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NATURAL 2–FORMS ON THE TANGENT BUNDLE OF A RIEMANNIAN MANIFOLD

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ABSTRACT. 1-order natural differential operators from metrics to 2-forms on the tangent bundle are classified. Some natural transformations from TT to T^*T for Riemannian manifolds are described.

Introduction

It is very well known that on the cotangent bundle $q_M : T^*M \to M$ of a manifold M there is the canonical symplectic 2-form given by the exterior differential of the Liouville 1-form. Similar canonical construction on the tangent bundle $p_M : TM \to M$ is not possible. But we can construct the canonical symplectic form on the tangent bundle of a Riemannian manifold. Namely, let (M, g) be a Riemannian manifold and let $h(u) = \frac{1}{2}g(u,u)$, $u \in TM$, be the induced function on TM. The canonical symplectic 2-form on TM is given by

$$\Omega(g) = dd_v h,$$

where d_v denotes the vertical differential, [G].

From the point of view of natural geometry, [N], [KMS], [KJ], Ω is a natural 1-oder differential operator, over the identity of T, from the natural bundle functor of Riemannian metrics to the natural bundle functor of exterior 2-forms on the tangent bundle. 2-forms on the tangent bundle of a Riemannian manifold which arise as the results of natural operators from metrics will be called *natural 2-forms* on TM. The aim of this paper is to give the full classification of natural 2-forms of order 1 on TM. We deduce that the family of natural 2-forms on TM depends on some smooth functions of one variable.

Kowalski and Sekizawa, [KS], gave the full classification of natural symmetric (0,2)-tensor fields of order 1 on TM which, together with our results, gives the complete classification of natural (0,2)-tensor fields on TM.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Kolář and Radziszewski, [KR], classified all natural transformations $TT^*M \to T^*TM$. They pointed out that there is no natural equivalence $TTM \to T^*TM$. It corresponds to the fact that there is no natural symplectic form on TM. But in the case of Riemannian manifolds metrics admit wide possibility to construct natural transformations of TTM to T^*TM. In Section 3 we use the natural transformations by Kolář and Radziszewski, [KR], and natural (0,2)-tensor fields on TM described in Section 2 to show some families of natural transformations $TTM \to T^*TM$ for Riemannian manifolds.

All manifolds and mappings are assumed to be infinitely differentiable.

1. The canonical example

Let M be a manifold with a Riemannian metric g and (x^i) be local coordinates on M. Then

$$g_x = g_{ij}(x)dx^i \odot dx^j, \quad g_{ij}(x) = g_{ji}(x), \quad \det(g_{ij}(x)) \neq 0.$$

We consider the induced function h on TM, $h(u) = \frac{1}{2}||u||^2 = \frac{1}{2}g_x(u,u)$, $u \in T_xM$. The vertical differential of h is a 1-form on TM with the coordinate extression

$$d_vh(u) = \frac{\partial h(u)}{\partial u^i}dx^i = g_{im}(x)u^m dx^i,$$

where (x^i, u^i) are the induced fibred coordinates on TM. The canonical symplectic 2-form on TM is then defined by

$$\Omega_u(g) = dd_vh(u)$$

with coordinate expression

$$\Omega_u(g) = dd_vh(u) = \partial_i g_{mj}(x)u^m dx^i \wedge dx^j - g_{ij}(x)dx^i \wedge du^j.$$

In what follows we shall write $g_{jk,l}$ instead of $\partial_l g_{jk}(x)$. We shall also use the matrix notation

$$\Omega_u(g) = \begin{bmatrix} (g_{mj,i} - g_{mi,j})u^m & -g_{ij} \\ g_{ij} & 0 \end{bmatrix}. \tag{1.1}$$

Now, we shall give another description of the canonical symplectic form, which will be more convenient for our purposes. Let Γ be the Levi-Civita connection on M, i.e. its Christoffel symbols are given by

$$\Gamma^i_{jk} = \frac{g^{im}}{2}(g_{mj,k} + g_{mk,j} - g_{jk,m}). \tag{1.2}$$

Then for any $u \in TM$ the tangent space T_uTM splits with respect to Γ into the horizontal and the vertical subspaces, i.e.

$$T_uTM = H_u \oplus V_u.$$

The connection Γ defines the isomorphism between the vector spaces T_xM and H_u, $p_M(u) = x$. This isomorphism is called the *horizontal lift* and for $\xi_x \in T_xM$ the horizontal lift will be denoted $\xi^H_u \in H_u$.

