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A K-theoretic approach to Chern-Cheeger-Simons invariants


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Nous construisons une application de la K-théorie multiplicative définie par Karoubi vers la cohomologie impaire à coefficients $\mathbb{C}^*$ sur une variété différentielle ce qui permet d'associer à tout fibré vectoriel complexe plat là-dessus des classes caractéristiques analogues aux classes étudiées par Chern, Cheeger et Simons.

1. Preliminaries. This paper is an extended version of [7] where all the proofs were suppressed. We construct a natural mapping from the multiplicative K-theory due to Karoubi [4] to the odd cohomology with coefficients $\mathbb{C}^*$ on a differentiable manifold $X$ which allows us to associate to any flat complex vector bundle $E$ on $X$ characteristic classes $\tilde{C}_k(E) \in H^{2k-1}_{dR}(X; \mathbb{C}^*)$ analogous to the classes studied by Chern, Cheeger, and Simons [1, 2].

Let $X$ be a differentiable manifold, $E$ a complex vector bundle on $X$, $D$ a connection on $E$, and $R$ the associated curvature. The differentiable Chern characters

$$\text{ch}_k^{(d)}(E, D) = \left(\frac{i}{2\pi}\right)^k \frac{1}{k!} \text{Trace}(R^k)$$
define de Rham cohomology classes $\text{Ch}_k^{d}(E) \in H_{dR}^{2k}(X)$, $k = 1, 2, 3, \ldots$, which coincide by the de Rham isomorphism with the "topological" [6] Chern characters $\text{Ch}_k^{(t)}(E) \in H^*_{\text{sg}}(X; \mathbb{Q})$ in the singular cohomology. Moreover, the integral Chern classes $C_k(E) = [c_k(E)]$ can be expressed as universal polynomials $M_k$ (inverses of the Newton polynomials) with rational coefficients of the Chern characters.

We now briefly recall the definition of the multiplicative $K$-theory $\mathcal{K}(X)$ of $X$ (associated to the trivial filtration of the de Rham complex) as defined by Karoubi in [4]. A multiplicative fibre bundle is a triplet $\xi = (E, D, \omega)$ where $\omega$ is a graded odd differential form, $\omega \in \Omega^{\text{odd}}(X)$, whose boundary is the reduced geometric Chern character, $d\omega = \text{ch}(E, D) = \sum_{k=1}^{\infty} \text{ch}_k(E, D)$. Two multiplicative fibre bundles $\xi = (E, D, \omega)$ and $\xi' = (E', D', \omega')$ are said to be equivalent if there exists an isomorphism $\sigma : E \rightarrow E'$ such that

$$\omega' - \omega = \text{C-S}(D, D')$$

where C-S stands for the canonical graded odd Chern-Simons transgression form [2].

Multiplicative $K$-theory inserts into the exact sequence

$$(1) \quad K^\text{top}_1(X) \xrightarrow{\sigma_1} \bigoplus_{r=1}^{\infty} H^{2r-1}_{dR}(X) \xrightarrow{\partial} \mathcal{K}(X) \xrightarrow{\text{u}} K^\text{top}(X) \xrightarrow{\sigma} \bigoplus_{r=1}^{\infty} H^r_{dR}(X).$$

Here $K^\text{top}_1(X) = [X, GL(C)]$ or the group of homotopy classes of differentiable maps from $X$ to $GL(C)$, and $K^\text{top}(X)$ is the Grothendieck - Atiyah - Hirzebruch group of $X$ [3].
In the exact sequence (1), $\sigma$ is induced by the differentiable Chern character and $u$ is the forgetful homomorphism. The homomorphism $\partial$ is defined by associating to an odd closed differential form $\omega$ the difference of two multiplicative vector bundles $\partial[\omega] = [T, d, \omega] - [T, d, 0]$ where $T$ denotes a trivial vector bundle endowed with the trivial connection $d$.

