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Gamma-function and Gaussian-sum-function

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Let us recall that Euler gave the following integral presentation of the gamma-function

$$\Gamma(a) = \int_0^\infty e^{-t^a}dt$$

and that the Gaussian sum of a multiplicative character A of a finite field \(F_q\) is defined by

$$G(A) = \sum_{t \neq 0} E(t)A(t)$$

where \(E\) is the non-trivial additive character of \(F_q\) defined by:

$$E(t) = e^{\frac{2\pi i Tr(t)}{\sqrt{p}}}$$

for \(t\) element of \(F_q\), where \(Tr(t) = t + t^p + \ldots + t^{p^{n-1}}\) is the trace of \(t\) in the subfield \(F_p\) of \(p\) elements, \(p\) prime and \(q = p^n\).

Let us set

$$X = \{A : F_q^* \rightarrow C^*| A(t_1t_2) = A(t_1)A(t_2) \text{ for } t_1, t_2 \in F_q^*\}$$

where \(F^*\) and \(C^*\) denote the multiplicative groups of \(F_q\) and of \(C\), the field of complex numbers, respectively. The Gaussian-Sum-function \(G\) is defined on \(X\) and takes its values in \(C\), or more precisely in the extension field of \(Q\), the field of rational numbers, obtained by adjunction of \(e^{\frac{2\pi i}{p}}\) and \(e^{\frac{2\pi i}{q-1}}\).

**Lemma 1** Let \(G\) be a finite abelian group, \(|G|\) the number of its elements, and \(A : G \rightarrow C^*\) a group morphism, then

$$\sum_{g \in G} A(g) = \delta(A)|G|,$$

where \(\delta(A) = 1\), if \(A\) is constant of value 1, and \(\delta(A) = 0\), if not.
The proof is easy and well-known: If $A$ is not constant of value 1, then there exists $g_0$ in $G$ such that $A(g_0) \neq 1$; it follows that

$$
\sum_g A(g) = \sum_g A(g_0 g) = A(g_0) \sum_g A(g)
$$

so

$$(1 - A(g_0)) \sum_g A(g) = 0,
$$

so

$$
\sum_g A(g) = 0.
$$

The following properties of Gaussian sums are consequences of Lemma 1:

(P1) For $A$ not constant of value 1, the absolute value of $G(A)$ equals the square-root of $q$; for $A$ constant of value 1 it is equal to 1.

(P2) For $A$ in $X$ not constant of value 1, we have

$$(G(A))^{-1} = q^{-1} A(-1) G(A^{-1}),$$

or more generally: for every $A$ in $X$ we have

$$G(A) G(A^{-1}) = q A(-1) - (q - 1) \delta(A).$$

This can be generalized to

(P3) For $A_1$ and $A_2$ in $X$, we have

$$G(A_1) G(A_2) = b(A_1, A_2) G(A_1 A_2) + (q - 1) A_1(-1) \delta(A_1 A_2)$$

where $b(A_1, A_2) = \sum_{t \neq 0, 1} A_1(t) A_2(1 - t)$ is so the called Jacobi sum.

The analogy between $G$ and $\Gamma$ appears by considering that

$$E(t_1 + t_2) = E(t_1) E(t_2)$$

for $t_1, t_2 \in F_q$ and

$$e^{-(t_1 + t_2)} = e^{-t_1} e^{-t_2}$$

for $t_1, t_2$ in the real interval from 0 to $\infty$, and that

$$A(t_1 t_2) = A(t_1) A(t_2).$$
for $A$ in $X$ and $t_1, t_2$ in $F_q^*$ just as

$$(t_1t_2)^a = t_1^a t_2^a$$

for $t_1, t_2$ reals, $a$ in $C$.

The property (P3) of the Gaussian-sum-function is analogous to the well-known relation between gamma and beta function

$$\Gamma(a_1)\Gamma(a_2) = B(a_1, a_2)\Gamma(a_1 + a_2),$$

where $a_1, a_2$ are complex numbers of real part greater than zero and

$$B(a_1, a_2) = \int_0^1 t^{a_1-1}(1 - t)^{a_2-1}dt$$

is the first Eulerian integral; the multiplication $A_1A_2$ in the character-group $X$ translates into the addition of the complex numbers $a_1$ and $a_2$ just as

$$t^{a_1}t^{a_2} = t^{a_1+a_2}$$

for $t$ a real number and

$$A_1(t)A_2(t) = (A_1A_2)(t)$$

for $t$ in $F_q$.

The classical First Barnes' Lemma

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \Gamma(a_1+s)\Gamma(a_2-s)\Gamma(a_3+s)\Gamma(a_4-s)ds = \frac{\Gamma(a_1+a_2)\Gamma(a_2+a_3)\Gamma(a_3+a_4)\Gamma(a_4+a_1)}{\Gamma(a_1+a_2+a_3+a_4)}$$

(see for instance [6] for the hypothesis concerning the complex numbers $\{a_k, k = 1, 2, 3, 4\}$ and the path of integration) translates into the following identity for Gaussian sums, for which we will give an easy direct proof.

