}$-module L is called unitarizable if it admits a positive definite u (p,q)-invariant Hermitien form. The Lie algebra u(p,q) is considered as a real form of $\mathfrak{g}$.

Suppose now Δ^+ contains at most two noncompact simple roots.

THEOREM 1. The module $D(\lambda)$ is unitarizable if and only if for every noncompact simple root γ the following condition holds: $(\lambda+\rho, -\gamma) \leq \max\{a, b\}$ or $(\lambda+\rho, -\gamma)$ is an integer and $(\lambda+\rho, -\gamma) \leq a+b-1$:



where for $i \leq p \leq k-1$ and $\gamma = \varepsilon_i - \varepsilon_k$

$a = \operatorname{card} \{ s \mid 1 \leq s \leq i, \ \lambda_s = \lambda_i \}$

$b = \operatorname{card} \{ s \mid k \leq s \leq n, \ \lambda_s = \lambda_k \}$

and for $\gamma = \varepsilon_k - \varepsilon_i$

$a = \operatorname{card} \{ s \mid i \leq s \leq p, \ \lambda_s = \lambda_i \}$

$b = \operatorname{card} \{ s \mid p < s \leq k, \ \lambda_s = \lambda_k \}$.

We remark that the discrete series representations correspond to those λ, for which $(\lambda+\rho, -\gamma) < 0$ for every noncompact simple root γ.

If there is only one simple noncompact root then $D(\lambda)$ is a highest weight module. For this case the proof is contained in [2,7].

The module $D(\lambda)$ is called non-degenerate if for every noncompact simple root γ we have $(\lambda+\rho, -\gamma) < \max\{a, b\}$.

Each unitarizable representation of u(p,q) which had been constructed by I.M. Gelfand and M.I. Graev (see [4]) is determined by the set of integers $(m_{1n}, \ldots, m_{nn})$, where $m_{1n} \geq \cdots \geq m_{nn}$, and by the pair (α, β), where α and β are nonnegative integers, such that $p = \alpha + \beta$. For these α and β we fix now $\Delta^+ = \Delta^+_{\alpha,\beta}$ chosen by the following way:

$$\varepsilon_i - \varepsilon_k \in \Delta^+ \quad \text{for } 1 \leq i \leq \alpha \text{ and } p < k \leq n$$

and $\varepsilon_k - \varepsilon_i \in \Delta^+$ for $\alpha < i \leq p$ and $p < k \leq n$.

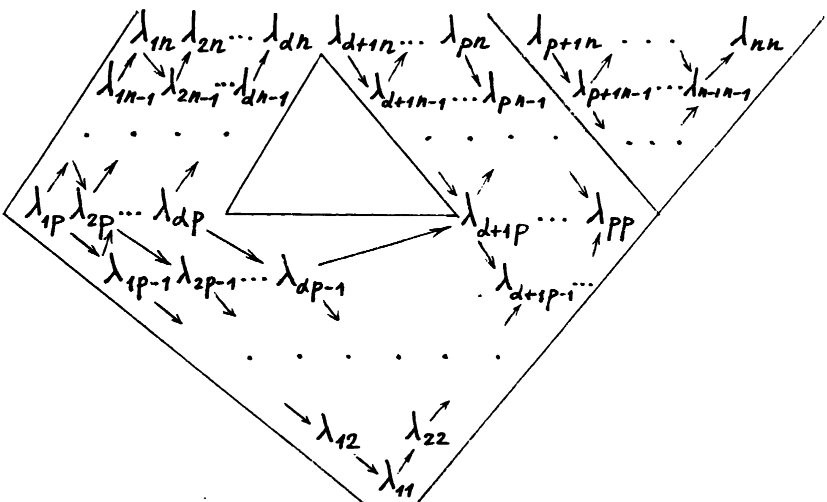
THEOREM 2. The Gelfand-Graev representation with the parameters $(m_{1n}, \ldots, m_{nn})$ and (α, β) is equivalent to Enright-Varadarajan module $\mathcal{D}(\lambda)$ with $\Delta^+ = \Delta^+_{\alpha,\beta}$ where

$$\lambda_i = \begin{cases} m_{in} + q & \text{for} \quad 1 \leq i \leq \alpha \\ m_{i+q,n} - q & \text{for} \quad \alpha < i \leq p \\ m_{i-\beta,n-\alpha+\beta} & \text{for} \quad p < i \leq n \end{cases}$$

As it follows from Theorem 2, all Gelfand-Graev representations belong to the discrete series.

