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EMBEDDING OF A URYSOHN DIFFERENTIABLE MANIFOLD WITH CORNERS IN A REAL BANACH SPACE[1]

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Abstract

In this paper we prove a characterization of the immersions in the context of infinite dimensional manifolds with corners, we prove that a Hausdorff paracompact C^p-manifold whose charts are modelled over real Banach spaces which fulfil the Urysohn C^p-condition can be embedded in a real Banach space, E, by means of a closed embedding, f, such that, locally, its image is a totally neat submanifold of a quadrant of a closed vector subspace of E and finally we prove that a Hausdorff paracompact topological space, X, is a Hilbert C^∞-manifold without boundary if and only if X is homeomorphic to A, where A is a C^∞-retract of an open set of a real Hilbert space.

1. INTRODUCTION

Introductions and notations can be found in [3] and [4].

The paragraph 2 includes the injective version of the implicit mapping theorem in quadrants of Banach spaces, the definition of index of an inner tangent vector and the general characterization of the C^p-immersions. In the paragraph 3, we prove the main theorem 3.1, about embeddings of Urysohn manifolds into Banach spaces. Finally, using corollary 3.2 and the theorem of existence of tubular neighbourhoods, we prove the proposition 3.3, about the existence of a Hilbert differentiable C^∞-structure over a Hausdorff paracompact topological space.

We note that the study of the Hilbert manifolds has increased last years. The antecedents may be found in the period 1960-1970 in which a lot of results about

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Hilbert manifolds whose charts are modelled over separable Hilbert spaces had been discovered. Later others results were developed without the condition of separability, i.e, in 1973, H. Torunczyk, [7], proved that every paracompact and Hausdorff Hilbert C^p-manifold admits differentiable partitions of unity of class p. This result generalizes the well known one about Hilbert manifolds modelled over separable real Hilbert spaces. Recently, C.L. Terng, [6], introduced a special type of submanifolds of a Hilbert space such that the basic properties of the hypersurfaces of the euclidean spaces $\mathbb{R}^n$ can be generalized to these submanifolds.

Our results 3.1 and 3.3 are in this context. The first generalizes the J. McAlpin's theorem [1] and the second generalizes the S.B. Nadler's theorem, [5].

Proposition 1.1. *Let X be a C^p-manifold, x an element of X and $c=(U,\varphi,(E,\Lambda))$ a chart of X such that $x\in U$ and $\varphi(x)=0$. Then we have that* $\theta^x_c(E^+_\Lambda) = (T_xX)^i = (T_xX)^+_{\Lambda(\theta^x_c)^{-1}}$, *where* $\theta^x_c:E \longrightarrow T_xX$ *is the natural linear homeomorphism and* $(T_xX)^i$ *is the set of the inner tangent vectors of* T_xX.□

Definition 1.2. *Let X' be a submanifold of class p of X. Then:*

a) One says that X' is a neat submanifold of X if $\partial X' = (\partial X) \cap X'$.

b) One says that X' is a totally neat submanifold of X if for all $x'\in X'$, $ind_{X'}(x') = ind_X(x')$.

We note that, $\partial X = \{x\in X \ / \ ind(x)\geq 1\}$, $B_k(X) = \{x\in X \ / \ ind(x)=k\}$ for all $k\in\mathbb{N}\cup\{0\}$, where, if $(U,\varphi,(E,\Lambda))$ is a chart of X with $x\in U$, then $ind(x) = ind(\varphi(x))$ and $ind(\varphi(x)) = card\{\lambda\in\Lambda \ / \ \lambda(\varphi(x))=0\}$.

Definition 1.3. *Let E be a real Banach space. We say that E satisfies the Urysohn C^p-condition if for all pair of closed sets of E, A and B, such that $A\neq\emptyset$, $B\neq\emptyset$ and $A\cap B=\emptyset$, there is a C^p-function $f:E\longrightarrow[0,1]$ such that $f(A)=\{0\}$ and $f(B)=\{1\}$. A C^p-manifold whose charts are modelled over real Banach spaces which fulfil the Urysohn C^p-condition will be called Urysohn C^p-manifold.*

Definition 1.4. *Let $f:X\longrightarrow X'$ be a C^p-map and $x\in X$. We say that f is an immersion at x if there is a chart of X, $c=(U,\varphi,(E,\Lambda))$, with $\varphi(x)=0$ and there is a chart of X', $c'=(U',\varphi',(E',\Lambda'))$, with $\varphi'(f(x))=0$ such that $f(U)\subset U'$, E is a closed linear subspace of E' which admits topological supplement in E', $\varphi(U)\subset\varphi'(U')$ and $\varphi'\circ f_{|U}\circ\varphi^{-1}:\varphi(U)\longrightarrow\varphi'(U')$ is the inclusion map.*

