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COMPARISON OF DIRAC OPERATORS ON MANIFOLDS WITH BOUNDARY

Ulrich Bunke

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Abstract

We introduce boundary conditions for Dirac operators D giving selfadjoint extensions such that the Hamiltonians $H := D^2$ define elliptic operators . Assuming bounded geometry we estimate the trace of the difference of two heat operators e^{-tH} associated to a pair of Dirac operators coinciding on cocompact sets.

Contents

1 Introduction

The aim of the present paper is to extend some results obtained in [3] to the case of manifolds with boundary. Let D be the Dirac operator associated to a Clifford bundle E over a Riemannian manifold (M,g). It is an elliptic formally selfadjoint differential operator of first order. If M is complete, D is known to be essentially selfadjoint on the domain $C_0^\infty(M,E)$ giving a supercharge (see [2],[7]) on the Hilbert space $L^2(M,E)$. If M has a boundary and is hence noncomplete this fails and one has to require suitable boundary conditions. In order to have good estimates on the kernel of the heat operator e^{-tH} the Hamiltonian $H := D^2$ has to be elliptic at the boundary too . This means that some boundary value problem constructed from H and dom$H$ must satisfy the condition of Lopatiskij-Ŝhapiro [15] , [14]. That makes finding such a boundary condition more sophisticated. In section 3 we discuss global boundary conditions (as e.g. introduced in [1]) which exist in any case and also local boundary conditions which require a further structure of the Clifford bundle at the boundary as considered in [13].

The main result in [3] was the trace class estimate of the difference of heat operators associated to Dirac operators on manifolds and Clifford bundles coinciding outside of compact sets. Such results are important e.g. for applying scattering theory to compare the absolute continuous spectrum, computing asymptotic constants [5] e.t.c. In section 5 we extend this result to manifolds with compact boundaries. The technique of the proof is slightly different from that used in [3]. In the present article we employ finite propagation speed methods [6],[7] and have to assume bounded geometry of the manifolds and bundles. In section 4 we collect some results on local Sobolev embedding theorems and elliptic regularity under the bounded geometry assumption. These results seem to be folklore but we do not know references.

Complete proofs including the results of section 4 will appear in [4].

2 Generalized Dirac operators

Let E be a Hermitian vector bundle with connection ∇ over a Riemannian manifold (M, g) of dimension n. We assume that E has the structure of a Clifford bundle. Thus there is a multiplication $TM \otimes E \to E$ denoted by $\circ$ and an involution τ satisfying:

$$\tau^* = \tau, \qquad \tau^2 = 1 \tag{1}$$

$$\nabla\tau := [\nabla, \tau] = 0 \tag{2}$$

$$[\tau, X\circ]_+ = 0 \qquad \forall X \in T_xM, \qquad x \in M \tag{3}$$

$$(X\circ)^* = -X\circ \qquad \forall X \in T_xM, \qquad x \in M \tag{4}$$

$$X \circ X\circ = -\|X\|^2 \qquad \forall X \in T_xM, \qquad x \in M \tag{5}$$

Here $*$ denotes the adjoint with respect to the Hermitian metric in E. For example $\bigwedge^* T^*M$ or the spinor bundle over a spin manifold have a Clifford bundle structure [11]. More examples can be produced using tensor product and direct sum. The Dirac operator D is associated with the Clifford bundle structure. It is given in terms of a local orthonormal frame $\{X_i\}_{i=1}^n$ of TM by

$$D = \sum_{i=1}^n X_i \circ \nabla_{X_i} \tag{6}$$

The Dirac operator is an elliptic formally selfadjoint differential operator of first order.

3 Boundary conditions

We are going to consider selfadjoint extensions of D acting on the Hilbert space $L^2(M, E)$ by specifying the domain dom$D$ of D. If M is complete we take dom$D := C_0^\infty(M, E)$ where D is essentially selfadjoint. If M has a boundary we have to choose a suitable subspace of $C_0^\infty(M, E)$ defined by conditions at the boundary and then have to take the closure. We will not indicate the domain in our notation D since it is clear from the context. We assume that M has a compact boundary ∂M and is complete outside the boundary. By this we mean that $M \setminus \partial M$ is an open subset of a complete Riemannian manifold such that the closure in that manifold is M. First we state the partial integration formula for the Dirac operator [8].

Lemma 3.1 *For $\phi, \psi \in C_0^\infty(M, E)$*

$$(D\psi, \phi) = (\psi, D\phi) + \int_{\partial M} < \mathrm{n} \circ \psi, \phi > \tag{7}$$

Here $(\, , \,)$ is the L^2 scalar product, $< , >$ is the Hermitian metric and n is the inner unit normal vector field on ∂M.