Finally, if $\alpha : X \to GL(C)$ is differentiable, $\sigma_1(\alpha)$ is represented by the closed differential form

$$\sum_{r=1}^{\infty} \frac{i^{3r-2}(r-1)!}{(2\pi)^r(2r-1)!} \text{Trace}(\alpha^{-1}d\alpha)^{2r-1}.$$ 

2. Chern-Cheeger-Simons invariant. Our aim is to combine the exact sequence (1) with the Bockstein exact sequence associated to the exponential exact sequence $0 \to \mathbb{Z} \to C \to C^* \to 0$ in order to find a commutative diagram

$$\begin{align*}
K_{\text{top}}^1(X) &\xrightarrow{\sigma_1} \bigoplus_{r=1}^{\infty} H_{dR}^{2r-1}(X) \xrightarrow{\partial} K(X) \xrightarrow{u} K^\text{top}(X) \xrightarrow{\sigma} \bigoplus_{r=1}^{\infty} H_{dR}^{2r}(X) \\
H_{*}^{2k-1}(X; \mathbb{Z}) &\xrightarrow{q_k} H_{*}^{2k-1}(X) \xrightarrow{p_k} H_{*}^{2k-1}(X; C^*) \xrightarrow{\beta_k} H_{*}^{2k}(X; \mathbb{Z}) \xrightarrow{M_k} H_{*}^{2k}(X)
\end{align*}$$

Here $q_k$ is minus the suspension of $C_k^{(i)}$, and $p_k$ is the obvious projection multiplied by the coefficient of the homogenous term of $M_k$, that is to say $(-1)^{k-1}(k-1)!$. The natural map $\bar{C}_k$ has the property that one recovers $C_k^{(i)}$ when composing it with the Bockstein homomorphism $\beta_k$.

The definition of $\bar{C}_k$ necessitates a universal construction. Any vector bundle $E$ of
rank $n$ over $X$ with connection $D$ can be pulled back via some connection preserving map $h : X \to \tilde{X}$ from the tautological bundle with universal connection $\tilde{D}$ over the Grassmannian manifold $\tilde{X} = G_n(C^m)$ where $m$ is large enough. The map $h$ is unique up to homotopy. The differentiable and topological Chern classes are the same; in particular, on $\tilde{X}$ there exists some singular $(2k - 1)$-cochain $\theta_{2k-1}$ such that

$$(2) \quad \int_\gamma c_k^d(\tilde{E}, \tilde{D}) - c_k^d(E, D) = \theta_{2k-1}(\partial \gamma)$$

for every singular chain $\gamma \in \Sigma_{2k-1}(\tilde{X})$. We now pull back each term of (2) via $h$. By the naturality of Chern classes, we find

$$h^*(c_k^d(\tilde{E}, \tilde{D})) = c_k^d(E, D) = M_k(\omega_1, \omega_2, \omega_3, ..., \omega_{2k-1}) = d\psi_{2k-1}$$

where the $(2k-1)$-form $\psi_{2k-1}$ on $X$ is defined modulo an exact form. On the other hand, topological Chern classes have integral periods so that a cocycle $\check{c}_k(E, D, \omega)$ will be associated to each multiplicative fibre bundle $(E, D, \omega)$ by the following definition. For a singular $(2k-1)$-chain $\lambda$ on $X$, $\lambda \in \Sigma_{2k-1}(X)$ set

$$\check{c}_k(E, D, \omega) = \int_\lambda \psi_{2k-1} - \theta_{2k-1}(h \circ \lambda) \mod \mathbb{Z}.$$ 