**Proposition 1** For $A_1, A_2, A_3, A_4$ in $X$ we have

$$\frac{1}{q-1} \sum_{A \in X} G(A_1A)G(A_2A^{-1})G(A_3A)G(A_4A^{-1}) =$$

$$= q(q - 1)(A_1A_3)(-1)\delta(A_1A_2A_3A_4) + \frac{G(A_1A_2)G(A_2A_3)G(A_3A_4)G(A_4A_1)}{G(A_1A_2A_3A_4)}$$
Proof: Using the property (P3) we get

\[ G(A_1 A)G(A_2 A^{-1}) = G(A_1 A_2) b(A_1 A, A_2 A^{-1}) + (q - 1) (A_1 A) (-1) \delta(A_1 A_2) \]

and

\[ G(A_3 A)G(A_4 A^{-1}) = G(A_3 A_4) b(A_3 A, A_4 A^{-1}) + (q - 1) (A_3 A) (-1) \delta(A_3 A_4); \]

therefore the lefthand side of the identity becomes

\[ \frac{1}{q - 1} (S_1 + S_2 + S_3 + S_4) \]

with

\[ S_1 = G(A_1 A_2)G(A_3 A_4) \sum_{A \in \mathcal{X}} b(A_1 A, A_2 A^{-1}) b(A_3 A, A_4 A^{-1}), \]

\[ S_2 = (q - 1) \delta(A_1 A_2) A_1 (-1) G(A_3 A_4) \sum_{A \in \mathcal{X}} A (-1) b(A_3 A, A_4 A^{-1}), \]

\[ S_3 = (q - 1) \delta(A_3 A_4) A_3 (-1) G(A_1 A_2) \sum_{A \in \mathcal{X}} A (-1) b(A_1 A, A_2 A^{-1}), \]

\[ S_4 = (q - 1)^3 \delta(A_1 A_2) \delta(A_3 A_4) (A_1 A_3) (-1); \]

now we compute the summation term in \( S_1 \):

\[ \sum_{A \in \mathcal{X}} b(A_1 A, A_2 A^{-1}) b(A_3 A, A_4 A^{-1}) = \sum_{s \neq 0, 1; t \neq 0, 1} A_1 (s) A_2 (1 - s) A_3 (t) A_4 (1 - t) \sum_{A} A \left( \frac{st}{(1 - s)(1 - t)} \right) \]

and using a dual form of Lemma 1 we get

\[ \sum_{A} A \left( \frac{st}{(1 - s)(1 - t)} \right) = \begin{cases} q - 1 & \text{if } st = (1 - s)(1 - t) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} q - 1 & \text{if } t = 1 - s \\ 0 & \text{otherwise} \end{cases} \]

therefore we get

\[ \frac{1}{q - 1} (S_1 + S_2 + S_3 + S_4) = G(A_1 A_2) G(A_3 A_4) \sum_{s \neq 0, 1} (A_1 A_4) (s) (A_2 A_3) (1 - s) s_2 + s_3 + s_4, \]

where

\[ s_2 = \delta(A_1 A_2) A_1 (-1) G(A_3 A_4) \sum_{A \in \mathcal{X}, s \neq 0, 1} A (-1) A_3 (s) A_4 (1 - s) A \left( \frac{s}{1 - s} \right) \]
\[ s_3 = \delta(A_3 A_4) A_3 (-1) G(A_1 A_2) \sum_{A \in \mathcal{X}, A \neq 0, 1} A(-1) A_1(s) A_2(1 - s) A\left(\frac{s}{1 - s}\right) \]

\[ s_4 = (q - 1)^2 \delta(A_1 A_2) \delta(A_3 A_4) (A_1 A_3)(-1); \]

using the fact that \( \frac{s}{1 - s} \neq 1 \) we see that

\[ \sum_{A \in \mathcal{X}} A\left(\frac{-s}{1 - s}\right) = 0 \]

applying a dual version of Lemma 1; therefore \( s_2 = s_3 = 0 \); inverting the property (P3) we get

\[ b(A_1 A_4, A_2 A_3) = \frac{G(A_1 A_4) G(A_2 A_3)}{G(A_1 A_2 A_3 A_4)} + (q - 1) \delta(A_1 A_2 A_3 A_4)(A_1 A_4)(-1) \]

and substituting the summation in the first term \( S_1 \) by this expression we obtain altogether

\[ \frac{1}{q - 1} (S_1 + S_2 + S_3 + S_4) = \frac{G(A_1 A_2) G(A_2 A_3) G(A_3 A_4) G(A_4 A_1)}{G(A_1 A_2 A_3 A_4)} \]

\[ + (q - 1) \delta(A_1 A_2 A_3 A_4)(A_1 A_4)(-1) G(A_1 A_2) G(A_3 A_4) \]

\[ + (q - 1)^2 \delta(A_1 A_2) \delta(A_3 A_4)(A_1 A_3)(-1); \]

property (P2) allows easily to see that this expression is equal to the righthand side of the announced identity.

This simple and elementary proof seems to be new; at least no reference to it is known to the author who thanks Patrick Solé and Frédéric Testard for helpful comments during her seminar talks on the subject at Nice.

A different proof, using Mellin-transforms, has been found by Patrick Solé and the author, and has been adapted to the classical case by P.Solé; this seems to constitute a new proof of the classical Barnes' identity avoiding the use of the theorem of residues [2].

For other proofs, historical remarks, related questions and further references see [1, 2, 3, 4].

For elementary background concerning Gaussian sums see [5].
References


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