From this lemma it is clear that in order to obtain a selfadjoint extension of D one has to introduce boundary conditions to make the last term vanish. In addition to this we require that the boundary problem of finding ϕ for given ψ such that

$$H^m \phi = \psi, \qquad \phi \in \mathrm{dom} H^m \tag{8}$$

satisfies the condition of Lopatinskij-Ŝapiro [14] ,[15] for all $m \in \mathrm{N}$ where H is the Hamiltonian of the extension of D. Further more we want to satisfy

$$\tau\, \mathrm{dom} D = \mathrm{dom} D, \qquad [D, \tau] = 0. \tag{9}$$

We are going to describe two such boundary conditions. A global one can be defined without further structure while the local boundary condition requires a section $\epsilon \in \Gamma(End(E_{|\partial M}))$ satisfying

$$\epsilon^2 = 1, \qquad \epsilon^* = 1, \tag{10}$$

$$[\tau, \epsilon] = 0, \qquad [\mathbf{n}\circ, \epsilon]_+ = 0 \tag{11}$$

$$[X\circ, \epsilon] = 0 \qquad \forall X \in T_x\partial M, x \in \partial M. \tag{12}$$

In the special case of $E = \bigwedge^* T^*M$ one can construct such ϵ out of the decomposition of forms into tangential and normal part part leading eventually to the relative and the absolute boundary conditions of [13].

We split $E = E^+ \oplus E^-$ according to the ± 1 eigenspaces of τ and $E^\pm_{|\partial M} = E^\pm_+ \oplus E^\pm_-$ with respect to the ± 1 eigenspaces of ϵ. For $\psi \in \Gamma(E)$ we denote by $\psi^\pm$ the parts of ψ in $E^\pm$ and by $\psi_\pm$ the parts in $E_\pm$ of the restriction of ψ to ∂M. The Dirac operator is represented by the matrix

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}. \tag{13}$$

We will define a domain for D^+, such that it becomes closable. Then we define D^- as the adjoint of D^+ and

$$D = \begin{pmatrix} 0 & (D^+)^* \\ \mathrm{closure} D^+ & 0 \end{pmatrix} \tag{14}$$

becomes selfadjoint. (9) will be satisfied automatically. It remains to verify (8).

Definition 3.2 *The local boundary condition is defined by giving the domain of* D^+

$$\mathrm{dom} D^+ := \{\psi \in C_0^\infty(M, E^+) \,|\, \psi_- = 0\} \tag{15}$$

For defining the global boundary condition we introduce the Dirac operator A associated to the Clifford bundle $E_{|\partial M}$ and $B := -(\tau\, \mathbf{n} \circ A + A\, \tau\, \mathbf{n}\circ)$. By a local computation using (1) ... (6) one obtains

Lemma 3.3

$$B^* = B, \tag{16}$$

$$\mathbf{n} \circ B = B\, \mathbf{n}\circ \tag{17}$$

$$\tau B = B \tau \tag{18}$$

B is an elliptic selfadjoint operator on $L^2(\partial M, E_{|\partial M})$. Let P_+ be the projection onto the space spanned by the eigenvectors of B associated to the positive eigenvalues, K be the projection onto $\ker B$ and $P_- := 1 - P_+ - K$.

Definition 3.4 *The global boundary condition is defined by giving the domain of D^+*

$$\mathrm{dom}D^+ := \{\psi \in C_0^\infty(M, E^+) \mid P_+\psi_{|\partial M} = 0\} \tag{19}$$

In the remainder of this section we verify the ellipticity condition (8) for these two boundary conditions. For this we describe the domain of H.

Lemma 3.5 *$\psi \in \mathrm{dom}H \cap C_0^\infty(M, E)$ iff $\psi \in C_0^\infty(M, E)$ and*

1. *local boundary condition*

$$\psi_+^+ = 0, \quad (D\psi^+)_+ = 0 \tag{20}$$

$$\psi_+^- = 0, \quad (D\psi^-)_+ = 0 \tag{21}$$

2. *global boundary condition*

$$P_+\psi^+_{|\partial M} = 0, \quad (P_- + K)(D^+\psi^+)_{|\partial M} = 0 \tag{22}$$

$$(P_- + K)\psi^-_{|\partial M} = 0, \quad P_+(D^-\psi^-)_{|\partial M} = 0 \tag{23}$$

This is proved using Lemma 3.1, 3.3 and 3.5 simply by employing the definition of the adjoint operator.

