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Let us recall that Euler gave the following integral presentation of the gamma-function

$$\Gamma(a) = \int_0^\infty e^{-t} t^a \frac{dt}{t}$$

and that the Gaussian sum of a multiplicative character A of a finite field $\mathbf{F}_q$ is defined by

$$G(A) = \sum_{t \neq 0} E(t)A(t)$$

where E is the non-trivial additive character of $\mathbf{F}_q$ defined by:

$$E(t) = e^{\frac{2\pi i Tr(t)}{p}}$$

for t element of $\mathbf{F}_q$, where $Tr(t) = t + t^p + \ldots + t^{p^{n-1}}$ is the trace of t in the subfield $\mathbf{F}_p$ of p elements, p prime and $q = p^n$.

Let us set

$$X = \{A : F_q^* \to C^* | A(t_1 t_2) = A(t_1)A(t_2) \text{ for } t_1, t_2 \in F_q^*\}$$

where F^* and C^* denote the multiplicative groups of F_q and of C, the field of complex numbers, respectively. The Gaussian-Sum-function G is defined on X and takes its values in C, or more precisely in the extension field of Q, the field of rational numbers, obtained by adjunction of $e^{\frac{2\pi i}{p}}$ and $e^{\frac{2\pi i}{q-1}}$.

Lemma 1 *Let G be a finite abelian group, $|G|$ the number of its elements, and $A : G \to C^*$ a group morphism, then*

$$\sum_{g \in G} A(g) = \delta(A)|G|,$$

where $\delta(A) = 1$, if A is constant of value 1, and $\delta(A) = 0$, if not.

The proof is easy and well-known: If A is not constant of value 1, then there exists g_0 in G such that $A(g_0) \neq 1$; it follows that

$$\sum_g A(g) = \sum_g A(g_0 g) = A(g_0) \sum_g A(g)$$

so

$$(1 - A(g_0)) \sum_g A(g) = 0,$$

so

$$\sum_g A(g) = 0.$$

The following properties of Gaussian sums are consequences of Lemma 1 :

(P1) For A not constant of value 1, the absolute value of G(A) equals the square-root of q; for A constant of value 1 it is equal to 1.

(P2) For A in X not constant of value 1, we have

$$(G(A))^{-1} = q^{-1} A(-1) G(A^{-1}),$$

or more generally: for every A in X we have

$$G(A)G(A^{-1}) = qA(-1) - (q-1)\delta(A).$$

This can be generalized to

(P3) For A_1 and A_2 in X, we have

$$G(A_1)G(A_2) = b(A_1, A_2)G(A_1 A_2) + (q-1)A_1(-1)\delta(A_1 A_2)$$

where $b(A_1, A_2) = \sum_{t \neq 0,1} A_1(t) A_2(1-t)$ is so the called Jacobi sum.

The analogy between G and Γ appears by considering that

$$E(t_1 + t_2) = E(t_1)E(t_2)$$

for $t_1, t_2 \in F_q$ and

$$e^{-(t_1+t_2)} = e^{-t_1} e^{-t_2}$$

for t_1, t_2 in the real interval from 0 to ∞, and that

$$A(t_1 t_2) = A(t_1) A(t_2)$$

for A in X and t_1, t_2 in F_q^* just as

$$(t_1 t_2)^a = t_1^a t_2^a$$

for t_1, t_2 reals, a in C.

The property (P3) of the Gaussian-sum-function is analogous to the well-known relation between gamma and beta function

$$\Gamma(a_1)\Gamma(a_2) = B(a_1, a_2)\Gamma(a_1 + a_2),$$

where a_1, a_2 are complex numbers of real part greater than zero and

$$B(a_1, a_2) = \int_0^1 t^{a_1-1}(1-t)^{a_2-1} dt$$

is the first Eulerian integral; the multiplication $A_1 A_2$ in the character-group X translates into the addition of the complex numbers a_1 and a_2 just as

$$t^{a_1} t^{a_2} = t^{a_1+a_2}$$

for t a real number and

$$A_1(t)A_2(t) = (A_1 A_2)(t)$$

for t in F_q.

The classical First Barnes' Lemma

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \Gamma(a_1+s)\Gamma(a_2-s)\Gamma(a_3+s)\Gamma(a_4-s)ds = \frac{\Gamma(a_1+a_2)\Gamma(a_2+a_3)\Gamma(a_3+a_4)\Gamma(a_4+a_1)}{\Gamma(a_1+a_2+a_3+a_4)}$$

(see for instance [6] for the hypothesis concerning the complex numbers $\{a_k, k = 1,2,3,4\}$ and the path of integration) translates into the following identity for Gaussian sums, for which we will give an easy direct proof.

