Zdzisław Pogoda
Γ-foliations and Weil prolongations


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In the paper we present a construction of the prolongation of a $\Gamma$-foliation on a manifold $X$ to $X^A$ - the Weil prolongation (the $A$-prolongation of the manifold $X$). Moreover, using the construction of Bott-Haefliger of the characteristic classes for $\Gamma$-foliations, we study relationships between the characteristic classes of $\Gamma$-foliations on $X$ and the characteristic classes of Weil prolongations.

1. Basic remarks about Weil prolongations.

Let $A$ be an algebra with 1 over $\mathbb{R}$. We say that $A$ is local if it is associative, commutative and of finite dimension over $\mathbb{R}$. Furthermore, in $A$ there exists the unique maximal ideal $m$ such that:

a) $\dim A/m = 1$.

b) there exists a number $h \in \mathbb{N}$ for which $m^{h+1} = 0$.

The smallest such $h$ will be called the height of $A$. One can prove ([6]) that any local algebra is of the form $\mathbb{R}[p]/a$, where $\mathbb{R}[p] = \mathbb{R}[X_1, \ldots, X_p]$ is the algebra of all formal power series of $p$ indeterminantes and $a$ an ideal of $\mathbb{R}[p]$ such that

$$\dim \mathbb{R}[p]/a < \infty$$

Let $A$ be a local algebra with the maximal ideal $m$ and $C^\infty(M)$ be the space of $C^\infty$ functions on a manifold $M$. Let $\varphi$ and $\psi$ be two $C^\infty$ maps from $\mathbb{R}^p$ to $M$. We say that these maps are $A$-equivalent if

$$\xi_A(\tau(\varphi \circ \varphi)) = \xi_A(\tau(\varphi \circ \psi)) \quad \text{for any } f \in C^\infty(M)$$

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where \( r = r_p \) is a map of the form

\[
\tau : C^\infty(R^p) \rightarrow R[p]
\]

\[
\tau(g) = \sum_{y \in N^p} \frac{1}{y!} [D^y g](0) X^y
\]

and \( \xi_A \) is the canonical projection of \( R[p] \) on \( A \). The equivalence class of \( \varphi \) in this relation we denote by \( [\varphi]_A \). By \( M^A \) we denote the set of all equivalence classes of \( C^\infty \) maps \( \varphi : R^p \rightarrow M \). We have the natural projection \( \pi : M^A \rightarrow M \) defined by

\[
\pi_A([\varphi]_A) = \varphi(0)
\]

The structure of a manifold on a \( M^A \) we introduce in a natural way ([5], [6]). If \( F : M \rightarrow N \) is a \( C^\infty \)-map., then we define \( F^A : M^A \rightarrow N^A \) by the formula

\[
F^A([\rho]_A) = [F \circ \rho]_A \quad \text{for } [\rho]_A \in M^A
\]

The correspondence \( M \rightarrow M^A \) is a functor which has many important properties (see [5], [6]).

The following proposition gives a topological relation between a manifold \( M \) and its \( A \)-prolongation \( M^A \).

**Proposition.** 1. If \( A \) is a local algebra, then \( M \) and \( M^A \) have the same homotopy type.

**Proof.** Denote by \( i \) the canonical imbedding of \( M \) in \( M^A \) defined by the formula

\[
i(x) = [\gamma_x]_A \quad \text{where } \gamma_x : R^p \rightarrow M, \gamma_x(t) = x \quad \text{for each } t \in R^p.
\]

Now we define a map

\[
F : M^A \times R \rightarrow M^A
\]

\[
F([\varphi]_A, s) = [\varphi_s]_A
\]

where \( [\varphi_s]_A \in M^A \) is represented by a map \( \varphi_s \) and

\[
\varphi_s(t) = \varphi((1-s)t) \quad \text{for } t \in R^p
\]

The map \( F \) is, of course, continuous, and

\[
F|_{M^A \times \{0\}} = id_{M^A} \quad F|_{M^A \times \{1\}} = i \circ \pi_A
\]

Q.E.D.

Immediately, we have the following
Corollary. 1. If $A$ is a local algebra, then the de Rham cohomology complexes $H^*(M)$ and $H^*(MA)$ are canonically isomorphic.

