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In: Jarolím Bureš and Vladimír Souček (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1993. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 30. pp. [125]–131.

Persistent URL: http://dml.cz/dmlcz/701511

FORMAL COMPUTATIONS IN LOW-DIMENSIONAL TOPOLOGY: LINKS AND GROUP PRESENTATIONS

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> ***...uzel...pokládají moderní matematikové za jedno z největších tajemství. Je možný jen při lichém počtu dimenzí, nemožný na úrovni a v prostorech párových, o čtyřech, šesti, dvou dimenzích, a topologisté studovali úspešně jen nejjednodušší uzly.****

Presented to the 11th Winter School "Geometry and Physics", January 1991, Srní, Czechoslovakia, as the last part of the series of three lectures "Formal computations in low-dimensional topology" given by B. Berceanu, Şt. Papadima and the author.

The aim of this short note is to show how the deformation theory for the fundamental group, developed in [MP], can be used for the computation of the associated graded $\mathrm{gr}^*\pi_1$ of the fundamental group of link complements and spaces, associated with group presentations.

Let us open this introductory paragraph with some remarks on 2-*skeletal spaces*, where by a 2-skeletal space is meant a (connected) space S satisfying $H^{\geq 3}(S;\mathbf{Q}) = 0$. Let $m = \dim H^1(S;\mathbf{Q})$ and $l = \dim H^2(S;\mathbf{Q})$, we always suppose that these dimensions are finite. Let X be an m-dimensional rational vector space, considered as a graded vector space concentrated in degree 0. Similarly, let Y be and l-dimensional graded vector space concentrated in degree 1.

Let $\mathsf{L}(X,Y)$ be the free Lie algebra on $X \oplus Y$. It is naturally graded as $\mathsf{L}(X,Y) = \bigoplus_{i\geq 1, j\geq 0} \mathsf{L}^i_j(X,Y)$, where i = the lenght, and the lower grading is induced from the grading on $X \oplus Y$. Put $\widehat{\mathsf{L}}(X,Y) = \prod_{i\geq 1} \mathsf{L}^i(X,Y)$. This is a graded Lie algebra filtered by $F_n\widehat{\mathsf{L}}(X,Y) = \{(\lambda_1,\lambda_2,\dots) \in \widehat{\mathsf{L}}(X,Y);\ \lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0\}$. Let U denote the group of unipotent automorphisms of $\widehat{\mathsf{L}}(X,Y)$ and A be the group of automorphisms of the graded vector space $X \oplus Y$. Our first result reads:

THEOREM 1. *The moduli space of models of 2-skeletal spaces S having fixed the Betti numbers $m = \dim H^1(S;\mathbf{Q})$ and $l = \dim H^2(S;\mathbf{Q})$ can be described as*

$$\mathrm{Der}^{\geq 1}_{-1}(\widehat{\mathsf{L}}(X,Y))/U \ltimes A$$

where $\mathrm{Der}^{\geq 1}_{-1}(\widehat{\mathsf{L}}(X,Y)) = \{\theta \in \mathrm{Der}(\widehat{\mathsf{L}}(X,Y));\ \theta(F_n\widehat{\mathsf{L}}_i(X,Y)) \subset F_{n+1}\widehat{\mathsf{L}}_{i-1}(X,Y)\}$.

The proof can be obtained by dualizing the Félix bigraded model, as it is constructed in [Fé]. Note that the construction there is carried under the 1-connectivity assumption, but it is easy to see that the arguments remain valid also without this restriction. As I was informed by

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* *J. Bergier, L.Pauwels: Jitro kouzelníků, Editions Gallimard, Paris 1960, Translation: Jiří Konůpek, Praha 1969*

Ştefan Papadima, there exists a preliminary version of the Félix's paper which does not impose the 1-connectivity assumption.

Suppose that we have chosen a basis $\mathbf{y} = (y_1, \dots, y_l)$ of Y and denote by $t_1, \dots, t_l$, $t_i \in \mathsf{L}^{k_i}(X)$, the "leading terms" of $\theta(y_i)$, $1 \le i \le l$. Of course, $t_1, \dots, t_l$ depend on a particular choice of $\mathbf{y}$.