Let us fix $\lambda \in \mathbb{R}^n$ where $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+$ for $i \neq p$. A pattern Λ (of Gelfand-Tsetlin-Graev type) defined as a table of real numbers:

$\Lambda =$



where the upper row coincides with λ and a $\longrightarrow$ b means a-b $\in \mathbb{Z}_+$.

Put

$$l_{ik} = \begin{cases} \lambda_{ik} + p - i & \text{for} \quad 1 \leq i \leq \alpha\,;\ k > p \\ \lambda_{ik} + k + \alpha - i & \text{for} \quad p < i \leq k \leq n \\ \lambda_{ik} + k - i & \text{in other cases.} \end{cases}$$

Let $\theta_k = 1$ for $k \neq p$ and $\theta_p = -1$.

THEOREM 3. Any non-degenerate module $D(\lambda)$ admits an ortonormal basis $\{\xi_\Lambda\}$, which is parameterized by all patterns Λ, such that

$$e_k^k \xi_\Lambda = \Big(\sum_{i=1}^{k} \lambda_{ik} - \sum_{i=1}^{k-1} \lambda_{ik-1} \Big) \xi_\Lambda ,$$

$$e_{k+1}^k \xi_\Lambda = \sum_{\tau=1}^{k} \left(-\theta_k \frac{\prod_{i=1}^{k-1} (l_{\tau k} - l_{ik-1}) \prod_{i=1}^{k+1} (l_{\tau k} - l_{ik+1} + 1)}{\prod_{i=1, i\neq\tau}^{k} (l_{\tau k} - l_{ik})(l_{\tau k} - l_{ik} + 1)} \right)^{\frac{1}{2}} \xi_{\Lambda + \delta_{\tau k}}$$

$$e_k^{k+1} \xi_\Lambda = \theta_k \cdot \sum_{\tau=1}^{k} \left(-\theta_k \frac{\prod_{i=1}^{k-1} (l_{\tau k} - l_{ik-1} - 1) \prod_{i=1}^{k+1} (l_{\tau k} - l_{ik+1})}{\prod_{i=1, i\neq\tau}^{k} (l_{\tau k} - l_{ik} - 1)(l_{\tau k} - l_{ik})} \right)^{\frac{1}{2}} \xi_{\Lambda - \delta_{\tau k}}$$

where $\Lambda \pm \delta_{\tau k}$ is the table, obtained from Λ by replacing $\lambda_{\tau k}$ by $\lambda_{\tau k} \pm 1$.

An analogous theorem holds for the remaining unitarizable modules $D(\lambda)$. Here the ortonormal basis is parameterized by a certain part of the patterns Λ. The matrix elements of generators of $\mathfrak{g}$ are given by similar formulae. For highest weight modules $D(\lambda)$ such a theorem is contained in [6].

2. OUTLINE OF THE PROOF

The main tool of the proof of the theorems is Mickelsson $\mathcal{Z}$ - algebras [8]. Let $\mathfrak{g}_k = \mathfrak{gl}(k, \mathbb{C})$ and
$$\mathfrak{g}_k = \mathfrak{g}_k^- \oplus \mathfrak{h}_k \oplus \mathfrak{g}_k^+$$
be the triangular decomposition of $\mathfrak{g}_k$. Consider the following natural inclusions
$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_n .$$

In place of the universal enveloping algebra $U(\mathfrak{g}_n)$ we consider its extension
$$U'(\mathfrak{g}_n) = U(\mathfrak{g}_n) \otimes_{U(\mathfrak{h}_n)} R(\mathfrak{h}_n)$$
where $R(\mathfrak{h}_n)$ is the field of fractions of the commutative algebra $U(\mathfrak{h}_n)$. Let M be the quotient module of $U'(\mathfrak{g}_n)$ by the left ideal $U'(\mathfrak{g}_n)\mathfrak{g}_p^+$. Put
$$\mathcal{Z} = \{x \in M,\ \mathfrak{g}_p^+ x = 0\}.$$
The space $\mathcal{Z} = \mathcal{Z}(\mathfrak{g}_n, \mathfrak{g}_p)$ is an algebra over the field $\mathbb{C}$. It is called (extended) Mickelsson algebra (see [8]). This algebra is generated by elements z_k^i, z_i^k and e_m^k, where
$$z_k^i = P e_k^i, \quad z_i^k = P e_i^k,$$
$1 \le i \le p < k \le n$; $p < m \le n$, $m \ne k$ and P is the extremal projection for $\mathfrak{g}_p$ [8]. These elements satisfy the following relations.

