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The Penrose Transform for Dirac equation

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1 Introduction

The presented paper is a continuation of the paper ([3]) where the Penrose transform for solutions of the Laplace equation was described by means of Clifford analysis. Here we discuss the Penrose transform for solutions of the Dirac equation.

The description of the Penrose transform in a general situation can be found in the book by R.Baston and M.Eastwood ([5]). We shall discuss here the special case corresponding to the orthogonal group in even dimensions. The Penrose transform maps elements of a certain cohomology group to solutions of (complex) Dirac equation in this case. We are presenting here a simple approach to it using the Dolbeault realization of the cohomology groups, the construction of the Penrose transform and the inverse transform is quite explicit.

Contrary to the case of the Laplace equation treated in [3], to describe the Penrose transform in terms of a simple calculus with differential forms, it is necessary to make all calculations on the complex Spin groups instead of Stiefel manifolds. Advanced and sophisticated tools (such as B-G-G resolution, hypercohomology or spectral sequences) are avoided.

The presented paper contains a summary of the results, the full version ([4]) with all proofs will be published elsewhere.

2 The Penrose transform for complex Dirac equation

2.1 Basic twistor diagram

The Penrose transform is based on a diagram of homogeneous spaces (see [2] and [5]). In our case (i.e. for the description of solutions of the Dirac equation in higher even dimensions), we shall need the following homogeneous spaces of the group $\mathrm{SO}(2n+2,\mathbf{C})$.

Let us consider the quadratic form

$$Q(Z)=\sum_{1}^{n+1} Z'_j Z''_j;\ Z=[Z',Z'']^t;\ Z',Z''\in\mathbf{C}^{n+1}$$

on the vector space $\mathbf{C}^{2n+2}$. The corresponding bilinear form will be denoted by $\langle\ ,\ \rangle$. We shall need the following type of flag manifolds:

$$IG_{i_1,\dots,i_j;2n+2}:=\{[L_{i_1},\dots,L_{i_j}]|L_{i_1}\subset\dots\subset L_{i_j}\subset\mathbf{C}^{2n+2};\ Q|L_{i_j}=0\}.$$

We shall drop the dimension of the ambient space if it is clear from the context. In particular we shall use the complex quadric IG_1 (which can be considered as the compactification of the complex Minkowski or Euclidean space) and the spaces IG_{n+1} and $IG_{1,n+1}$. The last two spaces are not connected, we shall work always with one of their connected components.

The space IG_{n+1} can be interpreted either as the space of all maximal isotropic subspaces in the quadric IG_1 or as the space of all pure spinors.

Together with the natural forgetting maps, they form the basic diagram

$$\begin{array}{ccc} & IG_{1,n+1} & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ IG_{n+1} & & IG_1 \end{array} \tag{1}$$

It is always an advantage to have a possibility to describe global objects on the isotropic Grassmannians in coordinates similar to homogeneous coordinates on projective spaces. Such a role is played by isotropic Stieffel manifolds for isotropic Grassmannians. In our case (for Dirac operator), we use also elements from the group $\mathrm{Spin}(2n+2,\mathbf{C})$ as "homogeneous" coordinates.

The group $\mathrm{Spin}(2n+2,\mathbf{C})$ is not connected, let us denote by $\mathrm{Spin}_0(2n;\mathbf{C})$the connected component of the unit element in $\mathrm{Spin}(2n+2,\mathbf{C})$.

So let us consider the space

$$\begin{aligned} &ISt_{n+1}:= \\ &\quad\{\mathcal{Z}=[Z^0,\dots,Z^n]|Z^i\in\mathbf{C}^{2n+2},\ \mathrm{rank}\,\mathcal{Z}=n+1, \\ &\quad\langle Z^i,Z^j\rangle=0;\ i,j=0,\dots,n\} \end{aligned}$$

as a principal fibre bundle over IG_{n+1} with the group $G=\mathrm{GL}(n+1,\mathbf{C})$ acting from the right. The corresponding projection will be denoted by π. The same space will be considered as a principal fibre bundle over $IG_{1,n+1}$

$$\pi' : ISt_{n+1} \mapsto IG_{1,n+1}$$

with

$$\pi'(\mathcal{Z}) = [L_1, L_{n+1}],\ L_1 = \text{span}\{Z^0\}, L_{n+1} = \text{span}\{Z^0, \ldots, Z^n\}.$$

The group of the fibration consists of all regular matrices having the form

$$g = \begin{pmatrix} a & v \\ 0 & \gamma \end{pmatrix}, a \in \mathbf{C},\ v^t \in \mathbf{C}_n,\ \gamma \in \text{GL}(n, \mathbf{C}).$$

To define the Penrose transform for the Dirac equation, we need to consider bigger principal fibre bundles over IG_{n+1}.