The *vertical lift* of a vector $\xi_x \in T_xM$ is a vector $\xi_u^V \in V_u$ such that $\xi_u^V(df) = \xi_x f$ for all $f \in C^\infty M$. Here df is considered as a function on TM, i.e. $df(u) = uf$. The vertical lift defines an isomorphism between T_xM and V_u. Obviously, each vector $\zeta \in T_uTM$ can be written in the form $\zeta_u = \xi_u^H + \eta_u^V$, where $\xi, \eta \in T_xM$ are uniquely determined vectors.

Now we can define a 2-form on TM as follows

$$\begin{aligned} \Omega_u(g)(\xi^H, \eta^H) &= 0, & \Omega_u(g)(\xi^H, \eta^V) &= -g_x(\xi, \eta), \\ \Omega_u(g)(\xi^V, \eta^H) &= g_x(\eta, \xi), & \Omega_u(g)(\xi^V, \eta^V) &= 0. \end{aligned} \tag{1.3}$$

for all $\xi, \eta \in T_xM$. The matrix expression of (1.3) is

$$\Omega_u(g) = \begin{bmatrix} (g_{mj}\Gamma_{ai}^m - g_{mi}\Gamma_{aj}^m)u^a & -g_{ij} \\ g_{ij} & 0 \end{bmatrix}. \tag{1.4}$$

From (1.2) we can easily see that the matrix expressions (1.1) and (1.4) coincide and hence (1.3) defines the canonical symplectic form on TM.

Remark 1.1. From (1.3) we see that the canonical symplectic form $\Omega(g)$ is defined by the construction which is similar to the construction of the horizontal lift, [KS], of a metric to a symmetric (0,2)-tensor field on TM. It is why the construction (1.3) is called the *horizontal lift* of a metric to a 2-form on TM.

2. Natural 2-forms on TM

Let $S_+^2 T^* \subset T^* \otimes T^*$ be the natural bundle functor of Riemannian metrics. The canonical symplectic form described in Section 1 is a natural 1-order operator from $S_+^2 T^* \oplus T$ to $\wedge^2 T^*(T)$ over the identity of T. We shall classify all such 1-order (with respect to metric) operators. It is very well known that such operators are in a bijective corespondence with G_n^2-equivariant mappings from the standard fibre of the bundle functor $J^1(S_+^2) \oplus T$ to the standard fibre of $\wedge^2 T^*(T)$. To determine these equivariant mappings we use the infinitesimal method, [KS], [KJ].

Let us denote $Q = \odot^2 \mathbf{R}^{n*} \times (\odot^2 \mathbf{R}^{n*} \otimes \mathbf{R}^{n*}) \times \mathbf{R}^n$ the standard fibre of $J^1(S_+^2) \oplus T$ and $(g_{ij}, g_{ij,k}, u^i)$ the canonical coordinates on Q. The action of G_n^2 on Q is given by

$$\begin{aligned} \bar{g}_{ij} &= \tilde{a}_i^p \tilde{a}_j^q g_{pq} \\ \bar{g}_{ij,k} &= \tilde{a}_i^p \tilde{a}_j^q \tilde{a}_k^r g_{pq,r} + (\tilde{a}_{ik}^p \tilde{a}_j^q + \tilde{a}_i^p \tilde{a}_{jk}^q) g_{pq} \\ \bar{u}^i &= a_p^i u^p, \end{aligned}$$

where (a_j^i, a_{jk}^i) are the canonical coordinates on G_n^2 and tilde denotes the inverse element.