Indeed, $\check{c}_k$ is co-closed:

$$\delta \check{c}_k(E, D, \omega)(\lambda) = \int_\lambda d\psi_{2k-1} - d\theta_{2k-1}(h \circ \lambda) \mod \mathbb{Z}$$

$$= c_k^d(\tilde{E})(h \circ \lambda) \mod \mathbb{Z}$$

$$= 0 \mod \mathbb{Z}.$$
The cohomology class of $\tilde{c}_k(E, D, \omega)$, to be denoted by $\tilde{C}_k(E, D, \omega)$, is independent of the choice of $\theta_{2k-1}$ as, by Bott periodicity, the universal Grassmannian has no odd cohomology. By standard homotopy arguments, $\tilde{C}_k(E, D, \omega)$ is also seen to be independent of the choices of $h$ and $\psi_{2k-1}$. A similar homotopy argument also applies to the proof that $\tilde{C}_k(E, D, \omega)$ is independent of the choice of the representative of the multiplicative K-theory class, once one recalls from [4] the following alternative characterization of $\mathcal{K}(X)$: two multiplicative vector bundles $\xi_i = (E^i, D^i, \omega^i)$, $i = 0, 1$, are equivalent if and only if there exists a homotopy $(D_t, \omega_t)$ such that $D_0 = D^0$, $\omega_0 = \omega^0$, $D_1 = \alpha^*(D^1)$, $\omega_1 = \omega^1$ for an isomorphism $\alpha : E^0 \to E^1$.

Hence we have a natural well-defined map

$$\tilde{C}_k : \mathcal{K}(X) \to H^{2k-1}_*(X; \mathbb{C}^*).$$

It is appropriate to call the resulting characteristic class the Chern-Cheeger-Simons invariant as our construction is analogous to theirs [1], [2].

3. The commutative diagram. We now establish the commutativity of the above diagram. This was the main result announced in [7]. Let us number the squares of the diagram by I-IV from left to right.

Square I: In the de Rham cohomology the arrow $K^{\text{top}}_1(X) \to H^{2k-1}(X)$ is described by integration with respect to the suspension parameter $-1 \leq t \leq 1$. But to compute
in terms of differential forms, one needs to deal with the differentiable Chern class and, first of all, to endow with a connection the vector bundle \( \pi : E \to \Sigma X \) determined by \( \alpha : X \to GL(C) \) over the suspension \( \Sigma X \).

For this, let \( T = \{ U, V \} \) be a trivializing open cover of \( \Sigma X \) such that \( U \) (resp. \( V \)) is the contractible open set obtained by puncturing the suspension double cone at the south pole \( p_- (t = -1) \), resp. north pole \( p_+ (t = 1) \), i.e. \( U = \Sigma X \backslash \{ p_- \} \), \( V = \Sigma X \backslash \{ p_+ \} \). Let \( \{ \mu, \nu \} \) be a partition of the unity subordinate to \( T = \{ U, V \} \) such that \( \mu(p_+) = 1, \nu|_{[-1,1] \times X} \equiv 1 \).

Construct a connection \( D \) of the vector bundle \( \pi : E \to \Sigma X \) by choosing for the local connection 1-forms associated to the trivialization \( T \)

\[
\omega_U(x) = \nu(x) g^{-1}_{\nu U}(x) dg_{\nu U}(x) = \nu(x) \alpha^{-1}(x) da(x) , \ x \in U
\]
\[
\omega_V(x) = \mu(x) g^{-1}_{\mu V}(x) dg_{\mu V}(x) = -\mu(x) da(x) \alpha^{-1}(x) , \ x \in V
\]

Then

\[
g^{-1}_{\nu V} dg_{\nu V} + g^{-1}_{\mu V} \omega_U g_{\mu V} = \alpha da^{-1} + \alpha \nu \alpha^{-1} da \alpha^{-1}
\]
\[
= (-1 + \nu) da \alpha^{-1}
\]
\[
= - \mu da \alpha^{-1}
\]
\[
= \omega_V
\]

as wanted. The associated curvature 2-forms are

\[
\Omega_U = d\omega_U + \omega_U \wedge \omega_U
\]
\[
= d \nu \alpha^{-1} da - \nu (\alpha^{-1} da) ^2 + \nu ^2 (\alpha^{-1} da) ^2
\]
\[
= d \nu \alpha^{-1} da - \mu \nu (\alpha^{-1} da) ^2
\]
and

\[ \Omega_N = -d\mu.d.ua^{-1} - \nu(\alpha.a^{-1})^2. \]