Definition 1.5. *Let $f:X\longrightarrow X'$ be a C^p-map. We say that f is an C^p-embedding, if f is an immersion such that $f:X\longrightarrow f(X)$ is an homeomorphism.*

Theorem 1.6. *Let Y be a Hausdorff paracompact C^p-manifold ($p\geq 3$) which admits differentiable partitions of unity of class p and let X be a closed submanifold of class p of Y with $\partial X=\phi$ and $X\subset int(Y)$. Then there is a tubular neighbourhood of class (p-2) from X to Y.*□

Proposition 1.7 *([2]). Let X be a C^p-manifold with $\partial X=\emptyset$ and let $f:X\to X$ be a C^p-map such that $f\circ f=f$. Then we have that: $f(X) = \{x\in X \mid f(x)=x\}$ is a submanifold of class p without boundary of X and if X is a Hausdorff manifold, then f(X) is a closed submanifold of X.*□

2. CHARACTERIZATION OF THE IMMERSIONS

Theorem 2.1. *Let E, F_1, F_2 be real Banach spaces, $\Lambda\subset L(E,\mathbb{R})$, $\Lambda_1\subset L(F_1,\mathbb{R})$ and $\Lambda_2\subset L(F_2,\mathbb{R})$ finite linearly independent systems, U an open set of E^+_Λ with $0\in U$ and $h:U\to(F_1xF_2)^+_M$ a C^p-map with $h(0)=(0,0)$, where $M=\Lambda_1\circ p_1\cup\Lambda_2\circ p_2$. Suppose that $Dh(0)(E) = F_1x\{0\}$, $Dh(0):E \to F_1x\{0\}$ is a linear homeomorphism,*

$$(p_1\circ h)(\partial U)\subset\partial\left[(F_1)^+_{\Lambda_1}\right] \text{ and } (p_2\circ h)\left[h^{-1}\left(int\left[(F_1)^+_{\Lambda_1}\right]x(F_2)^+_{\Lambda_2}\right)\cap int(U)\right]\subset(F_2)^\circ_{\Lambda_2}.$$

Then we have that there is an open neighbourhood, U_1; of (0,0) in $(F_1xF_2)^+_{M'}$, where $M'=\Lambda\circ\Psi^{-1}\circ p_1 \cup \Lambda_2\circ p_2$, $\Psi= p_1\circ Dh(0)$, there is an open neighbourhood of (0,0) in $(F_1xF_2)^+_M$, U', there is a C^p-diffeomorphism, $\mu:U'\to U_1$, and there is an open neighbourhood of 0 in U, U^*_1, such that $h(U^*_1) \subset U'$, $Dh(0)(U^*_1) \subset U_1$ and $\mu\circ h|_{U^*_1} = j\circ Dh(0)|_{U^*_1}$. Moreover $(F_1)^+_{\Lambda\circ\Psi^{-1}}\subset(F_1)^+_{\Lambda_1}$ and if $\Lambda_2=\phi$ then $(F_1)^+_{\Lambda\circ\Psi^{-1}}=(F_1)^+_{\Lambda_1}$.

Proof. Let us consider $\Psi:E\to F_1$ defined by $\Psi(x)=p_1\circ Dh(0)(x)$, where $p_1:F_1xF_2\to F_1$ is the projection map.

Then we have that $\Psi(U)x(F_2)^+_{\Lambda_2}$ is an open set of $(F_1xF_2)^+_{M'}=(F_1)^+_{\Lambda\circ\Psi^{-1}}x(F_2)^+_{\Lambda_2}$, where $M'= \Lambda\circ\Psi^{-1}\circ p_1 \cup \Lambda_2\circ p_2$, and $\varphi: \Psi(U) \times (F_2)^+_{\Lambda_2} \longrightarrow (F_1xF_2)^+_M = (F_1)^+_{\Lambda_1} \times (F_2)^+_{\Lambda_2}$ defined by $\varphi(y_1,y_2)=(h\circ\Psi^{-1})(y_1)+(0,y_2)$ is a C^p-map. Moreover one has that $\varphi(0,0)=(0,0)$, $D\varphi(0,0)=1_{F_1xF_2}$ and $\varphi\left(\partial(\Psi(U)x(F_2)^+_{\Lambda_2})\right)\subset\partial(F_1xF_2)^+_M$. Then, from the inverse mapping theorem, we have that there is an open neighbourhood of (0,0) in $\Psi(U)x(F_2)^+_{\Lambda_2}$, U_1, there is an open neighbourhood of (0,0) in $(F_1xF_2)^+_M$, U', such that $\varphi|_{U_1}:U_1\to U'$ is a C^p-diffeomorphism. Let us consider $\mu=(\varphi|_{U_1})^{-1}:U'\to U_1$ and the open set of U, $U^*_1=h^{-1}(U')\cap(Dh(0))^{-1}(U_1)$. Then it occurs that $h(x)=\varphi\circ Dh(0)(x)$ for all $x\in U^*_1$ or

$\mu\circ h(x)=Dh(0)(x)$ for all $x\in U_1^*$.