The fundamental vector fields on Q relative to this action are

$$\xi_p^q(Q) = u^q \frac{\partial}{\partial u^p} - 2g_{ap}\frac{\partial}{\partial g_{aq}} - (\delta_a^q g_{pb,c} + \delta_b^q g_{ap,c} + \delta_c^q g_{ab,p}) \frac{\partial}{\partial g_{ab,c}}, \tag{2.1}$$

$$\xi_p^{qr}(Q) = -g_{ap}\left(\frac{\partial}{\partial g_{aq,r}} + \frac{\partial}{\partial g_{ar,q}}\right). \tag{2.2}$$

Let us denote $S = \mathbf{R}^n \times \Lambda^2 \mathbf{R}^{2n*}$ the standard fibre of $\Lambda^2 T^*(T)$ with the canonical coordinates $(u^i, \begin{pmatrix} u^1_{ij} & u^2_{ij} \\ -u^2_{ji} & u^4_{ij} \end{pmatrix})$, $u^1_{ij} = -u^1_{ji}$, $u^4_{ij} = -u^4_{ji}$. The action of G^2_n on S is given by

$$\begin{aligned}
\bar{u}^i &= a^i_p u^p, \\
\bar{u}^1_{ij} &= \tilde{a}^p_i \tilde{a}^q_j u^1_{pq} + (\tilde{a}^p_i \tilde{a}^q_{jm} - \tilde{a}^p_j \tilde{a}^q_{im}) a^m_r u^r u^2_{pq} + \tilde{a}^p_{im} \tilde{a}^q_{jk} a^m_r a^k_s u^r u^s u^4_{pq}, \\
\bar{u}^2_{ij} &= \tilde{a}^p_i \tilde{a}^q_j u^2_{pq} + \tilde{a}^p_{im} \tilde{a}^q_j a^m_r u^r u^4_{pq}, \\
\bar{u}^4_{ij} &= \tilde{a}^p_i \tilde{a}^q_j u^4_{pq}.
\end{aligned}$$

The fundamental vector fields on S relative to this action are

$$\xi^q_p(S) = u^q \frac{\partial}{\partial u^p} - 2u^1_{ap} \frac{\partial}{\partial u^1_{aq}} - u^2_{pa} \frac{\partial}{\partial u^2_{qa}} - u^2_{ap} \frac{\partial}{\partial u^2_{aq}} - 2u^4_{ap} \frac{\partial}{\partial u^2_{aq}}, \tag{2.3}$$

$$\begin{aligned}
2\xi^{qr}_p(S) = {} & (\delta^q_a u^r u^2_{bp} + \delta^r_a u^q u^2_{bp} - \delta^q_b u^r u^2_{ap} - \delta^r_b u^q u^2_{ap}) \frac{\partial}{\partial u^1_{ab}} - \\
& - (\delta^q_a u^r u^4_{pb} + \delta^r_a u^q u^4_{pb}) \frac{\partial}{\partial u^2_{ab}}.
\end{aligned} \tag{2.4}$$

A mapping $F : Q \to S$ is G^2_n-equivariant iff the corresponding fundamental vector fields are F-related. If F has the coordinate expression

$$u^i = u^i, \quad u^\alpha_{ij} = F^\alpha_{ij}(g_{ab}, g_{ab,c}, u^a), \quad \alpha = 1, 2, 4,$$

then F^α_{ij} have to satisfy the following system of partial differential equations

$$2g_{ap} \frac{\partial F^\alpha_{ij}}{\partial g_{aq}} + (\delta^q_a g_{pb,c} + \delta^q_b g_{ap,c} + \delta^q_c g_{ab,p}) \frac{\partial F^\alpha_{ij}}{\partial g_{ab,c}} - u^q \frac{\partial F^\alpha_{ij}}{\partial u^p} = F^\alpha_{ip} \delta^q_j + F^\alpha_{pj} \delta^q_i, \tag{2.5}$$

$$g_{pa} \left(\frac{\partial F^4_{ij}}{\partial g_{aq,r}} + \frac{\partial F^4_{ij}}{\partial g_{ar,q}} \right) = 0, \tag{2.6}$$

$$g_{pa} \left(\frac{\partial F^2_{ij}}{\partial g_{aq,r}} + \frac{\partial F^2_{ij}}{\partial g_{ar,q}} \right) = F^4_{pj} u^r \delta^q_i + F^4_{pj} u^q \delta^r_i, \tag{2.7}$$

$$2g_{pa} \left(\frac{\partial F^1_{ij}}{\partial g_{aq,r}} + \frac{\partial F^1_{ij}}{\partial g_{ar,q}} \right) = F^2_{ip} u^r \delta^q_j + F^2_{ip} u^q \delta^r_j - F^2_{jp} u^r \delta^q_i - F^2_{jp} u^q \delta^r_i. \tag{2.8}$$