LEMMA 1. If $1 \le i, j \le p$ and $p < k, m \le n$
$$z_k^i z_m^j = z_m^j z_k^i + z_k^j z_m^i \frac{1}{h_i - h_j} \quad \text{for } i<j,$$
$$z_i^k z_j^m = z_j^m z_i^k - z_i^m z_j^k \frac{1}{h_i - h_j} \quad \text{for } i<j,$$
$$z_k^i z_m^i = z_m^i z_k^i, \qquad z_i^k z_i^m = z_i^m z_i^k,$$
$$z_k^i z_j^m = z_j^m z_k^i \quad \text{for } i \ne j,$$
$$z_k^i z_i^m = \sum_{j=1}^{p} z_j^m z_k^j\, b_{ij} + \left(\delta_k^m h_i - e_k^m\right) c_i^-,$$
where
$$c_i^{\pm} = \prod_{r=i+1}^{p} \frac{h_i - h_r \pm 1}{h_i - h_r}; \quad b_{ij} = \frac{c_i^- c_j^+}{h_j - h_i + 1}; \quad h_i = e_i^i + p - i.$$

Moreover for $p < r \leq n$

$$[z^i_r, e^k_m] = \delta^k_r z^i_m, \quad [z^r_i, e^k_m] = -\delta^r_m z^k_i.$$

We shall give the definition of the modules $D(\lambda)$ by using Mickelsson Z-algebra. Put

$$S^i_k = z^i_k (h_i - h_1) \cdots (h_i - h_{i-1})$$

$$S^k_i = z^k_i (h_i - h_{i+1}) \cdots (h_i - h_p)$$

where $1 \leq i \leq p < k \leq n$ and

$$t^k_m = \sum_{i=1}^{\alpha} e^i_m e^k_i + \sum_{j=\alpha+1}^{p} e^k_j e^j_m + \delta^k_m d\beta$$

where $p < k, m \leq n$.

Then S^i_k, S^k_i can be regarded as elements of $U(g_n)$.

Let $M'(\lambda) = U(g_n)/I_\lambda$ where I_λ is the left ideal generated by the following elements:

$$
\begin{array}{lll}
e^i_i - \lambda_i & \text{for} & i = 1, 2, \ldots, n \\
e^i_j & \text{for} & \varepsilon_i - \varepsilon_j \in \Delta^+_c, \\
S^i_j & \text{for} & \varepsilon_j - \varepsilon_i \in \Delta^+_n \text{ and} \\
t^k_m & \text{for} & p < k, m \leq n.
\end{array}
$$

LEMMA 2. The module $M'(\lambda)$ is equivalent to $M(\lambda)$ (see Introduction).

One can see that $M'(\lambda)$ admits a g-invariant Hermitian form B_λ and the Enright-Varadarajan module $D(\lambda)$ is a factor of $M'(\lambda)$:

$$D(\lambda) = M'(\lambda) / \operatorname{Ker} B_\lambda.$$

The scheme of the proof of Theorem 1 is as follows: if the weight λ doesn't satisfy the conditions of the Theorem 1, one can find a non-zero vector $v \in M'(\lambda)$, such that $B_\lambda(v, v) < 0$; if the weight λ satisfies them we give an explicit construction of $D(\lambda)$ by using the Gelfand-Tsetlin basis (Theorem 3). This construction is quite similar to the construction of the Gelfand-Tsetlin basis for the finite-dimensional representations of $gl(n, \mathbb{C})$ (see, for example [5,9]). Here one use a calculations in the Mickelsson algebra (Lemma 1). But instead of the chain of the Lie algebras it is useful to take one of the Mickelsson algebras:

$$\mathcal{Z}(\mathfrak{g}_{p+1}, \mathfrak{g}_p) \subset \mathcal{Z}(\mathfrak{g}_{p+2}, \mathfrak{g}_p) \subset \cdots \subset \mathcal{Z}(\mathfrak{g}_n, \mathfrak{g}_p).$$

The proof of the Theorem 2 follows from the explicit accordance between the patterns Λ and Gelfand-Graev schemes.

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