2. $A$-prolongations of pseudogroups and foliations.

Let $\Gamma$ be a pseudogroup of local diffeomorphisms of a manifold $M$. For any $g \in \Gamma$ we denote by $O_g$ a family of local diffeomorphisms of $MA$, which cover $g$. Then the set

$$A\Gamma = \bigcup_{g \in \Gamma} O_g$$

is a pseudogroup of local diffeomorphisms of $MA$.

Before we define the $A$-prolongation of a foliation, we recall a definition of a foliation, which we shall use. Let $M$ be a differentiable manifold and $\Gamma$ a pseudogroup of diffeomorphisms acting transitively on $M$. Suppose, that $A\Gamma$ is a transitive Lie pseudogroup.

Actually we shall consider $M = \mathbb{R}^n$ and $\Gamma$ a pseudogroup of local diffeomorphisms of $\mathbb{R}^n$.

To define a $\Gamma$-foliation on a manifold $X$ we need the following data:
1) an open covering $\{U_i\}_{i \in I}$ of $X$.
2) a family $\mathcal{F}$ of submersions ("local projections") $f_i : U_i \to M$,
3) a family of local diffeomorphisms $g_{ij} \in \Gamma$ such that

$$g_{ij} : f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)$$

and

$$g_{ij} \circ f_j|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$ 

A map $f : X' \to X$ is transverse to $\mathcal{F}$ if the maps $f_i \circ f$ are submersions. In this case the maps $f_i \circ f$ are local projections of a $\Gamma$-foliation on $X'$. This foliation is called the inverse image $f^{-1}\mathcal{F}$ of $\mathcal{F}$ via $f$. The map $f$ is a morphism from $f^{-1}$ to $\mathcal{F}$. We can say that $\Gamma$-foliations form a category $\text{Fol}(\Gamma)$.

Now we shall construct the $A$-prolongation of a $\Gamma$-foliation $\mathcal{F}$.

Proposition. 2. Let $\mathcal{F}$ be a $\Gamma$-foliation on $X$. There exists canonically defined a $A\Gamma$-foliation $\mathcal{F}^A$ such that the correspondence $\mathcal{F} \to \mathcal{F}^A$ is a contravariant functor from $\text{Fol}(\Gamma)$ to $\text{Fol}(A\Gamma)$.

Proof. Let $\{U_i\}_{i \in I}$ be an open covering of $X$, and $\{f_i\}_{i \in I}$ a family of submersions defining the foliation $\mathcal{F}$.
The family
\[ \{ U_i^A = \pi^{-1}_A(U_i) : U_i \in \{ U_i \}_{i \in I} \} \]
is an open covering of \( X^A \). Now we shall define the prolongation \( \mathcal{F}^A \) of \( \mathcal{F} \). As the family of submersions for \( \mathcal{F}^A \) we can take the family \( \{ f_i^A \} \) where \( f_i^A : U_i^A \rightarrow M^A \)
The compatibility condition is fulfilled. Q.E.D.

If \( f : X' \rightarrow X \) is a regular map transversal to \( \mathcal{F} \), then \( f^A : X'^A \rightarrow X^A \) is transversal to \( \mathcal{F}^A \) and
\[ (f^{-1}\mathcal{F})^A = f^A(f^{-1}(\mathcal{F}^A)) \]

Thus we can give the following definition:

**Definition.** 1. Let \( \mathcal{F} \) be a \( \Gamma \)-foliation on \( X \). The \( \mathcal{A}_\Gamma \)-foliation \( \mathcal{F}^A \) on \( X^A \) we call the \( \Lambda \)-prolongation or the Weil prolongation of \( \mathcal{F} \).