Theorem 2.1. *All G^2_n-equivariant mappings $F : Q \to S$ are given by the formulas*

$$\begin{aligned} F^1_{ij} &= u^b u^c \Gamma^r_{bi}\Gamma^s_{cj}\alpha_{rs} + u^b\Gamma^r_{bj}\beta_{ri} - u^b\Gamma^r_{bi}\beta_{rj} + \gamma_{ij}, \\ F^2_{ij} &= u^b\Gamma^r_{bi}\alpha_{rj} + \beta_{ij}, \\ F^4_{ij} &= \alpha_{ij}, \end{aligned} \tag{2.9}$$

where Γ^i_{jk} are the formal Christoffel symbols and $\alpha_{ij}, \beta_{ij}, \gamma_{ij}$ are functions on Q which are solutions of the following system of differential equations

$$\frac{\partial \zeta_{ij}}{\partial g_{pq,r}} = 0, \tag{2.10}$$

$$2g_{ap}\frac{\partial \zeta_{ij}}{\partial g_{aq}} - u^q\frac{\partial \zeta_{ij}}{\partial u^p} = \zeta_{ip}\delta^q_j + \zeta_{pj}\delta^q_i. \tag{2.11}$$

Moreover, $\alpha_{ij} = -\alpha_{ji}, \gamma_{ij} = -\gamma_{ji}$.

Proof. We have to show that all solutions of (2.5) - (2.8) are of the form (2.9). We contract both sides of (2.6)-(2.8) by g^{pq} and use the cyclic permutation of the indices p, q, r. We get

$$\frac{\partial F^4_{ij}}{\partial g_{pq,r}} = 0, \tag{2.12}$$

$$4\frac{\partial F^2_{ij}}{\partial g_{pq,r}} = F^4_{mj}g^{ma}u^b(\delta^{pgr}_{abi} + \delta^{pgr}_{aib} + \delta^{pgr}_{iab} - \delta^{pgr}_{iba} - \delta^{pgr}_{bia} + \delta^{pgr}_{bai}), \tag{2.13}$$

$$\begin{aligned} 4\frac{\partial F^1_{ij}}{\partial g_{pq,r}} &= F^2_{jm}g^{ma}u^b(\delta^{pgr}_{iba} + \delta^{pgr}_{bia} - \delta^{pgr}_{abi} - \delta^{pgr}_{aib} - \delta^{pgr}_{iab} - \delta^{pgr}_{bai}) + \\ &\quad + F^2_{im}g^{ma}u^b(\delta^{pgr}_{abj} + \delta^{pgr}_{ajb} + \delta^{pgr}_{jab} - \delta^{pgr}_{jba} - \delta^{pgr}_{bja} + \delta^{pgr}_{baj}), \end{aligned} \tag{2.14}$$

which can be rewrite, by using (1.2), in the form

$$\frac{\partial F^2_{ij}}{\partial g_{pq,r}} = F^4_{mj}u^s\frac{\partial \Gamma^m_{si}}{\partial g_{pq,r}}, \tag{2.15}$$

$$\frac{\partial F^1_{ij}}{\partial g_{pq,r}} = F^2_{im}u^s\frac{\partial \Gamma^m_{sj}}{\partial g_{pq,r}} - F^2_{jm}u^s\frac{\partial \Gamma^m_{si}}{\partial g_{pq,r}}. \tag{2.16}$$