Proposition 1 *For A_1, A_2, A_3, A_4 in X we have*

$$\frac{1}{q-1}\sum_{A\in X} G(A_1 A)G(A_2 A^{-1})G(A_3 A)G(A_4 A^{-1}) =$$

$$= q(q-1)(A_1 A_3)(-1)\delta(A_1 A_2 A_3 A_4) + \frac{G(A_1 A_2)G(A_2 A_3)G(A_3 A_4)G(A_4 A_1)}{G(A_1 A_2 A_3 A_4)}$$

Proof: Using the property (P3) we get

$$G(A_1A)G(A_2A^{-1}) = G(A_1A_2)b(A_1A, A_2A^{-1}) + (q-1)(A_1A)(-1)\delta(A_1A_2)$$

and

$$G(A_3A)G(A_4A^{-1}) = G(A_3A_4)b(A_3A, A_4A^{-1}) + (q-1)(A_3A)(-1)\delta(A_3A_4);$$

therefore the lefthand side of the identity becomes

$$\frac{1}{q-1}(S_1 + S_2 + S_3 + S_4)$$

with

$$S_1 = G(A_1A_2)G(A_3A_4)\sum_{A\in X} b(A_1A, A_2A^{-1})b(A_3A, A_4A^{-1}),$$

$$S_2 = (q-1)\delta(A_1A_2)A_1(-1)G(A_3A_4)\sum_{A\in X} A(-1)b(A_3A, A_4A^{-1}),$$

$$S_3 = (q-1)\delta(A_3A_4)A_3(-1)G(A_1A_2)\sum_{A\in X} A(-1)b(A_1A, A_2A^{-1}),$$

$$S_4 = (q-1)^3\delta(A_1A_2)\delta(A_3A_4)(A_1A_3)(-1);$$

now we compute the summation term in S_1 :

$$\sum_{A\in X} b(A_1A, A_2A^{-1})b(A_3A, A_4A^{-1}) = \sum_{s\neq 0,1;t\neq 0,1} A_1(s)A_2(1-s)A_3(t)A_4(1-t)\sum_A A(\frac{st}{(1-s)(1-t)})$$

and using a dual form of Lemma 1 we get

$$\sum_A A\left(\frac{st}{(1-s)(1-t)}\right) = \left\{\begin{array}{ll} q-1 & \text{if } st = (1-s)(1-t) \\ 0 & \text{otherwise} \end{array}\right\} = \left\{\begin{array}{ll} q-1 & \text{if } t = 1-s \\ 0 & \text{otherwise} \end{array}\right\}$$

therefore we get

$$\frac{1}{q-1}(S_1+S_2+S_3+S_4) = G(A_1A_2)G(A_3A_4)\sum_{s\neq 0,1}(A_1A_4)(s)(A_2A_3)(1-s)+s_2+s_3+s_4,$$

where

$$s_2 = \delta(A_1A_2)A_1(-1)G(A_3A_4)\sum_{A\in X, s\neq 0,1} A(-1)A_3(s)A_4(1-s)A(\frac{s}{1-s})$$

$$s_3 = \delta(A_3A_4)A_3(-1)G(A_1A_2) \sum_{A\in X, s\neq 0,1} A(-1)A_1(s)A_2(1-s)A(\frac{s}{1-s})$$

$$s_4 = (q-1)^2\delta(A_1A_2)\delta(A_3A_4)(A_1A_3)(-1);$$

using the fact that $\frac{-s}{1-s} \neq 1$ we see that

$$\sum_{A\in X} A(\frac{-s}{1-s}) = 0$$

applying a dual version of Lemma 1; therefore $s_2 = s_3 = 0$; inverting the property (P3) we get

$$b(A_1A_4, A_2A_3) = \frac{G(A_1A_4)G(A_2A_3)}{G(A_1A_2A_3A_4)} + (q-1)\delta(A_1A_2A_3A_4)(A_1A_4)(-1)$$

and substituting the summation in the first term S_1 by this expression we obtain altogether

$$\frac{1}{q-1}(S_1 + S_2 + S_3 + S_4) = \frac{G(A_1A_2)G(A_2A_3)G(A_3A_4)G(A_4A_1)}{G(A_1A_2A_3A_4)}$$

$$+(q-1)\delta(A_1A_2A_3A_4)(A_1A_4)(-1)G(A_1A_2)G(A_3A_4)$$

$$+(q-1)^2\delta(A_1A_2)\delta(A_3A_4)(A_1A_3)(-1);$$

property (P2) allows easily to see that this expression is equal to the righthand side of the announced identity.

This simple and elementary proof seems to be new; at least no reference to it is known to the author who thanks Patrick Solé and Fréderic Testard for helpful comments during her seminar talks on the subject at Nice.

A different proof, using Mellin-transforms, has been found by Patrick Solé and the author, and has been adapted to the classical case by P.Solé; this seems to constitute a new proof of the classical Barnes' identity avoiding the use of the theorem of residues [2].

For other proofs, historical remarks, related questions and further references see [1, 2, 3, 4].

For elementary background concerning Gaussian sums see [5].

References

[1] Greene J. : Hypergeometric functions over finite fields, Trans.Amer.Math.Soc. 301 (1987), 77-101

[2] Helversen-Pasotto A., Sole P.: On Barnes' first lemma and its finite analoques, 1991, preprint

[3] Helversen-Pasotto A.: L' identité de Barnes pour les corps finis, C.R. Acad.Sci Paris, t.286 (1978), Série A, 297-300

[4] Helversen-Pasotto A.: On the structure constants of certain Hecke algebras, Proc 10th Winter School "Geometry and Physics" Srni, to appear in Rendiconti del Circolo Mat. di Palermo.

[5] Ireland K.,Rosen M.: A classical introduction to modern number theory, Springer-Verlag 1990, New York.

[6] Slater L.J. : Generalized hypergeometric functions, Cambridge University Press 1966.

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