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A Riemannian manifold (M, ρ) is called Einsteinian iff its Ricci tensor satisfies $Ric(\rho) = C \cdot \rho$ for some constant C.

The compact simply connected homogeneous spaces admitting homogeneous Riemannian metric of positive sectional curvature have been classified by M. Berger [B], N. Wallach [W] and B. Bergery [Be]. There are compact rank one symmetric spaces (CROSS) and also $SU(3)/i_{k,l}(S^1)$, $Sp(2)/SU(2)$, $SU(5)/Sp(2)\times T^1$, $SU(3)/T_{max}$, $Sp(3)/Sp(1)\times Sp(1)\times Sp(1)$, $F_4/Spin(8)$. The classification of all homogeneous Einstein metrics on the CROSS have been done by W. Ziller [Z]. The case of $SU(3)/i_{k,l}(S^1)$ was considered by M. Wang [Wa]. The homogeneous space $Sp(2)/SU(2)$ is isotropy irreducible and therefore it admits, up to a homothety, only one homogeneous Einstein metric.

The main purpose of the present paper is to obtain the classification of all invariant Einstein metrics on the remaining homogeneous spaces: $SU(3)/T_{max}$, $Sp(3)/Sp(1)\times Sp(1)\times Sp(1)$, $F_4/Spin(8)$, $SU(5)/Sp(2)\times T^1$.

Theorem. The classification mentioned above is given, up to a homothety, by the following table:

Space	The isotropy representation	Characteristic numbers of the metric	Sectional curvature
$SU(3)/T_{max}$	$\rho = \rho_1 \oplus \rho_2 \oplus \rho_3$	(1,1,1)	positive
		(2,1,1)	oscillate
		(1,2,1)	oscillate
		(1,1,2)	oscillate

$Sp(3)/Sp(1)\times Sp(1)\times Sp(1)$	$p=p_1\oplus p_2\oplus p_3$	(1,1,1) (3,1,1) (1,3,1) (1,1,3)	positive oscillate oscillate oscillate
$F_4/Spin(8)$	$p=p_1\oplus p_2\oplus p_3$	(1,1,1) (7,2,2) (2,7,2) (2,2,7)	positive oscillate oscillate oscillate
$SU(5)/Sp(2)\times T^1$	$p=p_1\oplus p_2$	$((20-4\sqrt{7})/9,1)$ $((20+4\sqrt{7})/9,1)$	positive oscillate

Sketch of the proof for the cases $SU(3)/T_{max}$, $Sp(3)/Sp(1)\times Sp(1)\times Sp(1)$, $F_4/Spin(8)$. Let χ be the isotropy representation of G/H and $g=h\oplus p$ is the reductive decomposition of G/H with respect of the minus Killing's form of g. Then $p=p_1\oplus p_2\oplus p_3$, where χ acts irreducibly on each p_i $(i=1,2,3)$ and $p_i \not\approx p_j$ for $i\neq j$. Hence a G-invariant metric on G/H is of the form: $(\cdot,\cdot) = c_1 B|p_1 + c_2 B|p_2 + c_3 B|p_3$, where $c_1, c_2, c_3 \in R^+$ and $B(X,Y) = -tr\, adX \circ adY$ for all $X,Y\in g$. We suppose, without the loss of generality, that $(c_1,c_2,c_3)=(1,t,s)$. Then non-trivial computations using the formula for the sectional curvature on homogeneous Riemannian spaces show that $(\cdot,\cdot)$ is Einstein iff

$$\begin{cases} 3st-t^2-3s+1=0 \\ (t-s)(t+s-3)=0 \end{cases} \quad \text{for } SU(3)/T_{max}$$

$$\begin{cases} 4st-t^2-4s+1=0 \\ (t-s)(t+s-4)=0 \end{cases} \quad \text{for } Sp(3)/Sp(1)\times Sp(1)\times Sp(1)$$

$$\begin{cases} 9st-9s-2t^2+2=0 \\ (t-s)(2t+2s-9)=0 \end{cases} \quad \text{for } F_4/Spin(8)$$

We obtain immediately the rezult of our theorem by solving these systems of quadratic equations. The case of $SU(5)/Sp(2)\times T^1$ is considered analogously.

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