Putting $F^4_{ij} = \alpha_{ij}$ and substituting it into (2.15) we get after the integration

$$F^2_{ij} = u^b\Gamma^r_{bi}\alpha_{rj} + \beta_{ij}, \tag{2.17}$$

where $\frac{\partial \beta_{ij}}{\partial g_{pq,r}} = 0$ and substituting (2.17) into (2.16) we get after the integration

$$F^1_{ij} = u^b u^c \Gamma^r_{bi}\Gamma^s_{cj}\alpha_{rs} + u^b\Gamma^r_{bj}\beta_{ri} - u^b\Gamma^r_{bi}\beta_{rj} + \gamma_{ij}, \tag{2.18}$$

where $\frac{\partial \gamma_{ij}}{\partial g_{pq,r}} = 0$. It is easy to see that $\alpha_{ij} = -\alpha_{ji}, \gamma_{ij} = -\gamma_{ji}$ and α_{ij}, β_{ij} and γ_{ij} satisfy (2.11). $\square$

Remark 2.1. From (2.10) and (2.11) it follows that $\alpha_{ij}, \beta_{ij}, \gamma_{ij}$ are the components of (0,2)-tensor fields on M which are given as 0-order natural differential operators from $S^2_+ \oplus T$ to $\otimes^2 T^*$. Such natural tensor fields are called *natural F-metrics*, [KS].

Now we can easily prove

Theorem 2.2. *All natural 2-forms $\Omega(g)$ of order 1 on TM are of the form*

$$\begin{aligned} \Omega_u(g)(\xi^H,\eta^H) &= \gamma_x(\xi,\eta), \quad \Omega_u(g)(\xi^H,\eta^V) = \beta_x(\xi,\eta), \\ \Omega_u(g)(\xi^V,\eta^H) &= -\beta_x(\eta,\xi), \quad \Omega_u(g)(\xi^V,\eta^V) = \alpha_x(\xi,\eta). \end{aligned} \tag{2.19}$$

where α, β, γ are natural F-metrics and moreover γ and α are skew-symmetric.

Proof. It is easy to see that the coordinate expression of (2.19) coincides with (2.9). □

We recall here the classifying theorem for natural 1-order symmetric (0,2)-tensors on TM by Kowalski and Sekizawa, [KS], (see also [KMS]).

Theorem 2.3. *All natural 1-order symmetric (0,2)-tensor fields $G(g)$ on TM are of the form*

$$\begin{aligned} G_u(g)(\xi^H,\eta^H) &= \gamma_x(\xi,\eta), \quad G_u(g)(\xi^H,\eta^V) = \beta_x(\xi,\eta), \\ G_u(g)(\xi^V,\eta^H) &= \beta_x(\eta,\xi), \quad G_u(g)(\xi^V,\eta^V) = \alpha_x(\xi,\eta). \end{aligned} \tag{2.20}$$

where α, β, γ are natural F-metrics and moreover γ and α are symmetric. □

Hence the problem of classifying natural (0,2)-tensor fields of order 1 on TM is reduced to the problem of classifying natural F-metrics. This problem was completely solved by Kowalski and Sekizawa, [KS].

Theorem 2.4. *Let (M,g) be an oriented Riemannian manifold of dimension n. Then all natural F-metrics on M derived from g are given as follows:*

i) For $n=1$, all natural F-metrics are of the form

$$\zeta_u(\xi,\eta) = \mu(||u||^2)g(\xi,\eta), \tag{2.21}$$

where μ is an arbitrary function of $||u||^2 = g(u,u)$.

ii) For $n=2$, all symmetric natural F-metrics are of the form

$$\begin{aligned} \zeta_u(\xi,\eta) = {} & \mu(||u||^2)g(\xi,\eta) + \nu(||u||^2)g(\xi,u)g(\eta,u) + \\ & + \kappa(||u||^2)[g(\xi,u)g(\eta,Ju) + g(\eta,u)g(\xi,Ju)], \end{aligned} \tag{2.22}$$

and all skew-symmetric natural F-metrics are of the form

$$\zeta_u(\xi,\eta) = \lambda(||u||^2)[g(\xi,u)g(\eta,Ju) - g(\eta,u)g(\xi,Ju)], \tag{2.23}$$

where $\mu, \nu, \kappa, \lambda$ are arbitrary functions of $||u||^2$ and J is one of the two canonical almost complex structures on (M,g).

iii) For $n=3$, all symmetric natural F-metrics are of the form

$$\zeta_u(\xi,\eta) = \mu(||u||^2)g(\xi,\eta) + \nu(||u||^2)g(\xi,u)g(\eta,u), \tag{2.24}$$

and all skew-symmetric natural F-metrics are of the form

$$\zeta_u(\xi,\eta) = \lambda(||u||^2)g(\xi\times\eta,u), \tag{2.25}$$

where μ, ν, λ are arbitrary functions of $||u||^2$ and $\xi\times\eta$ is the usual vector product of ξ and η.

iv) For $n>3$, all natural F-metrics are symmetric and are of the form (2.24). □

Remark 2.2. M was supposed to be oriented in Theorem 2.4. In the case of non-oriented Riemannian manifold all natural F-metrics are of the form (2.21), for $n = 1$, and (2.24), for $n \geq 2$.