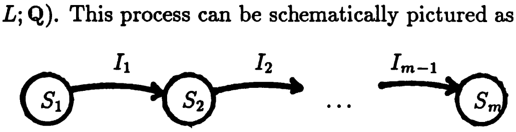
The following theorem, being a generalization of [**MP**; Theorem B'], is of a crucial importance for us.

THEOREM 2. *Let S be represented by a derivation θ in the orbit space of Theorem 1. Suppose that there exists a basis $\mathbf{y} = (y_1, \dots, y_l)$ of Y such that the sequence $t_1, \dots, t_l$ of leading terms of $\theta(y_1), \dots, \theta(y_l)$ is inert in $\mathsf{L}(X)$ (in the sense of* [**HL**; Definition 2.1]*). Then*

$$gr^*\pi_1 S \otimes \mathbf{Q} \cong \mathsf{L}^*(X)/(t_1, \dots, t_l).$$

The most natural examples of 2-skeletal spaces are link complements (in $\mathbf{S}^3$) and spaces associated with group presentations. We will usually work with links for which an *order* and *orientation* of components were chosen. As for presentations, we restrict our attention to presentations of the form $\{A|R\}$, $A = \{a_1, \dots, a_m\}$, $R = \{r_1, \dots, r_m\}$ with $r_i \in [F(A), F(A)]$ for each i, $1 \le i \le n$, where $F(A)$ denotes the free group on A. The last condition means topologically that the CW-complex $X_{\{A|R\}}$, associated with this presentation, is *in the normal form* ([**FS**; page 80]).

So, let $S = \mathbf{S}^3 \setminus L$ for an m-component ordered and oriented link, or $S = X_{\{A|R\}}$ for a presentation $\{A|R\} = \{a_1, \dots, a_m | r_1, \dots, r_n\}$ as above. Then plainly $H^1(S)$ is free Abelian of rank m and $H^2(S)$ is free Abelian of rank $m-1$ (links) or n (presentations), respectively. Notice that both $H^1(S; \mathbf{Q})$ and $H^2(S; \mathbf{Q})$ always have a preferred basis which is given by the very geometrical nature of S. In the link case, choose a spanning surface S_i of the i-th component and orient it compatibly with a given orientation of this component. The duals of $S_1, \dots, S_m$ form then a basis of $H^1(\mathbf{S}^3 \setminus L; \mathbf{Q})$. Next, for each $1 \le j \le m-1$, choose an oriented segment I_j connecting the j-th and $(j+1)$-th component. The duals of these segments will then represent a basis for $H^2(\mathbf{S}^3 \setminus L; \mathbf{Q})$. This process can be schematically pictured as



The existence of a preferred basis for $S = X_{\{A|R\}}$ is clear from the construction – elements of A (resp. of R) correspond to oriented 1-cells (resp. 2-cells) of S.

The existence of a distinguished basis in $H^*(S; \mathbf{Q})$, $S = \mathbf{S}^3 \setminus L$ or $S = X_{\{A|R\}}$, enables one to be left in Theorem 1 only with the unipotent part of the automorphism group.

THEOREM 3. *The moduli space of models of spaces $S = \mathbf{S}^3 \setminus L$, where L is an (ordered and oriented) m-component link, or $S = X_{\{A|R\}}$, where $\{A|R\} = \{a_1, \dots, a_m | r_1, \dots, r_n\}$ is a group presentation with $R \subset [F(A), F(A)]$, is isomorphic to*

$$\mathrm{Der}^{\ge 1}_{-1}(\mathsf{L}(X, Y))/U,$$

where $\dim X = m$ *and* $\dim Y = m-1$ *(the case of a link) or* $\dim Y = n$ *(the case of a group presentation).*

Notice that the moduli space of the previous theorem, however most unfortunate from the geometrical point of view, has the following surprising feature: there exists an *inductive* (though possibly infinite) process of recognizing whether two derivations from $\mathrm{Der}^{\geq 1}_{-1}(\hat{\mathsf{L}}(X,Y))$ represent the same point of the moduli space or not.

The results above enable us, as we will see in the next paragraphs, to obtain–on the rational level–both [**H**; theorem on page 57] and [**L1**; Theorem 1], in a unique fashion. As we have seen in [**MP**; § 5], the integral results are, in some cases, also available.