Finally since $h(0)=(0,0)$, it occurs that $Dh(0)(E_\Lambda^+)\subset(F_1\times F_2)_M^+$, $\Psi(E_\Lambda^+)\subset(F_1)_{\Lambda_1}^+$ and $(F_1)^+_{\Lambda\circ\Psi^{-1}}\subset(F_1)^+_{\Lambda_1}$. If $\Lambda_2=\phi$, then $\varphi:\Psi(U)\times F_2\to(F_1)^+_{\Lambda_1}\times F_2$ is a local diffeomorphism at (0,0) and $card(\Lambda) = card(\Lambda_1)$, $(F_1)^\circ_{\Lambda_1} = (F_1)^\circ_{\Lambda\circ\Psi^{-1}}$ and $(F_1)^+_{\Lambda\circ\Psi^{-1}}=(F_1)^+_{\Lambda_1}$.□

Definition 2.2. *Let X be a C^p-manifold, $x\in X$ and $v\in(T_xX)^i$. We call index of v, and we denote ind(v), to the index of $(\theta_c^x)^{-1}(v)$ in E_Λ^+, where $c=(U,\varphi,(E,\Lambda))$ is a chart of X with $x\in U$ and $\varphi(x)=0$.*

Theorem 2.3. *Let $f:X\to X'$ be a C^p-map and $x\in X$ such that: 1°) there is an open neighbourhood of x in X, V^x, with $f(V^x\cap\partial X)\subset\partial X'$, 2°) $ind(v) = ind((T_xf)(v))$ for all $v\in(T_xX)^i$. Then we have that:*

a) If T_xf is an injective map and $im(T_xf)$ is a closed set, then
$$T_xf((T_xX)^i) = (T_{f(x)}X')^i\cap T_xf(T_xX).$$

b) f is an immersion at x if and only if T_xf is an injective map and $im(T_xf)$ admits a topological supplement in $T_{f(x)}X'$.

Proof. a) Let $c=(U,\varphi,(E,\Lambda))$ be a chart of X with $x\in U$ and $\varphi(x)=0$ and let $c'=(U',\varphi',(E',\Lambda'))$ be a chart of X' with $f(x)\in U'$, $\varphi'(f(x))=0$ and $f(U)\subset U'$.

Then $D(\varphi'\circ f\circ\varphi^{-1})(0):E\to E'$ is a linear continuous injective map and $im(D(\varphi'\circ f\circ\varphi^{-1})(0))=F_1$ is a closed subspace of E'. Moreover $u=D(\varphi'\circ f\circ\varphi^{-1})(0):E\to F_1$ is a linear homeomorphism.

Since $\theta_c^x(E_\Lambda^+) = (T_xX)^i$ and $\theta_{c'}^{f(x)}((E')^+_{\Lambda'}) = (T_{f(x)}X')^i$ we have that $u(E^+)_\Lambda \subset (E')^+_{\Lambda'}$, or $(F_1)^+_{\Lambda\circ u^{-1}}\subset (E')^+_{\Lambda'}$, and $ind_{(F_1)^+_{\Lambda\circ u^{-1}}}(v) = ind_{(E')^+_{\Lambda'}}(v)$ for all $v\in(F_1)^+_{\Lambda u^{-1}}$. Therefore

$card(\Lambda) = card(\Lambda') = k$, $F_1= (F_1)^\circ_{\Lambda\circ u^{-1}}\oplus_T L\{x_1,\dots,x_k\}$, $E' = (E')^\circ_{\Lambda'}\oplus_T L\{y_1,\dots,y_k\}$ and $(F_1)^\circ_{\Lambda\circ u^{-1}}\subset(E')^\circ_{\Lambda'}$, with $(\lambda_i\circ u^{-1})(x_j) = \delta_{ij}$, $\lambda'_i(y_j) = \delta_{ij}$. Then $x_i=e_i+a_{j_i}y_{j_i}$ where $e_i\in(E')^\circ_{\Lambda'}$ and $a_{j_i}>0$ and if $i\neq i'$ then $j_i\neq j_{i'}$. Let us consider $x\in(E')^+_{\Lambda'}\cap F_1$. Then $x = t + \beta_1x_1+ \dots + \beta_kx_k = t + \beta_1(e_1+ a_{j_1}y_{j_1}) + \dots + \beta_k(e_k+ a_{j_k}y_{j_k})$ where $t\in(F_1)^\circ_{\Lambda\circ u^{-1}}$ and $e_1,\dots,e_k\in(E')^\circ_{\Lambda'}$. Of course we have that $\beta_1\geq0,\dots,\beta_k\geq0$ and $x\in(F_1)^+_{\Lambda\circ u^{-1}}$, $(F_1)^+_{\Lambda\circ u^{-1}}=(E')^+_{\Lambda'}\cap F_1$, $u(E_\Lambda^+)=(E')^+_{\Lambda'}\cap F_1$ and $(T_xf)(T_xX)^i = (T_{f(x)}X')^i \cap (T_xf)(T_xX)$.