To integrate \( c_k^{(d)}(E, D) \) from \(-1\) to \(+1\) with respect to \( t \) we note that the terms of bidegree \((m, 2k - m), m = 2, 3, 4, \ldots, 2k, \) of the \( k \)’th power of the curvature trivially vanish; the term of bidegree \((0, 2k)\) or \((-1)^k(\mu \nu)(\alpha.a^{-1})^{2k-1} \) is traceless, and there only remains the term of bidegree \((1, 2k - 1)\) or \((-1)^{k-1}k(\mu \nu)^{k-1}\nu(\alpha.a^{-1})^{2k-1} \). Consequently, all the products of Chern characters vanish in the universal polynomials \( M_k \), and there only remains the term in \( \text{Ch}_k^{(d)}(E, D) \) whose coefficient is \((-1)^{k-1}(k - 1)!\). We thus compute that

\[
- \int_{-1}^{1} c_k^{(d)}(E, D) = - \left( \frac{i}{2\pi} \right)^k \frac{1}{k!} \int_{-1}^{1} \text{Tr}(\Omega^k)
\]

\[
= - \left( \frac{i}{2\pi} \right)^k \frac{1}{k!} \text{Tr}(\alpha^{-1}d\alpha)^{2k-1}(-1)^{k-1}k \int_{-1}^{1} (\mu \nu)^{k-1} d\mu
\]

\[
= \frac{i^{3k-2}}{(2\pi)^k} \frac{1}{(k-1)!} \text{Tr}(\alpha^{-1}d\alpha)^{2k-1} \int_{0}^{1} (\mu(t) - \mu(t)^2)^{k-1} d\mu(t)
\]

\[
= \frac{i^{3k-2}}{(2\pi)^k} \frac{1}{(k-1)!} \text{Tr}(\alpha^{-1}d\alpha)^{2k-1} \int_{0}^{1} (t - t^2)^{k-1} dt
\]

\[
= \frac{i^{3k-2}}{(2\pi)^k} \frac{(k-1)!}{(2k-1)!} \text{Tr}(\alpha^{-1}d\alpha)^{2k-1}
\]

and

\[
- \int_{-1}^{1} c_k^{(d)}(E, D) = (-1)^{k-1}(k - 1)! \left( - \int_{-1}^{1} \text{ch}_k^{(d)}(E, D) \right)
\]

that is, the representative of \((p_k \circ \sigma_1)[\alpha] \).

**Square II:** For \([\omega] \in \bigoplus_{r=1}^{\infty} H^{2r-1}_{dR}(X)\) we find

\[
(\bar{\mathcal{C}}_k \circ \partial)[\omega] = \bar{\mathcal{C}}_k[T, d, \omega] - \bar{\mathcal{C}}_k[T, d, 0] = [f] \in H^{2k-1}_{d}(X; \mathbb{C}^*)
\]
where modulo $\mathbb{Z}$

$$f(\lambda) = \int_\lambda \psi_{2k-1} - \theta_{2k-1}(h \circ \gamma) + \theta_{2k-1}(h \circ \gamma) = \int_\lambda \psi_{2k-1}.$$ 

Now, for $\omega$ closed, we see that only the homogenous term will survive in the definition of $\psi_{2k-1}$, that is,

$$\psi_{2k-1} = (-1)^{k-1}(k-1)! \omega_{2k-1}.$$ 

But

$$f(\lambda) = (-1)^{k-1} k! \int_\lambda \omega_{2k-1}, \quad \lambda \in \Sigma_{2k-1}(X)$$

is exactly the cochain needed for the square II to be commutative.

**Square III:** One only needs to recall the definition of the Bockstein homomorphism.

**Square IV:** It is trivial.

We have thus established the main theorem of [7].

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**References**