Now we shall define a homotopy of foliations. Let \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) be two \( \Gamma \)-foliations on \( X \). We denote by
\[ i_t : X \rightarrow X \times \mathbb{R} \]
\[ z \mapsto (z, t) \]
the canonical inclusion. Two \( \Gamma \)-foliations are homotopic if there exists a \( \Gamma \)-foliation \( \mathcal{F} \) on \( X \times \mathbb{R} \) such that \( i_0 \) and \( i_1 \) are transversal to \( \mathcal{F} \) and
\[ i_0^{-1}\mathcal{F} = \mathcal{F}_0 \quad i_1^{-1}\mathcal{F} = \mathcal{F}_1 \]

The homotopy relation is in natural way, an equivalence relation. Denote by \( Htp_{\Gamma}(X) \) the set of homotopy classes of \( \Gamma \)-foliations on \( X \). If \( f : X' \rightarrow X \) is a morphism in \( \text{Fol}(\Gamma) \), then we obtain the induced map
\[ Htp(f) : Htp_{\Gamma}(X) \rightarrow Htp_{\Gamma}(X') \]

It is easy to prove that \( Htp_{\Gamma}(\bullet) \) is a contravariant functor.
We still have some remarks about homotopy of foliations.

**Proposition.** 3. Let \( A \) be a local algebra. If \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) are two homotopic \( \Gamma \)-foliations on \( X \), then \( \mathcal{F}_0^A \) and \( \mathcal{F}_1^A \) are homotopic.

The proof is easy consequence of definitions and properties of the Weil functor.

On the other hand we can define
Definition. 2. Let $A$ be a local algebra. Two foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ are $A$-homotopic if their $A$-prolongations $\mathcal{F}_0^A$ and $\mathcal{F}_1^A$ are homotopic.

This relation is an equivalence relation. Let $Htp(A)(X)$ be the family of $A$-homotopy classes. As previously, $Htp(A)(\bullet)$ is a contravariant functor. The following simple proposition is true.

Proposition. 4. Let $f_0, f_1 : X' \to X$ be two homotopic maps. Then for a local algebra $A$, the maps $f_0^A$ and $f_1^A$ are homotopic.

3. Characteristic classes of $\Gamma$-foliations and their prolongations.

Now we recall briefly the Bott-Haefliger construction of characteristic classes of $\Gamma$-foliations ([2], [4]). Let $\Gamma$ be a Lie pseudogroup acting transitively on $M$. A vector field on $M$ is called a $\Gamma$-field, if its local one parameter group consists of elements of $\Gamma$. Let $o \in M$ be a fixed point in $M$. The set of $k$-jets at $o$ of $\Gamma$-fields is a vector space denoted by $\Gamma^k$ i.e.

$$\Gamma^k = \{ j^k_0 \varphi : \varphi \in \mathcal{X}_\Gamma(M) \}$$

where $\mathcal{X}_\Gamma(M)$ is the space of $\Gamma$-fields on $M$.

Now $\Gamma = \lim \Gamma^k$ is a Lie algebra called the Lie algebra of formal $\Gamma$-fields.

Let us denote by $\mathcal{A}(\Gamma)$ the inductive limit of the algebras $\mathcal{A}(\Gamma^k)$ of multilinear antisyymmetric forms on $\Gamma^k$. The bracket on $\Gamma$ induces a differential on $\mathcal{A}(\Gamma)$, and we obtain the cohomology groups $H^*(\Gamma)$.

Let

$$J^k_0(\Gamma) = \{ j^k_0 \varphi : \varphi \in \Gamma \}$$

and

$$\Gamma^k_0 = \{ j^k_0 \in J^k_0(\Gamma) : \varphi(0) = 0 \}$$

$\Gamma^k_0$ acts on the right on $J^k_0(\Gamma)$, and $J^k_0(\Gamma)$ is a principal fibre bundle with base $M$ and structure group $\Gamma^k_0$. Take

$$J^\infty_0(\Gamma) = \lim J^k_0(\Gamma)$$

On $J^\infty_0(\Gamma)$ we can introduce a structure of a differentiable manifold: the map $f : X \to J^\infty_0(\Gamma)$ is regular i.e. $C^\infty$ if for any $k$, $\pi_k \circ f$ is regular. where

$$\pi_k : J^\infty_0(\Gamma) \to J^k_0(\Gamma)$$

is the canonical projection.
$J^\infty_0(\Gamma)$ has a structure of a principal fibre bundle with the structure group $\Gamma^\infty_0$. Let $\mathcal{A}(J^\infty_0(\Gamma))$ be an algebra of differential forms on $J^\infty_0(\Gamma)$ defined as

$$\lim \mathcal{A}(J^\infty_0(\Gamma))$$

Then we have the following

**Proposition. 5.** $\mathcal{A}(\Gamma)$ is canonically isomorphic to the algebra of differential forms on $J^\infty_0(\Gamma)$, which are invariant under the action of $\Gamma$. This isomorphism commutes with the differential operator (4).