Proposition 3.6 *Let $m \in \mathbb{N}$. For both the local and the global boundary condition the boundary value problem finding ϕ for given ψ such that*

$$H^m\phi = \psi, \qquad \phi \in \mathrm{dom}H^m \tag{24}$$

satisfies the condition of Lopatinskij-Ŝhapiro.

Proof : The condition of Lopatinskij-Ŝapiro can be verified by local calculations with the symbols of H^m and the boundary operators corresponding to the condition $\phi \in \mathrm{dom}H^m$. We introduce normal coordinates at $x \in \partial M$ $\{x^1, \ldots, x^{n-1}, t\}$ where t is the distance to the boundary. Let $\{\xi_1, \ldots, \xi_{n-1}, \eta\}$ be the corresponding covariables. The symbol of H^m is $(\|\xi\|^2 + \eta^2)^m$. We replace η by $-\imath\frac{\partial}{\partial t}$ and hence have to find the solutions of

$$(\|\xi\|^2 - (\frac{\partial}{\partial t})^2)^m f(t) = 0 \tag{25}$$

in the Schwarz space $\mathcal{S}(\mathbb{R}_+, E_x) =: \mathcal{S}$ of functions with values in the spinor fibre E_x. Let s be the symbol of the boundary operator. We are done if we show that $s\, f(0) = 0$ implies $f \equiv 0$. The general solution of (25) in S is

$$p(t)e^{-\|\xi\|^2 t} \tag{26}$$

where p is a polynomial of order $m-1$ with coefficients in E_x.

First we consider the local boundary condition. From (20),(21) we obtain the conditions

$$\left\{\left[\|\xi\|^2 - \left(\frac{\partial}{\partial t}\right)^2\right]^l \left\{p(t)e^{-\|\xi\|^2 t}\right\}\right\}_{+\;|t=0} = 0 \tag{27}$$

$$\left\{\left[\mathrm{n}\circ\frac{\partial}{\partial t}+\sum_{i=1}^{n-1}\partial x^i\circ\imath\xi_i\circ\right]\left[\|\xi\|^2-\left(\frac{\partial}{\partial t}\right)^2\right]^l\left\{p(t)e^{-\|\xi\|^2t}\right\}\right\}_{+\,|t=0}=0 \tag{28}$$

for $l=0,\dots m-1$. By induction in l and the order of p using the commutation relations (10) ...(12) one obtains $p=0$ proving the claim for the local boundary condition.

To discuss the global boundary condition we have to calculate the symbol of the boundary operator. Let $r(\xi)$ be the positive spectral projection of the symbol of B. Then r is the symbol of P_+ while the symbol of K vanishes since K is smoothing. The conditions read off from (22),(23) are

$$r(\xi)\left\{\left[\|\xi\|^2-\left(\frac{\partial}{\partial t}\right)^2\right]^l\left\{p(t)^+e^{-\|\xi\|^2t}\right\}\right\}_{|t=0}=0 \tag{29}$$

$$(1-r(\xi))\left\{\left[\|\xi\|^2-\left(\frac{\partial}{\partial t}\right)^2\right]^l\left\{p(t)^-e^{-\|\xi\|^2t}\right\}\right\}_{|t=0}=0 \tag{30}$$

$$(1-r(\xi))\left\{\left[\mathrm{n}\circ\frac{\partial}{\partial t}+\sum_{i=1}^{n-1}\partial x^i\circ\imath\xi_i\circ\right]\left[\|\xi\|^2-\left(\frac{\partial}{\partial t}\right)^2\right]^l\left\{p(t)^+e^{-\|\xi\|^2t}\right\}\right\}_{|t=0}=0 \tag{31}$$

$$r(\xi)\left\{\left[\mathrm{n}\circ\frac{\partial}{\partial t}+\sum_{i=1}^{n-1}\partial x^i\circ\imath\xi_i\circ\right]\left[\|\xi\|^2-\left(\frac{\partial}{\partial t}\right)^2\right]^l\left\{p(t)^-e^{-\|\xi\|^2t}\right\}\right\}_{|t=0}=0 \tag{32}$$

for $l=0,\dots m-1$. Again using induction in the order of p and the relations (16) ...(18) we obtain $p=0$. This proves the assertion in the case of local boundary conditions. □

Assuming one of the above boundary conditions we can apply elliptic theory to obtain a parametrix of H^m near the boundary.