The isotropic Grassmannian IG_{n+1} is a homogeneous space of $\text{SO}(2n+2, \mathbf{C})$, i.e. $IG_{n+1} \simeq SO(2n+2, \mathbf{C})/P$,

$$P := \left\{ \begin{pmatrix} a & b \\ 0 & (a^t)^{-1} \end{pmatrix} \middle| a \in \text{GL}(n+1, \mathbf{C}), (ba^t)^t = -ba^t \right\}.$$

Similarly, $IG_{n+1} \simeq \text{Spin}_0(2\text{n}+2; \mathbf{C})/\widetilde{P}$, where the group $\widetilde{P}$ is the preimage of P in $\text{Spin}_0(2\text{n}+2; \mathbf{C})$.

Now, the group $\text{GL}(n+1, \mathbf{C})$ can be imbedded into $\text{SO}(2n+2, \mathbf{C})$ by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & (a^t)^{-1} \end{pmatrix}$$

and then, in fact $\text{GL}(n+1, \mathbf{C}) \subset P \subset \text{SO}(2n+2, \mathbf{C})$.

We can denote by $\widetilde{\text{GL}}$ its preimage in $\text{Spin}_0(2\text{n}+2; \mathbf{C})$and we have

$$\widetilde{\text{GL}} \subset \widetilde{P} \subset \text{Spin}_0(2\text{n}+2; \mathbf{C}).$$

Moreover, we have the projections $\pi : P \mapsto \text{GL}(n+1, \mathbf{C})$ and $\tilde{\pi} : \widetilde{P} \mapsto \widetilde{G}$.

Putting everything together, we have the diagram:

$$\begin{array}{ccc} \text{Spin}_0(2\text{n}+2; \mathbf{C}) & \xrightarrow{\widetilde{P}} & IG_{n+1} \\ \downarrow {\scriptstyle 2:1} & & \downarrow = \\ \text{SO}(2n+2, \mathbf{C}) & \xrightarrow{P} & IG_{n+1} \\ \downarrow & & \downarrow = \\ ISt_{n+1} & \xrightarrow{\text{GL}} & IG_{n+1} \end{array}$$

2.2 Line bundles on IG_n

Considering the twistor space IG_n as a homogenenous space $\mathrm{Spin}_0(2n;\mathbf{C})/\tilde{P}$, then for every one-dimensional representation ρ of the isotropic group $\tilde{P}$ there is a line bundle L_ρ on IG_n associated to the representation ρ.

If the representation ρ of $\tilde{P}$ is holomorphic, then L_ρ is a holomorphic line bundle.

There are some interesting one-dimensional representations on $\tilde{P}$. For example, considering the map $\tilde{\pi} : \tilde{P} \to GL(n,\mathbf{C})$ given by the composition of two projections $\tilde{P} \to P \rightsquigarrow GL(n,\mathbf{C})$, where the last map is defined by

$$\begin{pmatrix} A & B \\ 0 & (A^t)^{-1} \end{pmatrix} \to A,$$

then clearly the map $\rho : \tilde{p} \mapsto \det A = \det(\tilde{\pi}(\tilde{p}))$ is a homomorphism of $\tilde{P}$ into the nonzero complex numbers.

We have even more. The group $\tilde{P}$ being a $2:1$ covering of P, we can construct a square root of the representation $\det(\tilde{\pi}(\tilde{p}))$. To this end, let us first recall that if $f_1,\dots,f_n,\bar{f}_1,\dots,\bar{f}_n$ is a canonical basis in $\mathbf{C}^{2n}$, then the idempotent $I = I_1\dots I_n$, $I_j = \bar{f}_j f_j$, was used for the realization of the spinor space S^+_{2n} as $(\bigwedge^{\text{even}} W)I$ in $\mathbf{C}_{2n}$, where W = span $\{f_1,\dots,f_n\}$ and $\mathbf{C}_{2n}$ is the complex Clifford algebra for $\mathbf{C}^{2n}$ (see [7]).