Now, let $K^*$ be a maximal compact subgroup in $\Gamma^\infty_0$ and let

$$K = \lim K^*$$

Then $\mathcal{A}(\Gamma, K)$ is a subcomplex of $K$-basic forms in $\mathcal{A}(\Gamma)$, and its cohomology group we denote by $H^*(\Gamma, K)$.

The following theorem is true

**Theorem. 1.** Let $\mathcal{F}$ be a $\Gamma$-foliation on $X$. There exists a homomorphism of algebras $\varphi_\mathcal{F} : H^*(\Gamma, K) \rightarrow H^*(X)$ such that if $f : X' \rightarrow X$ is transversal to $\mathcal{F}$ then

$$f^* \circ \varphi_\mathcal{F} = \varphi_{f^{-1}}\mathcal{F}$$

**Definition. 3.** The set $\text{im} \varphi_\mathcal{F}$ is called the set of characteristic classes of a foliation $\mathcal{F}$.

**Proposition. 6.** If $\mathcal{F}_0$ and $\mathcal{F}_1$ are homotopic $\Gamma$-foliations on $X$, then

$$\text{im} \varphi_{\mathcal{F}_0} = \text{im} \varphi_{\mathcal{F}_1}$$

Now we can formulate the main theorem of this paper.

**Theorem. 2.** Let $A$ be a local algebra and $\mathcal{F}_0, \mathcal{F}_1$ two $\Gamma$-foliations on $X$. If $\mathcal{F}_0$ and $\mathcal{F}_1$ are $A$-homotopic, then

$$\text{im} \varphi_{\mathcal{F}_0} = \text{im} \varphi_{\mathcal{F}_1}$$

This theorem is the generalisation of the analogous theorem due to L. A. Cordero in [3]. It is a consequence of the following theorem.
Theorem. 3. Let $\mathcal{F}$ be a $\Gamma$-foliation on $X$, and $A$ a local algebra, then

$$im\varphi_\mathcal{F} = i^*im\varphi_{\mathcal{F}^A}$$

where $i^* = i_X^*$ is the isomorphism induced by the inclusion

$$i : X \to X^A$$

To prove this theorem we use the following technical Lemma:

Lemma. 1. Let $A$ be a local algebra, $\mathcal{F}$ a $\Gamma$-foliation on $X$ and $\mathcal{F}^A$ that of its $A$-prolongation, then

(a) there exists a canonical homomorphism

$$\sigma : H^*(\Gamma, K) \to H^*(A\Gamma^A, K)$$

such that the diagram

$$\begin{array}{ccc}
H^*(A\Gamma^A, K) & \xrightarrow{\varphi_{\mathcal{F}^A}} & H^*(X^A) \\
\sigma \uparrow & & \downarrow i_X^* \\
H^*(\Gamma, K) & \xrightarrow{\varphi_{\mathcal{F}}} & H^*(X) \\
\end{array}$$

is commutative.

(b) there exists a canonical homomorphism

$$\tau : H^*(A\Gamma^A, K) \to H^*(\Gamma, K)$$

such that the diagram

$$\begin{array}{ccc}
H^*(A\Gamma^A, K) & \xrightarrow{\varphi_{\mathcal{F}^A}} & H^*(X^A) \\
\tau \downarrow & & \uparrow (\pi_A)^* \\
H^*(\Gamma, K) & \xrightarrow{\varphi_{\mathcal{F}}} & H^*(X) \\
\end{array}$$

is commutative.