b) If f is an immersion at $x\in X$, we have that T_xf is an injective map and $im(T_xf)$ admits topological supplement in $T_{f(x)}X'$.

To prove the converse we take the charts as in a) with $U\subset V^x$. In this case F_1 admits a topological supplement, F_2, in E'. Moreover $(F_1)^{o}_{\Lambda ou^{-1}}$ admits a topological supplement, G, in $E'^{o}_{\Lambda'}$ and $(E')^{o}_{\Lambda'}=(F_1)^{o}_{\Lambda ou^{-1}}\oplus_T G$, $E'=F_1\oplus_T F_2$.

If $y\in L(x_1,...,x_k)\cap(E')^{o}_{\Lambda'}$, then $y=\alpha_1x_1+...+\alpha_kx_k$ and we have the system $0=\alpha_1\lambda'_i(x_1)+...+\alpha_k\lambda'_i(x_k)$, for all $\lambda'_i\in\Lambda'$.

Let us consider the determinant $\Delta = \begin{vmatrix} \lambda'_1(x_1) & \dots & \lambda'_1(x_k) \\ \vdots & & \vdots \\ \lambda'_k(x_1) & \dots & \lambda'_k(x_k) \end{vmatrix}$

If $\lambda'_j(x_{i_1})>0$, $\lambda'_j(x_{i_2})>0$ and $i_1\neq i_2$, then $\lambda'_j(x_{i_1}+x_{i_2})>0$ and $\lambda'_t(x_{i_1}+x_{i_2})=0$ for all $t\neq j$ and $(\lambda_{i_1}ou^{-1})(x_{i_1}+x_{i_2})>0$ and $(\lambda_{i_2}ou^{-1})(x_{i_1}+x_{i_2})>0$, where $\Lambda=\{\lambda_1,...,\lambda_k\}$, which is a contradiction. Then $\Delta\neq 0$, $\alpha_1=...=\alpha_k=0$, $y=0$, $L(x_1,...,x_k)\cap(E')^{o}_{\Lambda'}=\{0\}$ and $E'=(E')^{o}_{\Lambda'}\oplus_T L(x_1,...,x_k)$. Moreover there are $\delta_1>0,...,\delta_k>0$ and there is a bijection $\tau:\{1,...,k\}\to\{1,...,k\}$ such that the system $\left\{w_1=\frac{x_{\tau(1)}}{\delta_{\tau(1)}},...,w_k=\frac{x_{\tau(k)}}{\delta_{\tau(k)}}\right\}$ verifies that $\lambda'_i(w_j)=\delta_{ij}$. We can suppose that $y_1=w_1$, ..., $y_k=w_k$. Then we have that $L(y_1,...,y_k)=L(x_1,...,x_k)\subset F_1$ and $E'=F_1+G$. If $y\in F_1\cap G$, then $y=x_1+\alpha_1x_1+...+\alpha_kx_k$, where $x_1\in(F_1)^{o}_{\Lambda ou^{-1}}$, $y-x_1=\alpha_1x_1+...+\alpha_kx_k\in(E')^{o}_{\Lambda'}\cap L(x_1,...,x_k)$ and hence $y-x_1=0$, $y=x_1$ and $y=0$. Therefore $E'=F_1\oplus_T G$, $G\subset(E')^{o}_{\Lambda'}$ and we take $G=F_2$. Now we have that $(F_1xF_2)^{+}_{\Lambda'o\theta}=(F_1)^{+}_{\Lambda'|F_1}xF_2=(F_1xF_2)^{+}_{\Lambda ou^{-1}op_1}=(F_1)^{+}_{\Lambda ou^{-1}}xF_2$ because of $(F_1)^{+}_{\Lambda ou^{-1}}=(E')^{+}_{\Lambda'}\cap F_1$, where $\theta:F_1xF_2\to E'$ is the map defined by $\theta(x_1,x_2)=x_1+x_2$. Of course, the set $(uo\varphi)(U)xF_2$ is an open set of $(F_1)^{+}_{\Lambda ou^{-1}}xF_2=(F_1xF_2)^{+}_{\Lambda ou^{-1}op_1}$ and we can consider the C^p-map

$$h:\varphi(U)\longrightarrow(F_1xF_2)^{+}_{\Lambda'o\theta}=(F_1xF_2)^{+}_{\Lambda ou^{-1}op_1} \quad \text{defined by } h(y)=(\theta^{-1}o\varphi'ofo\varphi^{-1})(y).$$