Corollary 3.7 *The operator H^m equiped with local or global boundary conditions admits a semilocal parametrix $W_m : L^2(M,E)\rightarrow W^{2,2m}_{loc}(M,E)\cap\mathrm{dom}H^m$ near the boundary where $W^{2,2m}_{loc}(M,E)$ is the local Sobolev space of order $2m$ defined with the connection ∇.*

Corollary 3.8 *For every compact $K\subset M$ containing ∂M and $l\in\mathbb{N}$ there exists an open neighbourhood U of K and $C>0$ such that for all $\psi\in\mathrm{dom}D^l$ holds*

$$\|\nabla^l\psi\|_{L^2(K,E)}\leq C\left[\,\|D^l\psi\|_{L^2(U,E)}+\|\psi\|_{L^2(U,E)}\,\right]. \tag{33}$$

Corollary 3.9 *H is the closure of the restriction of H to $\mathrm{dom}H\cap C_0^\infty(M,E)$.*

That 3.8 and 3.9 follows from 3.7 is shown in [12].

4 Analysis on manifolds of bounded geometry

In the next section we have to compare the Sobolev norm associated to the connection ∇ with the norm associated to the elliptic operator D. In general these norms are not equivalent [9]. Furthermore we need a uniform bound of local embedding constants. For this reason we introduce the notion of bounded geometry [9].

We say that a Riemannian manifold (M,g) has bounded geometry of order k if for every $x \in M$ there exists an open chart neighbourhood containing a r-ball $B(x,r)$ such that $\|g_{ij}\|_{C^{k+1}} < C$. Here g_{ij} are the coefficients of the metric in these coordinates and C, r are chosen uniformly with respect to $x \in M$. We call an atlas of such charts an atlas of bounded geometry. Analogously a Hermitian vector bundle E has bounded geometry of order k if the charts of an atlas of bounded gemetry of M are covered by trivializations of the bundle such that the connection coefficients satisfy $\|\Gamma^l_{i,k}\|_{C^k} < C$ where C is chosen uniformly on M. As it is shown in [10] the condition $\|\nabla^l R\| < C, l = 1,\ldots,k$ and a uniform lower bound of the injectivity radius is sufficient for bounded geometry of order k of M. If in addition $\|\nabla^l K\| < C, l = 1,\ldots,k$ then E has bounded geometry of order k. Here R and K are the curvatures of M and E.

For $s < r$ let $B(x,s)$ be the s-ball at x . Let $\delta(x) \in W^{2,k}(B(x,s),E)^* \otimes E_x$ be the delta distribution at x for $k > \frac{n}{2}$. Here $*$ denotes the dual.

Lemma 4.1 *If M and E have bounded geometry of order $k > \frac{n}{2}$ then there is a constant C such that for all $x \in M$*

$$\|\delta(x)\|_{W^{2,k}(B(x,s),E)^*\otimes E_x} < C \tag{34}$$

This lemma is proved by comparing the Sobolev norms induced by the connection with that given by the charts and trivialisations of bounded geometry. Of course near the boundary we have

Lemma 4.2 *For $k > \frac{n}{2}$ and $K \subset M$ compact such that $\partial M \subset U$ there exists a neigbourhood U of K and a constant C with*

$$\|\delta(x)\|_{W^{2,k}(U,E)^*\otimes E_x} < C, \quad \forall x \in U \tag{35}$$

Lemma 4.1 and 4.2 together give a uniform bound of $\delta(x)$ in the dual of the Sobolev spaces on s-balls (s fixed).

Theorem 4.3 *Let M, E have bounded geometry of order $k > 0$. Then there exists $s > 0$ and $C < \infty$ such that for all $l \leq k, \psi \in \mathrm{dom} D^l, x \in M$*

$$\|\psi\|_{W^{2,l}(B(x,s),E)} < C(\|D^l\psi\|_{L^2(M,E)} + \|\psi\|_{L^2(M,E)}) \tag{36}$$

The proof of this theorem is rather technical and will appear in [4]. Theorem 4.3 essentially states that the sobolev norm associated to the connection is equivalent to the norm associated to the elliptic operator D. In order to globalize (36) one has to employ a partition of unity with bounded derivatives up to the order k which exist under the assumptions of the theorem [9].

5 Finite propagation speed and comparison of heat kernels

In this section we compare two Dirac operators on Clifford bundles coinciding on cocompact subsets. For $i = 0,1$ let (M_i, g_i) be Riemannian manifolds of dimension n of bounded geometry of order $k > \frac{n}{2}$ and E_i be Clifford bundles on M_i of bounded geometry of order $k > \frac{n}{2}$. We assume that M_i decomposes as $K_i \cup U_i$ where K_i is compact and there exists an isometry between U_0 and U_1 covered by a bundle isomorphism $E_{0|U_0} \to E_{1|U_1}$ intertwining all structures. Let

$$\mathcal{H} = L^2(K_0,E_0) \oplus L^2(K_1,E_1) \oplus L^2(U_0,E_0). \tag{37}$$

Identifying sections over U_0 with those over U_1 there are natural embeddings $L^2(M_i, E_i) \to \mathcal{H}$. Let P_i be the projections onto these subspaces. We extend the Dirac operators to $\mathcal{H}$ by zero where they were not defined before. Let $H_i := D_i^2$ be the Hamiltonians.