(c)$$\tau \circ \sigma = id_{H^*(\Gamma, K)}$$

Proof. Let $i_M(o) = \bar{o} \in M^A$. For any $k \geq 0$ take

$$\sigma_k : J^k_\bar{o}(\Gamma) \to J^k_\bar{o}(\Gamma)$$
defined in the following way: if \( j^k_o(\mathcal{F}) \in J^k_o(\mathcal{F}) \), where \( \mathcal{F} \) is an element of \( \mathcal{L} \), which cover one \( \mathcal{F} \), then we put

\[
\sigma_k(j^k_o(\mathcal{F})) = j^k_o(\mathcal{F})
\]

It is easy to prove, that \( \sigma_k \) is well defined. This map induces a homomorphism of Lie groups

\[
\sigma_k : \mathcal{L}^k_o \rightarrow \mathcal{L}^k_o
\]

and further we have the morphism of fibre bundles

\[
\begin{array}{ccc}
J^k_o(\mathcal{L}) & \xrightarrow{\sigma_k} & J^k_o(\mathcal{F}) \\
\downarrow & & \downarrow \\
M^\mathcal{L} & \xrightarrow{\kappa_\mathcal{L}} & M
\end{array}
\]

For any \( \mathcal{F} \in \mathcal{L} \) such that \( \mathcal{F} \in \mathcal{O} \), let \( \lambda_\mathcal{F} \) and \( \lambda_f \) be differential transformations of \( J^k_o(\mathcal{L}) \) and \( J^k_o(\mathcal{F}) \) respectively, defined by the left action of \( \mathcal{F} \) and \( f \) respectively. The following equality is true

\[
\lambda_f \circ \sigma_k = \sigma_k \circ \lambda_\mathcal{F}
\]

Further, the induced homomorphism of algebras of differential forms we denote also by \( \sigma_k \)

\[
\sigma_k : \mathcal{A}(J^k_o(\mathcal{L})) \rightarrow \mathcal{A}(J^k_o(\mathcal{F}))
\]

which invariant forms under the action \( \mathcal{L} \) sends to forms invariant under the action \( \mathcal{F} \), and consequently we have got

\[
\sigma : \mathcal{A}(J^\infty_o(\mathcal{L})) \rightarrow \mathcal{A}(J^\infty_o(\mathcal{F}))
\]

which induces (by proposition 5)

\[
\sigma : \mathcal{A}(\mathcal{L}) \rightarrow \mathcal{A}(\mathcal{F})
\]

The mapping \( \sigma \) defines two new homomorphisms, denoted also by \( \sigma \).

\[
\sigma : \mathcal{A}(\mathcal{L}, K) \rightarrow \mathcal{A}(\mathcal{F}, \mathcal{L}, K)
\]

and

\[
\sigma : H^*(\mathcal{L}, K) \rightarrow H^*(\mathcal{F}, \mathcal{L}, K)
\]

For the proof of the commutativity of the diagram, it suffice to prove commutativity of the following diagram
\[ \mathcal{A}(J^k_o(\Lambda)) \xrightarrow{\Lambda \eta} \mathcal{A}(P^k(\mathcal{F}))|_{U_A} \xrightarrow{\Lambda p} \mathcal{A}(U^A) \]
\[ \sigma_k \uparrow \]
\[ \mathcal{A}(J^k_o(\Gamma)) \xrightarrow{\eta} \mathcal{A}(P^k(\mathcal{F})|_U) \xrightarrow{p} \mathcal{A}(U) \]

where \( U \) is an open set in \( X \), \( P^k(\mathcal{F})|_U \) and \( P^k(\mathcal{F}^A)|_{U^A} \) are restrictions to \( U \) and \( U^A \), respectively the fibre bundles of \( k \)-jets of local projections of \( \mathcal{F} \) and \( \mathcal{F}^A \), respectively, \( p \) and \( \Lambda p \) are the homomorphisms induced by local inclusions, and, at last, \( \eta \) and \( \Lambda \eta \) are the maps induced by the identification of \( J^k_o(\Gamma) \) and \( J^k_o(\Lambda) \) with \( P^k(\mathcal{F})|_U \) and \( P^k(\mathcal{F}^A)|_{U^A} \) respectively.