Since $h(\partial\varphi(U))\subset\partial(F_1xF_2)^{+}_{\Lambda'o\theta}$, $Dh(0)=\theta^{-1}ou$ and $h(0)=(0,0)$, from 2.1, we have that there is an open neighbourhood of (0,0) in $(F_1xF_2)^{+}_{\Lambda'o\theta}$, U_1, there is an open neighbourhood of (0,0) in $(F_1xF_2)^{+}_{\Lambda'o\theta}$, U', there is a C^p-diffeomorphism $\mu:U'\to U_1$ and there is an open neighbourhood of 0 in $\varphi(U)$, U_1^*, such that $h(U_1^*)\subset U'$, $Dh(0)(U_1^*)\subset U_1$ and $\mu oh|_{U_1^*}=joDh(0)|_{U_1^*}$, which ends the proof.□

Proposition 2.4. *Let* $f:X\to X'$ *be a* C^p*-immersion at* $x\in X$. *Suppose that* $f(x)\in int(X')$ *or that* f *verifies the following condition:* α) *there is a open neighbourhood of x in X,* V^x, *such that* $f(V^x\cap\partial X)\subset\partial X'$ *and* $ind(z)=ind(T_xf)(z)$ *for all* $z\in(T_xX)^i$.

Then we have that $(f,g):X\to X'xX''$ *is a* C^p*-immersion at x for all* C^p*-map,* $g:X\to X''$, *such that* $g(x)\in int(X'')$.□

3. EMBEDDING OF MANIFOLDS INTO BANACH SPACES.

We recall the construction of the Hilbert direct sum.

Let $\{(E_i,\| \ \|_i)\}_{i\in I}$ be a family of real Banach spaces. Consider the set $E = \{x=(x_i)_{i\in I}\in \prod_{i\in I} E_i \ / \ \sum_{i\in I} \|x_i\|_i^2<+\infty\}$. We note that the inequality $\sum_{i\in I} \|x_i\|_i^2<+\infty$ implies that the set $\{i\in I \ / \ x_i\neq 0\}$ is a countable set. We define, for all $x,y\in E$ and all $\lambda\in\mathbb{R}$, $x+y = (x_i+y_i)_{i\in I}$ and $\lambda.x = (\lambda.x_i)_{i\in I}$. Then E is a real vector space and the function $\| \ \|$, defined by $\|x\|=(\sum_{i\in I} \|x_i\|_i^2)^{1/2}$ for all $x\in E$, is a norm in E and $(E,\| \ \|)$ is a real Banach space that will be called Hilbert sum of the family of Banach spaces $\{(E_i,\| \ \|_i)\}_{i\in I}$. This Hilbert sum will be denoted by $(\bigoplus_{i\in I} E_i,\| \ \|)$.

We remark two important properties:

a) If J is a finite subset of I and $E_J = \{x=(x_i)_{i\in I}\in E \ / \ x_i=0 \text{ for all } i\in I\text{-}J\}$, *then* E_J *is a closed linear subspace of* $(E,\| \ \|)$ *and the real Banach spaces* $(E_J,\| \ \|_{|E_J})$ *and* $(\bigoplus_{i\in J} E_i,\| \ \|_J)$ *are isomorphic by the isomorphism* α_J *defined by* $\alpha_J((x_i)_{i\in J})=(y_i)_{i\in I}$, *where* $y_i=x_i$ *for all* $i\in J$ *and* $y_i=0$ *for all* $i\in I\text{-}J$.

b) If T is the topology of $(E,\| \ \|)$ *and* T_p *is the product topology of* $\prod_{i\in I} E_i$, *then* $T_p\subset T$.

Now we suppose that $(E_i,\| \ \|_i)$ is a real Hilbert space with $\|x_i\|_i=+\sqrt{<x_i,x_i>_i}$ for all $i\in I$ and all $x_i\in E_i$ and we define the product $<x,y>= \sum_{i\in I} <x_i,y_i>_i$ for all $x,y\in E$. It is clear that $< , >$ is an inner product in E and $\|x\|^2=<x,x>$ for all $x\in E$. Thus $(E,< , >)$ is a real Hilbert space that will be called Hilbert direct sum of the family of real Hilbert spaces $\{(E_i,< , >_i)\}_{i\in I}$. This Hilbert direct sum will be denoted by $(\bigoplus_{i\in I} E_i,< , >)$.