Remark 2.3. If we combine Theorem 2.2 and Theorem 2.4 we get the following: If $n = 1$, the family of all natural 2-forms depends on one arbitrary function of one variable, for $n = 2$ it depends on six arbitrary functions of one variable, for $n = 3$ on five functions and for $n > 3$ on two arbitrary functions of one variable.

Remark 2.4. In the general case (for oriented Riemannian manifolds if $n = 1$ or $n > 3$, for non-oriented Riemannian manifolds in all dimensions) all natural 2-forms are horizontal lifts of a natural F-metric to a 2-form on TM. The canonical symplectic form is then given for $\zeta = -g$.

Remark 2.5. The restriction of order of our operators is necessary. If we consider higher order operations we get many further natural (0,2)-tensor fields on TM. For instance let (M, ∇) be a manifold with a linear connection ∇. Let R be the Ricci tensor of ∇. Then

$$\begin{aligned} \Omega_u(g)(\xi^H, \eta^H) &= 0, & \Omega_u(g)(\xi^H, \eta^V) &= R_x(\xi, \eta), \\ \Omega_u(g)(\xi^V, \eta^H) &= -R_x(\eta, \xi), & \Omega_u(g)(\xi^V, \eta^V) &= 0. \end{aligned}$$

is a 2-form on TM which is naturally induced from ∇ and is of order 1 with respect to ∇. Hence, if M is a Riemannian manifold and ∇ is the Levi-Civita connection, we get 2-order (with respect to a metric) natural 2-form on TM.

3. Some natural transformations $TT \to T^*T$

Kolář and Radziszewski, [KR], classified all natural transformations of the bundle functor TT^* to T^*T. They pointed out that there is no natural equivalence of TT to T^*T. It is a consequence of different geometrical properties of these bundle functors and it corresponds to the fact that there is no canonical natural symplectic form on TM. In Section 1 we have constructed the canonical natural (with respect to a metric) symplectic form Ω on the tangent bundle of a Riemannian manifold, which gives a natural transformation $S_\Omega : TTM \to T^*TM$. This transformation is in fact a natural differential operator

$$S : S^2_+T^* \oplus TT \to T^*T$$

of order 1 in metrics. In this section we shall give some natural transformations $TT \to T^*T$ for Riemannian manifolds, which are of order 1 with respect to metrics.

First we recall the main result by Kolář and Radziszewski, [KR]. We give two canonical natural transformations $TT^* \to T^*T$. The first is the transformation $s_M : TT^*M \to T^*TM$ by Modugno and Stefani, [MS], which can be described geometrically as follows. Every $A \in TT^*M$ is a vector tangent to a curve $\gamma(t) : \mathbf{R} \to T^*M$ at $t = 0$. If $B \in T_{Tq_M(A)}TM$, then iB is tangent to the curve $\delta(t) : \mathbf{R} \to TM$ over the curve $q_M(\gamma(t))$ on M. $i : TTM \to TTM$ is the canonical involution. Hence we can evaluate $\langle \gamma(t), \delta(t) \rangle$ for every t and the derivative $\frac{d}{dt}|_0 \langle \gamma(t), \delta(t) \rangle =: \sigma(A, B)$ depends on A and B only. This determines a linear map $T_{Tq_M(A)}TM \to \mathbf{R}$, $B \mapsto \sigma(A, B)$, i.e. an element $s_M(A) \in T^*TM$.

