INVARIANT ORDERS IN LIE GROUPS

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One of the basic facts in the theory of Lie groups is Lie's third theorem which, given a Lie algebra L, guarantees the existence of a simply connected Lie group G with L(G) = L, which is unique up to isomorphism. It is clear from this theorem that all group theoretic properties of G are encoded in Lie algebraic properties of L(G). Therefore it should be possible to determine the property of L which corresponds to the existence of a continuous invariant order on G (see definition below). It turns out that the Lie theory of semigroups provides a suitable setting for this problem. Especially the branch of this theory which is concerned with invariant subsemigroups of Lie groups is very well developed and provides enough tools to solve the above problem.

Definition 1. Let us call a closed convex cone W in a finite dimensional vector space L a *wedge* and $H(W) = W \cap -W$ the *edge of the wedge*, i.e. the largest vector space contained in W. We say that W is *pointed* if $H(W)=\{0\}$ and that W is *generating* if $W - W = L$. The dual wedge W^* is the set of all linear functionals on L which are non-negative on W. Suppose, in addition, that L is a Lie algebra. Then W is said to be
1) a *Lie wedge* if $e^{\mathrm{ad}\, x} W = W$ for all $x \in H(W)$ and
2) an *invariant wedge* if $e^{\mathrm{ad}\, x} W = W$ for all $x \in L$.

One should notice that every pointed wedge is a Lie wedge and that a vector subspace of L is a Lie wedge if and only if it is a subalgebra of L.

The main objective of the Lie theory of semigroups is to study subsemigroups of Lie group via their tangent objects.

Definition 2. Let $S \subset G$ be a closed subsemigroup. Then we define the *tangent wedge* of S by

$$L(S) = \{ x \in L(G) : \exp(\mathbb{R}^+ x) \subset S \}.$$

We say that S is a *Lie semigroup* if it is reconstructable from its tangent wedge, i.e., if $S = \mathrm{cl}\langle \exp L(S) \rangle$. This is the class of subsemigroups of G for which a Lie theory of semigroups should work properly. The following Lemma connects these concepts.

Lemma 3. If S is a Lie semigroup in G then L(S) is a Lie wedge and S is invariant (under all inner automorphisms of G) if and only if L(S) is an invariant wedge.

In general, not every Lie wedge $W \subset L(G)$ is the tangent wedge of a Lie semigroup $S \subset G$. A simple example is $W = \mathbb{R}^+ \subset \mathbb{R} = L(\mathbb{R}/\mathbb{Z})$ with $G = \mathbb{R}/\mathbb{Z}$. This calls for a definition.

Definition 4. A Lie wedge $W \subset L(G)$ is said to be *global in G* if $W = L(S)$ for a Lie semigroup $S \subset G$.

It is easy to see that this is equivalent to

$$L(\mathrm{cl}\langle \exp W \rangle) = W.$$

There are two natural problems arising in this context:

The Globality problem : Let G be a connected Lie group and $W \subset L(G)$ a Lie wedge. When is W global in G?

This problem turns out to be very difficult in this generality. So it is reasonable to formulate an easier problem.

The Controllability problem : Let G be a connected Lie group and $W \subset L(G)$ a Lie wedge. When is $\langle \exp W \rangle = G$? We call those Lie wedges *controllable in G.* It is clear that a Lie wedge W different from L(G) which is controllable in G cannot be global in G and that a Lie wedge $W \subset L(G)$ is controllable if and only if it is contained in a global Lie wedge V different from L(G).

To see the connections between these problems and the existence of invariant orders in G we give the precise definition of a continuous order in G.

Definition 5. Let G be a Lie group. An *invariant order* in G is a partial order on G which is invariant under left and right shifts, i.e.

$$g \le g' \quad \Rightarrow \quad xg \le xg' \quad \text{and} \quad gx \le g'x \quad \text{for all } x \in G.$$

There is an easy connection between such orders and semigroups.

Theorem 6. a) Let $\le$ be an invariant order on G and $S_{\le} := \{ g \in G : \mathbb{1} \le g \}$.

Then $S_\leq$ is an invariant subsemigroup of G with trivial group of units H(S) = $S \cap S^{-1}$.

b) Conversely, if S is an invariant submonoid of G with $H(S) = \{ \mathbb{1} \}$ then the prescription $x \leq y \Leftrightarrow y \in xS$ defines an invariant order on G.

Definition 7. An invariant order $\leq$ on G is said to be *continuous* if the semigroup $S_\leq$ is closed and topologically generated by every neighborhood of $\mathbb{1}$ in S.

The following proposition builds the bridge to Lie semigroups.

Proposition 8. A connected Lie group G admits a continuous invariant order $\leq$ if and only if its Lie algebra L(G) contains a pointed invariant cone W which is global in G.

Now we formulate the main result on the existence of continuous invariant orders, it provides a characterization in terms of properties of the Lie algebra of those simply connected Lie groups which admit a continuous invariant order.

Theorem 9. (Existence of invariant orders) A simply connected Lie group G admits a continuous invariant order if and only if its Lie algebra L(G) contains a pointed invariant cone.

The proof of this theorem rests on the following two results and on the theory of invariant cones in [HHL89, III]. The complete proof of all theorems mentioned in this article can be found in [Ne90b].

Theorem 10. (The Reduction Theorem) Let G be a simply connected Lie group, suppose that L(G) is the semidirect product of a nilpotent ideal N and a subalgebra A and $W \subset L(G)$ is an invariant wedge. Then Wis global in G if

$$H\big(L(\langle \exp W \rangle \exp N) \big) \subset N.$$

Theorem 11. (The simple case) Let G be a simply connected simple Lie group and suppose that L(G) is a hermitean simple Lie algebra. Then L(G) contains a pointed generating invariant wedge which is global in G. ([Ol82, Vin80])

Y. M. Gichev has proved the Existence Theorem for solvable simply connected Lie groups in [Gi89]. If G is solvable and simply connected then all pointed invariant cones W in L(G) are global in G. This is false in general if G is a simple simply connected Lie group ([Ol82]) but it is possible to obtain sufficient conditions for such wedges to be global ([Ne90a]).

The following theorem gives more information about the global cone whose existence is guaranteed by Theorem 9.

Theorem 12. Let G be a simply connected Lie group and $W \subset L(G)$ a pointed generating invariant wedge. Then there exists a generating invariant wedge $V \subset W$ which is global in G.

We close this article with the solution of the controllability problem for invariant cones in the simply connected case.

Theorem 13. (The Controllability Theorem) Let G be a simply connected Lie group such that L(G) contains pointed generating invariant cones. Then there exists a nilpotent ideal $N \subset L(G)$, a Levi algebra S and an Iwasawa decomposition $S = K + T$, where K is compact and T is solvable, such that a pointed generating invariant cone $W \subset L(G)$ is controllable in G if and only if

$$W^* \cap N^\perp \cap T^\perp \cap [K,K]^\perp = \{ 0 \}.$$

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