Notice also that if the inertia condition of Theorem 2 is satisfied, the Hilbert series of $\mathrm{gr}^*\pi_1 S$ is determined only by $m = \dim(X)$ and by the degrees $k_1, \dots, k_l$ of the elements $t_1, \dots, t_l$.

We close this paragraph with a few comments on the homogeneous case of Theorem 2, i.e. on the case when $k_1 = k_2 = \cdots = k_l = N+1$ for a natural number $N \geq 1$. We show how in this case the elements $t_1, \dots, t_l$ can be computed using the Massey products of order $N+1$ (=the cup product if N=1). This will be of a basic importance in the next paragraphs; for links we get from this a formula computing $t_1, \dots, t_l$ via Milnor numbers, see Theorem 4 of the following paragraph. At first, it can be proved, using for example the computation developed in [**T** ;Chapitre V], that $k_i > N$ for all $1 \leq i \leq l$ means the vanishing of all (rational) Massey products of order $\leq N$. It is then well-known that the Massey product $\langle\ ,\dots,\ \rangle_{N+1}$ of degree $N+1$ is uniquely defined on the whole $\bigotimes^{N+1} H^*(S;\mathbf{Q})$, especially, it defines a map $\langle\ ,\dots,\ \rangle_{N+1}\colon \bigotimes^{N+1} H^1(S;\mathbf{Q}) \to H^2(S,\mathbf{Q})$ [**F**; 6.2]. Moreover, we have an identification of the space X of Theorem 1 and of the dual of $H^1(S;\mathbf{Q})$. Similarly, $Y \cong$ the dual of $H^2(S;\mathbf{Q})$. So, dualizing the map $\langle\ ,\dots,\ \rangle_{N+1}$ we obtain a map $\Omega\colon Y \to \bigotimes^{N+1} X$. It is also possible to show that the map $\langle\ ,\dots,\ \rangle_{N+1}$ is zero on the decomposables of the shuffle product in $\bigotimes^{N+1} H^1(S;\mathbf{Q})$ which, by the main result of [**R**], implies that, in fact, $\mathrm{Im}(\Omega) \subset \mathsf{L}(X) \subset \bigotimes X$. The element t_i is then equal (modulo a sign convention) to $\Omega(y_i) \in \mathsf{L}^{N+1}(X)$. Notice also that in the homogeneous case the inertia of $t_1, \dots, t_l$ does not depend on a particular choice of $\mathbf{y} = (y_1, \dots, y_l)$.

1. Link Examples. Let L be an m-component link. It can be shown that the sequence $(\overline{R}_1, \dots, \overline{R}_l)$ constructed in [**H**; page 57] using the intersection calculus of a suitable "defining system", corresponds to the sequence $(t_1, \dots, t_l)$ arising from the representation of $S = \mathbf{S}^3 \setminus L$ in our moduli space as in Theorem 2 (under a suitable choice of a basis $\mathbf{y}$). We see that Theorem 2 together with the characterization of the independence criterion of [**L1**; Theorem 1] via the inertia given in [**A1**; Theorem 1.5], gives a rational version of [**H**; theorem on page 57]. The integral form can be then obtained using the similar trick as in [**MP**; § 5]. The following theorem relates the ideal $(t_1, \dots, t_l)$ (where $l = m-1$; we are in the link case), with the Milnor numbers structure of the link L.

THEOREM 4. *Suppose that all Milnor numbers* $\overline{\mu}$ *of* L *of orders* $\leqslant N$ *are trivial. For* $1 \leqslant i \leqslant m-1$

denote

$$s_i = \sum_{a>i\geq b,I} \{\overline{\mu}(aIb)x_a x_I x_b - \overline{\mu}(bIa)x_b x_I x_a\} \in \mathsf{T}^{N+1}(X),$$

where the summation is taken over all $(N-1)$*-tuples* $I = (i_1, \dots, i_{N-1})$, $1 \leq i_j \leq m$, $(x_1, \dots, x_m)$ *is some basis of* X *and* x_I *abbreviates* $x_{i_1} \otimes \cdots \otimes x_{i_{N-1}} \in \mathsf{T}^{N-1}(X)$. *Then* $s_i \in \mathsf{L}^{N+1}(X) \subset \mathsf{T}^{N+1}(X)$ *and*

$$gr^*(\pi_1(\mathsf{S}^3 \setminus L)) \otimes \mathbf{Q} \cong \mathsf{L}^*(X)/(s_1, \dots, s_{m-1})$$

provided $s_1, \dots, s_{m-1}$ *is inert in* $\mathsf{L}(X)$.