Theorem 3.1. *Let X be a Hausdorff paracompact* C^p*-manifold whose charts are modeled over real Banach spaces which satisfy the Urysohn* C^p*-condition (1.3).*

Then there is a real Banach space, $(E,\| \ \|)$, *and there is a closed* C^p*-embedding, f, from X to E. Therefore the manifold X is* C^p*-diffeomorphic to a closed submanifold of E. Moreover it holds the following local property: For every* $x\in X$ *there is an open neighbourhood of x in X,* W^x, *there is a closed vector subspace of E,* E_1, *and there is a quadrant of* E_1, $(E_1)^+_{\Lambda_1}$, *such that* $f_{|W^x}:W^x\to(E_1)^+_{\Lambda_1}$ *is a* C^p*-embedding which fulfils that* $f(W^x)$ *is a totally neat submanifold of* $(E_1)^+_{\Lambda_1}$.

Proof. Since X is a Hausdorff paracompact space, it occurs that X is a regular space and we have that for every $x\in X$ there is a chart of X, $c_x=(U_x,\varphi_x,(E_x,\Lambda_x))$, with $x\in U_x$,

$\varphi_x(x)=0$ and $\varphi_x(U_x)$ is bounded in E_x and such that $F\subset U_x$ is a closed set of X if and only if $\varphi_x(F)$ is a closed set of $(E_x)^+_{\Lambda_x}$.

Since $\mathcal{U}=\{U_x \,/\, x\in X\}$ is an open covering of X, there is a locally finite open refinement, $\{U_i\}_{i\in I}$ of $\mathcal{U}$. For all $i\in I$ we take $x_i\in X$ such that $U_i\subset U_{x_i}$. Then $\mathcal{A}=\{c_i=(U_i,\varphi_i=\varphi_{x_i}|_{U_i},(E_i=E_{x_i},\Lambda_i=\Lambda_{x_i}))\}_{i\in I}$ is an atlas of class p of X such that $\{U_i\}_{i\in I}$ is a locally finite family and $F\subset U_i$ is a closed set of X if and only if $\varphi_i(F)$ is a closed set of E_i. Since X is a T_4 space there is a contraction, $\mathcal{V}=\{V_i\}_{i\in I}$, of $\{U_i\}_{i\in I}$. That is $\overline{V_i}\subset U_i$ for all $i\in I$, $V_i\neq\phi$ for all $i\in I$ and $\bigcup_{i\in I} V_i=X$. By the same argument there is a contraction of $\mathcal{V}$, $\mathcal{W}=\{W_i\}_{i\in I}$ with $W_i\neq\phi$ for all $i\in I$. Then for all $i\in I$, $\varphi_i(\overline{W_i})$ is a closed set of $(E_i)^+_{\Lambda_i}\subset E_i$ and $\varphi_i(\overline{W_i})\subset\varphi_i(V_i)$. Having in mind the Urysohn condition we have that for all $i\in I$ there is a C^p-function, $\lambda_i:E_i\to[0,1]$, such that $\lambda_i(\varphi_i(\overline{W_i}))=\{1\}$ and $\lambda_i((E_i)^+_{\Lambda_i}-\varphi_i(V_i))=\{0\}$. Then $A=\mathrm{Supp}(\lambda_i|_{(E_i)^+_{\Lambda_i}})\subset\varphi_i(\overline{V_i})\subset\varphi_i(U_i)$ for all $i\in I$.

Let us consider the map $\Psi_i:X\longrightarrow(E_i)^+_{\Lambda_i}\times[0,1]\subset E_i\times\mathbb{R}$ defined by

$$\Psi_i(x)=\begin{cases}(\lambda_i(\varphi_i(x)).\varphi_i(x),\lambda_i(\varphi_i(x))) & \text{if } x\in U_i\\ 0 & \text{if } x\notin U_i\end{cases}$$

It can be easily seen that $\Psi_i|_{U_i}$ is a C^p-map, $\Psi_i|_{X-\varphi_i^{-1}(A)}=0$ is a C^p-map, $X=U_i\cup(X-\varphi_i^{-1}(A))$ and $\varphi_i^{-1}(A)$ is a closed set of X. Therefore Ψ_i is a C^p-map. Let $\|\ \|_i$ be the norm of $E_i\times\mathbb{R}$ for all $i\in I$ and let $(E,\|\ \|)$ be the Hilbert sum of the family of real Banach spaces $\{(E_i\times\mathbb{R},\|\ \|_i)\}_{i\in I}$.