Theorem 2.1 *For each $\tilde{p} \in \tilde{P}$ there exists $f(\tilde{p}) \in \mathbf{C}$ such that $\tilde{p}I = f(\tilde{p})I$. Moreover, $f(\tilde{p})^2 = \det\tilde{\pi}(\tilde{p})$, where $\tilde{\pi} : \tilde{P} \to \mathrm{GL}(n,\mathbf{C})$ is the projection defined above and the map $f : \tilde{P} \to \mathbf{C}^* = \mathbf{C}\setminus\{0\}$ is a one-dimensional holomorphic representation of $\tilde{P}$.*

Note that the function f defined on $\tilde{P}$ is determined (up to a sign) by $\sqrt{\det\tilde{\pi}(\tilde{p})}$ and that the sign of the square root can be choosen consistently for points in $\tilde{P}$. This function will frequently appear in the sequel; we shall denote it simply by $\sqrt{det}$.

Definition 2.1 *The function $\sqrt{det}$ is defined on $\tilde{P}$ by the expression*

$$\tilde{p}I = \sqrt{det}(\tilde{p})I. \tag{2}$$

and by the condition $\sqrt{det}(e) = 1$.

By means of the function $\sqrt{det}$, we are now able to define important line bundles $\mathbf{L}^k$ on IG_n in the following way.

Definition 2.2 *The holomorphic line bundles $\mathbf{L}^k, k \in \mathbf{Z}$, are defined as the line bundles associated to the representation $(\sqrt{det})^k$ of $\tilde{P}$.*

The bundle $\mathbf{L}$ may be characterized in several other ways (see e.g. [4])

The space IG_n of maximal isotropic subspaces can be identified with the projective space of pure spinors $\mathbf{P}(S^{\mathrm{pure}})$, $S^{\mathrm{pure}} \subset S^+$. This identification can be alternatively described using Theorem 2.1.

Indeed, Theorem 2.1 implies that the map from $\mathrm{Spin}_0(2\mathrm{n};\mathbf{C})$ to S^+ given by the map $\mathcal{S} : g \mapsto gI$ induces a map from IG_n to $\mathbf{P}(S^+)$. The fact that the image of the induced map is the projective space of pure spinors and that the map is injective follows then from the properties of the map $\mathcal{S}$ listed in the following Lemma.

Lemma 2.1 *The map $\mathcal{S} : \mathrm{Spin}_0(2\mathrm{n};\mathbf{C}) \to S^+$ defined by $\mathcal{S}(g) = gI$, is a holomorphic map satisfying*

$$\mathcal{S}(g\tilde{p}) = \sqrt{det}(\tilde{p})\mathcal{S}(g).$$

for $\tilde{p} \in \tilde{P}, g \in \mathrm{Spin}_0(2\mathrm{n};\mathbf{C})$.

Moreover, if Z is a vector in the isotropic subspace $\pi(g) \in IG_n$, then $Z.\mathcal{S}(g) = 0$ in $\mathbf{C}_{2n}$ (where . denotes the Clifford multiplication).

The properties of the map $\mathcal{S}$ are quite important and will be substantially used below. Notice also that values of the map $\mathcal{S}$ belong to the space of pure spinors; even more, the value at a point g is exactly a pure spinor annihilated by the vectors in the isotropic subspace corresponding to g.

Given a pure spinor $s \in S^{\mathrm{pure}}$, we shall need an explicit formula for the projection from S^+ to the one-dimensional subspace generated by s in S^+.

Let us denote by $\bar{s}$ the element of Clifford algebra $\mathbf{C}_{2n}$ conjugate to s (see [7]).

Lemma 2.2 *Let us consider $[s] \in IG_n$ represented by $s \in \mathrm{Spin}_0(2\mathrm{n};\mathbf{C})$. Then the projection $\Pi([s])$ onto the line generated by the corresponding pure spinor defined by*

$$\Pi([s]) = \frac{1}{|s|^2} sI\bar{s}$$

does not depend on the choice of a representative s.

2.3 Invariant forms on isotropic Grassmannians

In the treatment of the Penrose transform for solutions of the Dirac equation, certain invariant forms on the isotropic Grassmanians IG_n will be quite useful.

The isotropic Grassmannian IG_n is, as we know, a homogeneous space of a compact group, namely $IG_n = \mathrm{Spin}(2n, \mathbf{R})/\tilde{\mathrm{U}}(\mathrm{n}, \mathbf{R})$. So there is a unique left invariant volume form κ_n on IG_n, normalized by the condition

$$\int_{IG_n} \kappa_n = 1.$$

Moreover, IG_n is also compact homogeneous Kaehler manifold with canonical invariant Kaehler metric g and Kaehler form ϕ (see e.g.[8]).