Theorem 5.1 *Let $(M_0, E_0), (M_1, E_1)$ have bounded geometry of order $k > \frac{n}{2}$. Then for all $t > 0$ and $l \in \mathbb{N}$ the differences*

$$D_0^l e^{-tH_0} P_0 - D_1^l e^{-tH_1} P_1 \tag{38}$$

are of trace class.

Proof: We represent the heat kernels by the Fourier transform

$$e^{-tH} = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{\lambda^2}{4t}} e^{i\lambda D} d\lambda \tag{39}$$

where the integral converges uniformly. The operators $D_i^l e^{-tH_i}$ are smoothing and we represent the kernels $D_i^l W_i$ by

$$D_i^l W_i(t, x, y) =< \delta(x), D_i^l e^{-tH_i} \delta(y) > \tag{40}$$

The following lemma is essential.

Lemma 5.2 [7] *The operators $e^{i\lambda H_i}$ extend to all Sobolev spaces $W^{2,k}(M_i, E_i)$ and for $\psi \in W^{2,k}(M_i, E_i)$ $\operatorname{supp} e^{i\lambda H_i}\psi$ is contained in a $|\lambda|$- neighbourhood of $\operatorname{supp}\psi$.*

Let $x, y \in U$ such that $\min(\operatorname{dist}(x, K_i), \operatorname{dist}(y, K_i)) =: u \geq 1$. Then

$$D_0^l W_0(t, x, y) - D_1^l W_1(t, x, y) =< \delta(x), \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}\setminus[-u,u]} e^{-\frac{\lambda^2}{4t}} \left[D_0^l e^{i\lambda D_0} - D_1^l e^{i\lambda D_1} \right] d\lambda \delta(y) > \tag{41}$$

We can exclude $[-u, u]$ from the integration since by lemma 5.2 the integrand vanishes in that intervall. Using partial integration the difference (41) can be estimated by

$$Ce^{-\frac{u^2}{8t}} \|\delta(x)\|_{W^{2,k}(B(x,s),E)^*\otimes E_x} \|\delta(y)\|_{W^{2,k}(B(y,s),E)^*\otimes E_y} \tag{42}$$

Here we have employed theorem 4.3 to compare the Sobolev norm defined by the elliptic operator D with the norm defined by the connection. Using lemma 4.1 and 4.2 we eventually obtain

$$\|D_0^l W_0(t, x, y) - D_1^l W_1(t, x, y)\| < Ce^{-\frac{u^2}{8t}}. \tag{43}$$

By essentially the same technique we have also

$$\|D_i^l W_i(t, x, y)\| < Ce^{-\frac{\operatorname{dist}(x,y)^2}{8t}} \tag{44}$$

for all $x, y \in M_i$. Let $A_\epsilon, \epsilon > 0$ be the multiplication operator with $e^{-\epsilon \operatorname{dist}(K,x)^2}$. Then by the estimates (43) and (44) the operators $A_\epsilon D_i^l e^{-sH_i} P_i$ and $(D_0^l e^{-sH_0} P_0 - D_1^l e^{-sH_1} P_1) A_\epsilon^{-1}$ are Hilbert-Schmidt for $s > 0$. Set $s := \frac{t}{2}$. Then the claim follows from

$$D_0^l e^{-tH_0} P_0 - D_1^l e^{-tH_1} P_1 = \left[e^{-sH_0} P_0 A_\epsilon \right] \left[A_\epsilon^{-1} (D_0^l e^{-sH_0} P_0 - D_1^l e^{-sH_1} P_1) \right] \tag{45}$$

$$+ \left[(e^{-sH_0} P_0 - e^{-sH_1} P_1) A_\epsilon^{-1} \right] \left[A_\epsilon D_1^l e^{-sH_1} P_1 \right]$$

since the product of two Hilbert-Schmidt operators is of trace class. □

Corollary 5.3 *The scattering operators*

$$W^{\pm} := s-\lim_{t\to\pm\infty} e^{itD_1} e^{-itD_0} P_{ac}(D_0) \tag{46}$$

exist and are complete.

That this follows from theorem 5.1 is proved in [2], Lemma 2.6 .

Corollary 5.4 *The restrictions of D_i to their absolute continuous subspaces are unitarily equivalent.*

The scattering operator W^+ gives the unitary equivalence.

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