The proof is based on the formula of [P] relating Milnor numbers and Massey products (see also [F; 6.4.2] and the results of the previous paragraph. These results enable us to show that s_i's are "initial terms" for some (in fact canonical) choice of $\mathbf{y}$.

The vanishing assumption on the Milnor numbers is always satisfied with $N = 1$. In this case the elements $(s_1, \dots, s_{m-1})$ can be easily computed as $\partial(y_i)$, $1 \leq i \leq m-1$, where $\partial \colon Y \to \mathsf{L}^2(X)$ is simply the dual of the cup product multiplication. As it is proved in [L2] (see also [MP; Theorem C] the sequence $(s_1, \dots, s_{m-1})$ is inert if and only if the linking diagram D of L is connected.

LINKS OF TWO CIRCLES. The fact that the one-element sequence (s_1) is inert if and only if $s_1 \neq 0$, together with Theorem 3, gives rise to the following corollary.

COROLLARY 5. *Let* L *be a two-component link. The group* $gr^*\pi_1(\mathsf{S}^3 \setminus L) \otimes \mathbf{Q}$ *can be computed as follows:*

- *If there exists some* $N \geq 1$ *such that all Milnor numbers of order* $\leq N$ *are zero, but there exists a nonzero Milnor number of order* $N+1$, *then*

$$gr^*\pi_1(\mathsf{S}^3 \setminus L) \otimes \mathbf{Q} \cong \mathsf{L}^*(x_1, x_2)/(s),$$

where

$$s = \sum_I \{\overline{\mu}(2I1)x_2 x_I x_1 - \overline{\mu}(1I2)x_2 x_I x_1\},$$

and the summation is taken over all $I = (i_1, \dots, i_{n-1})$, $i_j = 1$ *or* 2.

- *If all Milnor numbers are zero, then*

$$gr^*\pi_1(\mathsf{S}^3 \setminus L) \otimes \mathbf{Q} \cong \mathsf{L}^*(x_1, x_2).$$

Let us discuss the conclusion of the first part of Corollary 5 for small values of N.

$\boxed{N=1}$ This means exactly that the cup product is nontrivial, $\overline{\mu}(12) = l_{12}$ (the linking number), $s = l_{12}[x_1, x_2]$ and

$$gr^*\pi_1(\mathsf{S}^3 \setminus L) \otimes \mathbf{Q} \cong \mathsf{L}^*(x_1, x_2)/([x_1, x_2]).$$

$\boxed{N=2}$ The symmetry properties of Milnor numbers imply that this is impossible, see [C; Appendix B].

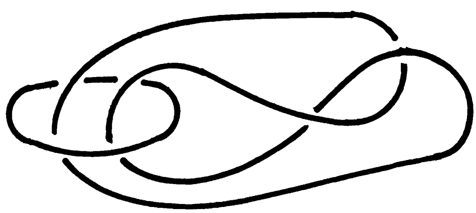
$\boxed{N=3}$ The following table shows the only possibly nontrivial Milnor numbers and relations among them (see again [**C**; Appendix B]):

$$\overline{\mu}(1122)=\overline{\mu}(2112)=\overline{\mu}(1221)=\overline{\mu}(2211),\ \overline{\mu}(1212)=\overline{\mu}(2121)=-2\overline{\mu}(1122).$$

Denoting for short $\overline{\mu}(1122)$ by A, we obtain

$$s=-2Ax_2x_1x_2x_1+2Ax_1x_2x_1x_2+Ax_2x_2x_1x_1-Ax_1x_1x_2x_2=A[x_1,[x_2,[x_1,x_2]]].$$

Therefore $\mathrm{gr}^*\pi_1(\mathbf{S}^3\setminus L)\otimes\mathbf{Q}$ is always isomorphic with $\mathsf{L}^*(x_1,x_2)/([x_1,[x_2,[x_1,x_2]]])$.
An example: The Whitehead link:



Links of Three Circles. The substantial difference from the previous case of two-component links is that we must verify the inertia condition for the sequence (s_1,s_2), which is generally a very difficult task. Nevertheless in some special cases the combinatorial criterion [**A2**; § 3] is applicable. Let us discuss Theorem 4 at least for small values of N.