Using the Kaehler form ϕ, we can construct the form

$$\underbrace{\phi \wedge \ldots \wedge \phi}_{n(n-1)/2},$$

which is a form of a top degree on IG_n, invariant with respect to $\mathrm{SO}(2n, \mathbf{R})$.

It comes as no suprise that it is a nonzero form (it is true in general that the top power of the Kaehler form is the volume form, see e.g. [8]) hence it is equal (up to a normalization) to the volume form κ_n.

So we have an explicit formula

$$\kappa_n = c_1 \cdot \underbrace{\phi \wedge \ldots \wedge \phi}_{n(n-1)/2},$$

where $c_1 = 1/(n!2^n \cdot vol(IG_n))$ and $vol(IG_n)$ is the volume of IG_n with respect to the Riemannian metric corresponding to the Kahler form.

Let us turn now the attention to another problem concerning invariant forms. The form κ_n was characterized as an invariant form of the top degree, an element of $\mathcal{E}^{(\binom{n}{2},\binom{n}{2})}(IG_n)$. The question is, if it is possible to find also an invariant form in the space $\mathcal{E}^{(\binom{n}{2},0)}(IG_n)$. Denoting the holomorphic cotangent space of IG_n by $T^{1,0*}$, we know that the top power $\Lambda^{\binom{n}{2}}(T^{1,0*})$ is a line bundle and forms in $\mathcal{E}^{(\binom{n}{2},0)}(IG_n)$ are sections of it. In fact, there is no such invariant form, but we can find an invariant form of bidegree $(\binom{n}{2},0)$, if we consider such forms with values in a suitable line bundle, as described in the following theorem.

Theorem 2.2 *There exist a unique (up to a nonzero multiple) holomorphic invariant form α_n of the bidegree $(\binom{n}{2},0)$ on IG_n with values in the line bundle $\mathbf{L}^{2(n-1)}$.*

The conjugated form $\overline{\alpha_n} \in \mathcal{E}^{(0,\binom{n}{2})}(IG_n, \overline{\mathbf{L}}^{2(n-1)})$, where $\overline{\mathbf{L}}$ is an anti-holomorphic line bundle given by the representation $\overline{\sqrt{det\tilde{p}}}$ can be constructed in the same way. It is an antiholomorphic form of the top degree on IG_n.

If now $M_I(\mathcal{Z})$, $I \subset \{1, \ldots, 2n\}, |I| = n$ denotes the determinant of the corresponding $n \times n$ minor of the matrix $\mathcal{Z}$, we get still another description of the volume form κ_n on IG_n. Under the substitution (5.1), the determinants are transforming as $M_I \mapsto M_I \det g$, hence the form $\frac{\alpha_n(\mathcal{Z}) \wedge \overline{\alpha_n(\mathcal{Z})}}{(\sum_I |M_I(\mathcal{Z})|^2)^{n-1}}$, is invariant with respect to the substitution. So it is a form on IG_n.

The forms α_n and $\overline{\alpha}_n$ are invariant with respect to SO$(2n, \mathbf{C})$ and the denominator is preserved under the action of the group SO$(2n, \mathbf{R})$. As a consequence, the form (2.3) coincides (after normalization) with the form κ_n.

The projection map Π defined in the preceeding section has the following property:

Theorem 2.3 *The map given by*

$$\int_{IG_n} \Pi([s])\kappa_n$$

is a (nonzero) multiple of the identity map on the spinor space S.

Finally we have a naturally defined fibration $IG_n \to S^{2n-2}$ with fibre IG_{n-1}, we shall need a form, say τ_n on IG_n, the degree of which is the (real) dimension of the fiber such that the integrals over any fiber is equal to 1. This form τ_n can be constructed using the Kaehler form ϕ on IG_n, namely

$$\tau_n = \tilde{c}\underbrace{\phi \wedge \ldots \wedge \phi}_{\frac{(n-1)(n-2)}{2}}.$$

2.4 The description of the Penrose transform

Our aim is to describe solutions of the Dirac equation only on subsets of the (complexified) Minkowski or Euclidean space

$$\mathbf{C}M \subset IG_1, \mathbf{C}M := \{L = \text{span}\{Z\} | Z \in \mathbf{C}^{2n+2},\ Z_{n+1} \neq 0\}.$$

It is an open dense subset of IG_1. So we shall consider nonhomogeneous coordinates (x, y) on $\mathbf{C}M$ by the identification

$$\mathbf{C}M = \{[x, 1, y, -x \cdot y]^t | x, y \in \mathbf{C}^n\},$$

where $x \cdot y := \sum_1^n x_j y_j$.