Let us consider the map $f:X\to E$ defined by:

$$f(x)=(\Psi_i(x))_{i\in I}\in\prod_{i\in I}\left[(E_i)^+_{\Lambda_i}\times[0,1]\right]\subset\prod_{i\in I}(E_i\times\mathbb{R}).$$

If $\sum_{i\in I}\|\Psi_i(x)\|_i^2<+\infty$, then $f(x)\in E$. In fact, we have that $\{U_i\}_{i\in I}$ is a locally finite family and therefore $A_x=\{i\in I \,/\, x\in U_i\}$ and $B_x=\{i\in I \,/\, \Psi_i(x)\neq 0\}$ are finite sets and $\sum_{i\in I}\|\Psi_i(x)\|_i^2<+\infty$.

Next we will see that $f:X\to E$ is a closed C^p-embedding.

1°) f is a C^p-map. Indeed, for all $x\in X$ there is an open neighbourhood of x in X, V^x such that $J_x=\{i\in I \,/\, V^x\cap V_i\neq\phi\}$ is a finite set. Then it happens that $\Psi_i(y)=0$ for all $i\in I-J_x$ and all $y\in V^x$. Therefore the map

$$f|_{V^x}:V^x \xrightarrow{(\Psi_i)_{i\in J_x}} \bigoplus_{i\in J_x}(E_i\times\mathbb{R}) \xrightarrow[\approx]{\alpha_{J_x}} E_{J_x} \xhookrightarrow{j} E$$

is a C^p-map and, since x is a arbitrary point, f is also a C^p-map in X.

We note that the map

$$f|_{V^x}:V^x \xrightarrow{(\Psi_i)_{i\in J_x}} \prod_{i\in J_x}\left[(E_i)^+_{\Lambda_i}\times\mathbb{R}\right]=\left[\prod_{i\in J_x}(E_i\times\mathbb{R})\right]^+_{\Lambda}$$

where $\Lambda = \bigcup_{i\in J_x}(\Lambda_i \circ p_1)\circ\bar{p}_i$, $p_1:E_i\times\mathbb{R} \to E_i$ and $\bar{p}_i: \prod_{i\in J_x}(E_i\times\mathbb{R}) \to E_i\times\mathbb{R}$, is a C^pmap and $f|_{V^x}(\partial V^x)\subset \partial\left[\prod_{i\in J_x}(E_i\times\mathbb{R})\right]^+_{\Lambda}$. Moreover $f|_{V^x}(\partial V^x)\subset \prod_{i\in J_x}(\partial(E_i)^+_{\Lambda_i}\times\mathbb{R})$.

2°) f is an injective map. Indeed, we have that $\bigcup_{i\in I}\overline{W_i} = X$ and $\bigcup_{i\in I}(\varphi_i^{-1}\circ\lambda_i^{-1})(1) = X$. Let us consider x,y $\in$ X with x $\neq$ y and $i_o\in I$ such that $y\in(\varphi_{i_o}^{-1}\circ\lambda_{i_o}^{-1})(1)$.

If $x\in(\varphi_{i_o}^{-1}\circ\lambda_{i_o}^{-1})(1)$, then $\Psi_{i_o}(x)=(\varphi_{i_o}(x),1)\neq\Psi_{i_o}(y)=(\varphi_{i_o}(y),1)$.

If $x\notin\varphi_{i_o}\lambda_{i_o}(1)$, then

$$\Psi_{i_o}(x) = \begin{cases} (\lambda_{i_o}(\varphi_{i_o}(x))\varphi_{i_o}(x),\lambda_{i_o}(\varphi_{i_o}(x))) & \text{if } x\in U_{i_o} \\ 0 & \text{if } x\notin U_{i_o} \end{cases}$$

and $\Psi_{i_o}(y)=(\varphi_{i_o}(y),1)$. But, if $x\in U_{i_o}$, we have that $\lambda_{i_o}(\varphi_{i_o}(x))<1$ and hence $\Psi_{i_o}(y)\neq\Psi_{i_o}(x)$ and $f(x)\neq f(y)$.

3°) $f:X\to E$ is an immersion of class p. Indeed, let $x\in X$. From 1°), we have the C^p-map $f|_{V^x}=(j\circ\alpha_{J_x})(\Psi_i)_{i\in J_x}:V^x\to E$. Let us consider $i_o\in J_x$ such that $x\in W_{i_o}$. Then the map $\Psi_{i_o}|_{W_{i_o}}:W_{i_o}\to(E_{i_o}^+)_{\Lambda_{i_o}}\times\mathbb{R}$ is given by $\Psi_{i_o}(y)=(\varphi_{i_o}(y),1)$ for all $y\in W_{i_o}$ and using 2.4, we have that $\Psi_{i_o}|_{W_{i_o}}:W_{i_o}\to(E_{i_o})^+_{\Lambda_{i_o}}\times\mathbb{R}$ is an immersion at x. From 2.4, we have that the map $\Psi_{i_o}|_{W_{i_o}}:W_{i_o}\to E_{i_o}\times\mathbb{R}$ is also an immersion at x.