The inclusion
\[ j_U : U \rightarrow P^k(\mathcal{F})|_U \]
we can define in the following way: if \( f_U : U \rightarrow M \) is a local submersion of \( \mathcal{F} \), then for each \( x \in U \)
\[ j_U(x) = j^k_g(f^{-1} \circ f_U) \]
where \( g \in \Gamma \) and \( g(o) = f_U(x) \). The map \( j_{U\Lambda} \) we define in the analogous way.

Let \( \omega \in J^k_o(\Gamma) \), thus
\[ p(\eta(\omega))|_x = \eta(\omega)|_{j^k_g(f^{-1} \circ f_U)} = \omega|_{j^k_g(x)} \]
If \( \tilde{x} = i_U(x) \) then
\[ i^k_o(\Lambda p(\Lambda \eta(\sigma_k(\omega))))|_{\tilde{x}} = \eta(\sigma_k(\omega))|_{j^k_g(f^{-1} \circ f_U)} = \]
\[ = \eta(\sigma_k(\omega))|_{j^k_g(f^{-1} \circ f_U)} = \]
\[ = \sigma_k(\omega)|_{j^k_g(f^{-1} \circ f_U)} = \omega|_{j^k_g(x)} \]

This finishes the proof of the point a).

b) In this case we construct the map \( \tau \). For \( k \geq 0 \) the map
\[ \tau_k : J^{k+r}_o(\Gamma) \rightarrow J^k_o(\Lambda) \]
is defined by the equality
\[ \tau_k(j^{k+r}_o(f)) = j^k_o(f^A) \]
for \( f \in \Gamma \), where \( r \) is the order of the natural bundle \( X \rightarrow X^A \). It is easy to prove that \( \tau_k \) is well defined. This \( \tau_k \) induces a homomorphism denoted also by \( \tau_k \):
\[ \tau_k : \mathcal{A}(J^k_o(\Lambda)) \rightarrow \mathcal{A}(J^{k+r}_o(\Gamma)) \]
Passing to limit, we get
\[ \tau : A(J^\infty_0(4\Gamma)) \to A(J^\infty_0(\Gamma)) \]

Analogously as previously \( \tau \) sends forms invariant under the action of \( 4\Gamma \) into forms invariant under the action of \( \Gamma \) because
\[ \lambda f A \circ \tau_k = \tau_k \circ \lambda f \]
The map \( \tau \) defines a homomorphism
\[ A(4\Gamma) \to A(\Gamma) \]
denoted for convenience also by \( \tau \) and \( \tau_k \) defines a morphism of principal fibre bundles
\[
\begin{array}{ccc}
J^k_0(\Gamma) & \xrightarrow{\tau_k} & J^k_0(4\Gamma) \\
\downarrow & & \downarrow \\
M & \xrightarrow{i_M} & M^A.
\end{array}
\]
Finally we take
\[ \tau : H^*(4\Gamma, K) \to H^*(\Gamma, K) \]
The proof of comutativity of the diagram is analogous as in the case of the morphism \( \sigma \).
c) To prove that
\[ \tau \circ \sigma = id_{H^*(4\Gamma, K)} \]
it suffices to remark that the map \( \mu_k = \tau_k \circ \sigma_k \) induces the identity if \( k \to \infty \). This is the consequence of definitions of \( \tau_k \) and \( \sigma_k \). Q.E.D.

Now we can prove the Theorem 3. From the first diagram of the lemma we have
\[ i^*_X(\text{im}\varphi_{\mathcal{F}A}) \supset \text{im}\varphi_{\mathcal{F}} \]
From the second
\[ \text{im}\varphi_{\mathcal{F}A} \subset (\pi_A)^*(\text{im}\varphi_{\mathcal{F}A}) \]
because \( \tau \) is a surjection. Since
\[ i^*_X \circ \pi_A^* = id_{H^*(X)} \]
we have
\[ \text{im}\varphi_{\mathcal{F}} = i^*_X \text{im}\varphi_{\mathcal{F}A} \]
Q.E.D.
References


INSTYTUT MATEMATYKI
UNIWERSYTET JAGIELŁONSKI
UL. REYMONTA 4
30-059 KRAKÓW
POLAND