$\boxed{N=1}$ This is the case of a nontrivial cup product multiplication. The sequence (s_1,s_2) is inert if and only if the linking diagram D is connected ([**L2**],[**MP**; Theorem C]). There still remain unpleasant cases as those having $\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\qquad\bullet$ as its linking diagram, but even these cases can be sometimes managed.

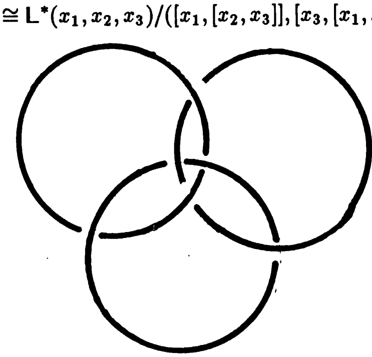
$\boxed{N=2}$ The only possibly nontrivial Milnor numbers and relations among them are (see [**C**; Appendix B]):

$$\overline{\mu}(123)=\overline{\mu}(231)=\overline{\mu}(312)=-\overline{\mu}(213)=-\overline{\mu}(132)=-\overline{\mu}(321).$$

Denoting $B=\overline{\mu}(123)$, Theorem 4 gives: $s_1=-B[x_1,[x_2,x_3]]$, $s_2=B[x_3,[x_1,x_2]]$. This sequence is always inert (see [**L1**; Example 3] and remember $B\neq 0$), hence

$$\mathrm{gr}^*\pi_1(\mathbf{S}^3\setminus L)\otimes\mathbf{Q}\cong\mathsf{L}^*(x_1,x_2,x_3)/([x_1,[x_2,x_3]],[x_3,[x_1,x_2]]).$$

An example: Borromean rings:



2. Group Presentations. Let us open this last paragraph with a brief exposition of the main result of [**L1**]. Let $\{A|R\} = \{a_1, \dots, a_m | r_1, \dots, r_n\}$ be a group presentation. Let $F(A)$ be the free group on A and $F(A) = F_1 \supset F_2 \supset \dots$ its lower central series. Denote $m_j = \sup\{k; r_j \in F_k\}$, the weight of r_j. Let ρ_j denote the image of r_j in $\mathrm{gr}^{m_j} F(A) = F_{m_j}/F_{m_j+1}$. Finally, denote by J the ideal generated by $\{\rho_1, \dots, \rho_n\}$ in the Lie algebra $\mathrm{gr}^* F(A) \cong \mathsf{L}^*(a_1, \dots, a_m)$ and let $G = F(A)/(r_1, \dots, r_n)$ be the associated group. The main result of [**L1**] then reads:

Suppose that

- *$gr^* F(A)/J$ is a free $\mathbb{Z}$-module and that*
- *$J/[J,J]$ is a free $gr^* F(A)/J$-module via the adjoint action, with the images $\overline{\rho}_j$ of ρ_j in $J/[J,J]$, $1 \le j \le m$, forming a basis.*

Then $gr^* G \cong gr^* F(A)/J$ as graded Lie algebras.

There is a striking resemblance of the Labute's condition as it is formulated above and the inertia condition as it is given in [**HL**; Theorem 3.3]. Anick in [**A1**] actually proves that the Labute's condition is (essentially) equivalent with the inertia of $(\rho_1, \dots, \rho_m)$ in the free Lie algebra $\mathrm{gr}^* F(A)$. We aim to explain here what topology lies behind this stunning coincidence.

Let $X = X_{\{A|R\}}$ be the space associated with our group presentation. Then $\pi_1(X) \cong G$, hence $\mathrm{gr}^* \pi_1 X \cong \mathrm{gr}^* G$ and the whole concept of our "deformation theory for the fundamental group" is applicable. The second piece of our mosaic is the fact that the whole information on the Massey products in X is contained in the initial terms of the relators, $\rho_1, \dots, \rho_n$, at least in the homogeneous case $m_1 = \dots = m_n$. This enables us to identify (over the rationals) the elements $\rho_1, \dots, \rho_n \in \mathrm{gr}^* F(A)$ and the "initial terms" $t_1, \dots, t_n \in \mathsf{L}^*(X) \cong_{\mathbb{Q}} \mathrm{gr}^* F(A)$. The topological by-pass is now evident.

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