Finally $f|_{V^x\cap W_{i_o}}:V^x\cap W_{i_o}\to E$ is an immersion at x and therefore $f:X\to E$ is an immersion at x.

4°) $f(X)$ is a closed set of E and $f:X\to f(X)$ is an homeomorphism. Indeed, let $x\in X$ and $i_o\in J_x$ such that $x\in W_{i_o}$. Then $\Psi_{i_o}(x)=(\varphi_{i_o}(x),1)$ and $\|f(x)\|\geq 1$. That is $f(X)\subset E-B_1(0)$, where $B_1(0)$ is the open ball of E.

Let C be a closed set of X and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of C such that $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $y_o\in E$. Since $E-B_1(0)$ is a closed set of E, it occurs that $y_o\notin B_1(0)$ and $y_o\neq 0$. Then there is $i_o\in I$ such that $\{\Psi_{i_o}(x_n)\}_{n\in\mathbb{N}}$ converges to $y_o^{i_o}=(a,b)\in E_{i_o}\times\overline{\mathbb{R}}-\{(0,0)\}$. (We have used the property b) of the Hilbert sum).

Of course, from the definition of Ψ_{i_o}, there is $n_o\in\mathbb{N}$ such that for all $n\geq n_o$ it happens that $x_n\in V_{i_o}$. Hence $\{(\lambda_{i_o}(\varphi_{i_o}(x_n)).\varphi_{i_o}(x_n),\lambda_{i_o}(\varphi_{i_o}(x_n)))\}_{n\in\mathbb{N},\, n\geq n_o}$ converges to (a,b), $\{\lambda_{i_o}\varphi_{i_o}(x_n)\}_{n\in\mathbb{N},\, n\geq n_o}$ converges to b and $b\neq 0$ because of $\{\varphi_{i_o}(x_n)\}_{n\in\mathbb{N},\, n\geq n_o}$ is a bounded set in E_{i_o}. Now we have that the sequence $\{\varphi_{i_o}(x_n)\}_{n\in\mathbb{N},\, n\geq n_o}\subset\varphi_{i_o}(V_{i_o})$ converges to $a.b^{-1}$ and $a.b^{-1}\in\overline{\varphi_{i_o}(U_{i_o})}$, because of $\overline{\varphi_{i_o}(V_{i_o})}\subset\varphi_{i_o}(U_{i_o})$. Finally the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to $x_o=(\varphi_{i_o})^{-1}(a.b^{-1})\in C$ and the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $y_o=f(x_o)\in f(C)$.

Then we have that $f:X\to E$ is a closed C^p-embedding.□

Corollary 3.2. *Let X be a Hausdorff paracompact Hilbert C^p-manifold. Then there is a real Hilbert space H,and there is a closed C^p-embedding, f, from X into H.*

Proposition 3.3. *Let X be a Haussdorff paracompact topological space. Then the following statements are equivalent:*

a) X admits a Hilbert differentiable C^∞-structure, without boundary.

b) There are a real Hilbert space, H, an open set of H, U, a subset of U, A, and a continuous map $r:U\to A$ such that $r\circ i=1_A$, where $i:A\to U$ is the inclusion map, the map $i\circ r:U\to U$ is a map of class ∞ and A, X are homeomorphic.

Proof. a)⇒b). From 3.2, there is a closed C^∞-embedding, $f:X\to H$. Then $f(X)$ is a closed submanifold of class ∞ of H, without boundary and $f:X\to f(X)$ is a C^∞-diffeomorphism. Thus, using 1.6, there is a vector bundle of class ∞, $(M,f(X),\pi)$, there is an open neighbourhood of $f(X)$ in H, U, there is an open neighbourhood of $\tau_M(f(X))$ in M, Z, and there is a C^∞-diffeomorphism, $h:Z\to U$ such that $h\circ\tau_M=j$, where $j:f(X)\hookrightarrow H$ is the inclusion map. Let us consider the C^{∞}-map, $r:U \longrightarrow f(X)$, defined by $r=\pi \circ h^{-1}$. It is clear that $r \circ i=1_{f(X)}$, where $i:f(X) \longrightarrow U$ is the inclusion map.

b)⇒a). Using 1.7 we have that $r(U)=A$ is a closed submanifold without boundary of U. Then a) follows from the fact that A and X are homeomorphic.□

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