The basic twistor diagram will hence be restricted to the double fibration

$$\begin{array}{ccc} & \mathbf{F} & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ \mathbf{T} & & \mathbf{C}\mathbf{M} \end{array} \tag{3}$$

where the spaces involved are defined as

$$\mathbf{F} := \nu^{-1}(\mathbf{C}M) = \{[L_1, L_{n+1}] | L_1 \in \mathbf{C}M\},$$

$$\mathbf{T} := \mu(\mathbf{F}) = \{L_{n+1} | L_{n+1} \cap \mathbf{C}M \neq \emptyset\}.$$

Consider an open domain $\Omega \subset \mathbf{C}M$ and the corresponding domains

$$\Omega' \subset \mathbf{F}, \Omega' := \nu^{-1}(\Omega) \text{ and } \Omega'' \subset \mathbf{T}, \Omega'' := \mu(\Omega').$$

Let $P = (P_1, \ldots, P_{2n})$ be holomorphic coordinates on CM, and let us consider the holomorphic form of maximal degree

$$dP = dP_1 \wedge \ldots \wedge dP_{2n}$$

on CM.

The domain Ω' is the base of a fibration

$$\widetilde{\Omega}' \subset \overset{s}{\longleftrightarrow} pino \mapsto \Omega'.$$

The holomorphic form of the top degree $\alpha_n \wedge dP$ can be lifted to $\widetilde{\Omega}'$ and it transforms under the substitution $s \mapsto s\tilde{p}$, $s \in \mathrm{Spin}(2n+2, \mathbf{C})$, $\tilde{p} \in \widetilde{P}$ as

$$\alpha_n \wedge dP \mapsto (\det(\widetilde{\pi}(\tilde{p}))^{n-1} \alpha_n \wedge dP.$$

We shall need the holomorphic section $\mathcal{S}(s)$ of $\mathbf{L}$, introduced in Lemma 2.1.

Theorem 2.4 *The map*

$$\begin{array}{rcl} \mathcal{E}^{(0,\frac{n(n-1)}{2})}(\widetilde{\Omega}') & \mapsto & \mathcal{E}^{n(n+1)}(\widetilde{\Omega}') \\ \beta & \mapsto & \mathcal{S} \cdot \beta \wedge \alpha_n \wedge dP \end{array}$$

induces a well-defined map

$$H^{(0,\frac{n(n-1)}{2})}(\Omega'', \mathbf{L}^{1-2n}) \mapsto H_{DR}^{n(n+1)}(\Omega', S).$$

Definition 2.1 *If* $[\beta] \in H^{(0,\frac{n(n-1)}{2})}(\Omega'', \mathbf{L}^{1-2n})$, *then the form*

$$\nu_*(\mathcal{S} \cdot \beta \wedge \alpha_n \wedge dP)$$

is a closed holomorphic form of top degree on $\mathbf{C}M$, *with values in* S, *hence can be written in the form* $\Phi(P)dP$, $\Phi(P) \in S$ *on* Ω. *The map* ν_* *is the integration along fibers, i.e.*

$$\Phi(P)dP = \int_{\nu^{-1}(P)} \mathcal{S} \cdot \beta \wedge \alpha_n \wedge dP.$$

The function Φ *defined in such a way will be called the Penrose transform of* β *and it will be denoted by* $\mathcal{P}(\beta)$.

Theorem 2.5 *The Penrose transform* $\Phi(P) = \mathcal{P}(\beta)(P)$ *is a holomorphic function on the domain* Ω *and it satisfies the (complex) Dirac equation there.*

2.5 Integral formulae

We shall need to use a special type of integral formulae for an inverse Penrose transform for solutions of (complex) Dirac equation.

Using them, it is possible to express the value of a solution in a point $P \in \mathbf{C}M$ using its value (and the values of its derivatives) on a suitable contour of integration *inside* the complex null cone $\mathbf{C}N_P$ of the point P.

These integral formulae are deduced from integral formulae used in Clifford analysis for solutions of Dirac equation ([2],[9]). The description of the Leray residue and Leray cobord can be found, e.g., in [10].

Theorem 2.6 *Let* Φ *be a solution of the (complex) Dirac equation on a null-convex domain* $\Omega \subset \mathbf{C}M$ *and let* $P \in \Omega$. *Let us suppose that the space* $\mathbf{C}M$ *is imbedded into the Clifford algebra* $\mathbf{C}_{2n}$ *and let us consider a Clifford algebra valued form* $DQ = \sum_{i=1}^{2n}(-1)^{i+1}e_i dQ_1 \wedge \ldots \wedge d\hat{Q}_i \wedge \ldots \wedge dQ_{2n}$.