The second construction is the following. We have the injection $\kappa_M : T^*M \to T^*TM$ given by the pullback with respect to the projection $p_M : TM \to M$. I.e. $\kappa_M(A)(B) = \langle A, Tp_M(B)\rangle$, $A \in T_x^*M$, $B \in T_uTM$. Then $\kappa_M \circ p_{T^*M} : TT^*M \to T^*TM$ is a natural transformation $TT^*M \to T^*TM$ such that the diagram

$$\begin{tikzcd}
 & TT^*M \arrow[dl, "p_{T^*M}"'] \arrow[d] \arrow[dr, "Tq_M"] & \\
T^*M \arrow[r, "\kappa_M"] \arrow[dr, "q_M"'] & T^*TM \arrow[r, "q_{TM}"] & TM \arrow[dl, "p_M"] \\
 & M &
\end{tikzcd}$$

commutes.

Finally we denote $Y \mapsto (k)_1Y$ and $Y \mapsto (k)_2Y$, $k \in \mathbf{R}$, the scalar multiplications in TT^*M with respect to two vector bundle structures $p_{T^*M} : TT^*M \to T^*M$ and $Tq_M : TT^*M \to TM$, respectively. In the notation $X \in TT^*M$, $p = p_{T^*M}(X) \in T^*M$, $\xi = Tq_M(X) \in TM$ we have, [KR], [KMS],

Theorem 3.1. *All natural transformations of TT^*M to T^*TM are of the form*

$$(F(\langle p,\xi\rangle))_1(G(\langle p,\xi\rangle))_2 s_M(X) + \kappa_M(H(\langle p,\xi\rangle)p), \tag{3.1}$$

where $F(t), G(t), H(t)$ are three arbitrary smooth functions of one variable. □

Let us express (3.1) in coordinates. If (x^i) are local coordinates on M, then we have the induced fibred coordinates (x^i, u^i) on TM, (x^i, u^i, ξ_i, U_i) on T^*TM and (x^i, p_i, ξ^i, π_i) on TT^*M. Then the coordinate expression of (3.1) is

$$\begin{aligned}
u^i &= F(p_m\xi^m)\xi^i, \\
\xi_i &= F(p_m\xi^m)G(p_m\xi^m)\pi_i + H(p_m\xi^m)p_i, \\
U_i &= G(p_m\xi^m)p_i.
\end{aligned} \tag{3.2}$$

The canonical transformation s_M is then given by $F = 1, G = 1, H = 0$, i.e.

$$u^i = \xi^i, \quad \xi_i = \pi_i, \quad U_i = p_i$$

and $\kappa_M \circ p_{T^*M}$ is given by $F = 1, G = 0, H = 1$, i.e.

$$u^i = \xi^i, \quad \xi_i = p_i, \quad U_i = 0.$$

Now we are in the position to describe some natural transformations of TTM to T^*TM for a Riemannian manifolds by using Theorem 3.1. Let us suppose that we have a (0,2)-tensor field ζ on M which defines the mapping $S_\zeta : TM \to T^*M$, over M, by

$$\langle S_\zeta(u), \xi\rangle_x = \zeta_x(\xi, u), \qquad \xi, u \in T_xM. \tag{3.3}$$

Then we can define two families of natural transformations (natural also with respect to ζ) of TTM to T^*TM. The first family is given by the commutativity of the diagram

$$\begin{tikzcd} TTM \arrow[rr, "TS_\zeta"] \arrow[dr, "Z_M"'] & & TT^*M \arrow[dl, "\Sigma_M"] \\ & T^*TM & \end{tikzcd}$$

where Σ_M are the natural transformations from Theorem 3.1. Transformations $Z_M : TTM \to T^*TM$ are over the natural transformation of TM given by the scalar multiplication in TM such that the diagram

$$\begin{tikzcd} TTM \arrow[r, "Z_M"] \arrow[d, "Tp_M"'] & T^*TM \arrow[d, "q_{TM}"] \\ TM \arrow[r, "F(\zeta_{km}\xi^k u^m)"] & TM \end{tikzcd}$$

commutes, where F is the function from Theorem 3.1. The coordinate expression of the family Z_M is

$$\begin{aligned} u^i &= F(\zeta_{km}\xi^k u^m)\xi^i, \\ \xi_i &= F(\zeta_{km}\xi^k u^m)G(\zeta_{km}\xi^k u^m)(\zeta_{im,k}u^m\xi^k + \zeta_{im}\Xi^m) + H(\zeta_{km}\xi^k u^m)\zeta_{im}u^m, \\ U_i &= G(\zeta_{km}\xi^k u^m)\zeta_{im}u^m, \end{aligned}$$

where (x^i, u^i, ξ^i, Ξ^i) are the induced fibred coordinates on TTM.

The second family of natural transformations of TTM to T^*TM is given by the following commutative diagram

$$\begin{tikzcd} TTM \arrow[r, "TS_\zeta"] & TT^*M \arrow[d, "\Sigma_M"] \\ TTM \arrow[u, "i_M"] \arrow[r, "\widetilde{Z}_M"] & T^*TM \end{tikzcd}$$

where i_M is the canonical involution of TTM. The family of natural transformations $\widetilde{Z}_M$ is over the scalar multiplication in TM via the commutative diagram

$$\begin{tikzcd} TTM \arrow[r, "\widetilde{Z}_M"] \arrow[d, "p_{TM}"'] & T^*TM \arrow[d, "q_{TM}"] \\ TM \arrow[r, "F(\zeta_{km}u^k\xi^m)"] & TM \end{tikzcd}$$

The coordinate expression of the family $\widetilde{Z}_M$ is

$$\begin{aligned} u^i &= F(\zeta_{km}u^k\xi^m)u^i, \\ \xi_i &= F(\zeta_{km}u^k\xi^m)G(\zeta_{km}u^k\xi^m)(\zeta_{im,k}u^k\xi^m + \zeta_{im}\Xi^m) + H(\zeta_{km}u^k\xi^m)\zeta_{im}\xi^m, \\ U_i &= G(\zeta_{km}u^k\xi^m)\zeta_{im}\xi^m. \end{aligned}$$

Now if ζ is a natural F-metric from Theorem 2.4 we get two families of natural transformations ot TTM to T^*TM for Riemannian manifolds. These families depend on functions of one variable via the natural F-metric and via Σ_M, where as arguments appear $g(u,u), g(\xi,\xi)$ and $g(u,\xi)$.

The third possibility how to construct natural transformations of TTM to T^*TM is to use a natural lift of a metric to (0,2)-tensor fields on TM. Namely, if $\Omega(g)$ is a natural lift described in Theorem 2.2 or 2.3, then

$$S_{\Omega(g)} : TTM \to T^*TM$$

defined by (3.3) is a natural transformation. All these transformations are over the identity of TM and some of them are contained in the family $\widetilde{Z}_M$.

References

[G] Godbillon G., *Géometrie Différentielle et Mécanique Analytique*, Hermann, Paris, 1969.

[KMS] Kolář I., Michor P. W. and Slovák J., *Natural Operations in Differential Geometry*, to appear.

[KR] Kolář I. and Radziszewski Z., *Natural transformations of second tangent and cotangent functors*, Czechoslovak Math. J., **38**(113)1988, 274–279.

[KS] Kowalski O. and Sekizawa M., *Natural transformations of Riemannian metrics on manifolds to metrics on tangent bundles - a classification*, Bull. Tokyo Gakugei Univ., Sect.IV, **40** (1988), 1–29.

[KJ] Krupka D. and Janyška J., *Lectures on Differential Invariants*, Folia Fac. Sci. Nat. Univ. Purkynianae Brunensis, Brno 1990.

[LM] Libermann P. and Marle Ch., *Symplectic Geometry and Analytical Mechanics*, Reidel Publ., Dortrecht, 1987.

[MS] Modugno M. and Stefani G., *Some results on second tangent and cotangent spaces*, Quaderni dell'Istituto di Matematica dell'Università di Lecce, Q. 16, 1984.

[N] Nijenhuis A., *Natural bundles and their general properties*, Diff. Geom., in honour of K. Yano, Kinokuniya, Tokyo 1972, 317–334.

[T] Terng C. L., *Natural vector bundles and natural differential operators*, Am. J. Math. **100** (1978), 775–828.

[YI] Yano K. and Ishihara S., *Tangent and Cotangent Bundles, Differential Geometry*, M. Dekker, New York, 1973.

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