The form

$$\omega = \omega(P,Q;\Phi) = \\ = \frac{1}{A_{2n-1}}\left\{\frac{Q-P}{|Q-P|^{2n}}DQ \cdot \Phi(Q)\right\},$$

where A_l *denotes the area of* l*-dimensional sphere, is a well-defined spinor valued* $(2n-2)$*-form on* $\Omega \setminus \mathbf{C}N_P$.

Then for every $(2n-1)$*-dimensional cycle* γ *in* $\mathbf{C}N_P \cap \Omega$ *we have the formula*

$$\mathrm{Ind}_{\delta\gamma} P \cdot \Phi(P) = 2\pi \mathrm{i} \int_\gamma \mathrm{Res}\,\omega, \tag{4}$$

where Res ω *denotes the Leray residue of the form* ω *and* $\delta\gamma$ *is the Leray cobord of the cycle* γ.

2.6 Surjectivity

Let $\Omega \subset \mathbf{C}M$ and

$$\mathcal{M} := \{[P,Q] \in \Omega \times \Omega | |P-Q|^2 = 0\}$$

To get back the twistor form representing the field Φ and to prove that the Penrose transform is surjective, we have to choose a map

$$\psi : \Omega \times ISt_n \mapsto (\mathcal{M} \setminus \Delta) \times ISt_n.$$

It has the following geometrical meaning: for any point

$$\mathcal{Z} = [Z^0; Z^1, \ldots, Z^n] \in \Omega \times ISt_n$$

we want to choose a point $\tilde{Z^0} \neq Z^0, \tilde{Z^0} \in \Omega$ in the α-plane spanned by vectors $\{Z^0, \ldots, Z^n\}$. So we are looking for a map ψ satisfying the following conditions:

$$\psi(Z) = [Z^0, \tilde{Z^0}; Z^1, \ldots, Z^n]$$

such that $\tilde{Z^0} \in \operatorname{span}\{Z^0, \ldots, Z^n\}, \tilde{Z^0} \neq Z^0, \tilde{Z^0} \in \Omega$. Such a map always exists.

Given a solution Φ of the complex Dirac equation on $\Omega \subset \mathbf{C}\mathbf{M}$, we can reconstruct the corresponding form on the twistor space $\Omega.''$

We shall first consider b the form ω (given by the Cauchy integral formula, see section 2.5) and its Leray residuum Res ω, which is a $(4n-2)$-form on $\mathcal{M} \setminus \Delta$.

If π denotes the natural projection from $(\mathcal{M} \setminus \Delta) \times ISt_n$ onto $(\mathcal{M} \setminus \Delta)$ and Π denotes the projection to the corresponding pure spinors, defined in Sect.2.2, then the form $\Pi \circ \phi^* \pi^*(\operatorname{Res}\omega) \wedge \tau_n$ represents a well-defined cohomology class in the (de Rham) cohomology group $H^{n(n+1)}_{DR}(\widetilde{\Omega}', S)$ (the form τ_n was defined in Sect.2.3.) . The cohomology class does not depend on the choice of the map ψ with the properties described above.

To show that the Penrose transform is onto, it is necessary to impose some restriction to the domain Ω. More details on these restrictions will be found in [4](see also [5]) We shall call a domain Ω satisfying these restrictions an admissible domain. For example, $\mathbf{C}M$ and Lie balls in $\mathbf{C}M$ are admissible domains.

Theorem 2.7 *Let* $\Omega \subset \mathbf{C}M$ *be an admissible domain, then there exists a map* ψ *such that the form* $\Pi \circ \bar{\psi}^* \pi^*(\operatorname{Res}\omega) \wedge \tau_n$ *belongs to the image of the map* $H^{(0,\frac{n(n-1)}{2})}(\mathbf{T}, \mathbf{L}^{(1-n)})$ *into* $H^{n(n+1)}_{DR}(\tilde{\mathbf{F}}, \mathbf{C})$ *induced by the correspondence*

$$\beta \mapsto \mathcal{S}\mu^*\beta \wedge \alpha_n \wedge dP.$$

Moreover, if β represents the preimage of the form $\Pi \circ \bar{\psi}^* \pi^*(\mathrm{Res}\,\omega) \wedge \tau_n$, *then*

$$\mathcal{P}([\beta]) = \Phi.$$

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