Jarolím Bureš

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HARMONIC SPINORS ON RIEMANN SURFACES.

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1 Introduction

The Dirac operator belongs to the most important objects in several parts of mathematics and mathematical physics.

From the classical point of view the Dirac operator D is defined on oriented riemannian manifold with a given spin structure and its properties and the dimension of the space of harmonic spinors (solutions of the equation $D\phi = 0$) depend in general on a given riemannian metric as well as on a spin structure. There are several generalization of the Dirac operator. One of them, which is defined for an arbitrary complex hermitian manifold is decribed here.

The comparison of the classical and generalized case is an interesting problem, and comes out from the complex geometry. In the paper a review of the theory of Dirac operator and Harmonic spinors on real and complex manifolds is presented and special attention is paid to the compact 2-dimensional case (compact Riemann surfaces). It is a part of general program of description of spin structures, Dirac operators and Harmonic spinors on different types of manifolds. More results for Riemann surfaces will be published in some of next paper.

2 Harmonic spinors on Riemannian manifolds.

2.1 Spin structures.

Let M be a differentiable manifold, denote by M (M) the set of all possible riemannian metrics on M and C (M) the set of conformal classes of riemannian metrics on M.

Then M (M) can be considered as a subspace of sections $\Gamma(M, S^2(T))$ of symmetric tensor fields on M, and we have the natural projection

$$c : \mathrm{M}(M) \longrightarrow \mathrm{C}(M)$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

which maps a riemannian metric to its conformal class.

Let M be an oriented manifold, let $B^+ \to M$ be the principal fibre bundle of oriented frames on M (with fibre $\mathrm{Gl}^+(n,\mathbf{R})$)

For any riemannian metric $g \in \mathbf{M}$ we have principal fibre bundle P_g of oriented orthonormal frames with respect to g and an embedding

$$P_g \to B^+$$

Spin structure on an oriented Riemannian manifold (M,g) is a principal fibre bundle $\tilde{P}_g$ on M with structural group Spin(n) which is $2-1$ covering of P_g. If $\iota : \tilde{P}_g \to P_g$ is the covering map and $\lambda_0 : Spin(n) \to SO(n)$ the standard covering map, the following diagram is commutative:

$$\begin{array}{ccccc} Spin(n)\times\tilde{P} & \longrightarrow & \tilde{P} & \overset{\tilde{\pi}}{\searrow} & \\ \downarrow\lambda_0\times\iota & & \downarrow & & \\ SO(n)\times P_g & \longrightarrow & P_g & \xrightarrow{\pi} & M \end{array}$$

In other words $\tilde{P}$ is 2-1 covering of P_g, such that after a restriction on fiber over any point $x \in M$ it is isomorphic to the standard map (on each fiber $\iota_x(r.g) = \iota_x(r).\lambda_0(g)$).

LEMMA 2.1 *Let $\tilde{P}_g \to P_g$ be a spin structure on a riemannian manifold (M,g), then there exists uniquely defined 2-1 covering $\tilde{B}^+$ of B^+, a principal fibre bundle on M with structural group $\widetilde{\mathrm{Gl}^+(n,\mathbf{R})}$ and embedding.*

$$\tilde{P}_g \to \tilde{B}^+$$

such that the following diagram is commutative.

$$\begin{array}{ccccc} \tilde{P}_g & \xrightarrow{id} & \tilde{B}^+ & \overset{\tilde{\pi}'}{\searrow} & \\ \downarrow_{2-1} & & & & \\ P_g & \xrightarrow{id} & B^+ & \xrightarrow{\pi'} & M \end{array}$$

Of course, S (M,g) can be empty. The existence of spin structure and cardinality of S (M,g) depend only on the topological properties of M, not on riemannian metrics. The following theorem is well-known:

THEOREM 2.1 *An oriented riemannian manifold admits a spin structure iff its second Stiefel-Whitney class w_2 is zero. Then the number of spin structures is card$H^1(M, \mathbf{Z_2})$.*

DEFINITION 2.1 *A riemannian manifold (M,g) with fixed spin structure is called spin manifold.*

Let S (M,g) be a set of spin structures on a riemannian manifold (M,g). It is an affine $\mathbf{Z_2}$-space with translation group $H^1(M, \mathbf{Z_2})$.

LEMMA 2.2 *Let g, g' be two riemannian metrics on M. Then there exists a natural one-to-one correspondence $S\ (M, g) \leftrightarrow S\ (M, g')$ which is an $\mathbf{Z_2}$-affine map.*

Proof: A spin structure $\tilde{P}_g$ corresponds to the spin structure $\tilde{P}_{g'}$ iff they are included in the same covering $\widetilde{B}^+$ of B^+.

Notation: In the following text the following notation will be used: The elements of S (M,g), will be denoted by small letters, say $s \in$ S (M, g), the corresponding principal spin bundle by $\tilde{P}^s_g$ or $\tilde{P}^s$ and also another objects constructed from s (as Spinor bundle,Dirac operator,etc) will have the index (upper or lower) s.

2.2 Dirac operator.

In this paper we suppose $Spin(n) \subset \mathbf{Cl_n}$ in a canonical way. (see e.g [9]).

To any given spin-structure $\tilde{P}^s$ on M corresponds so called fundamental spinor bundle S_s which is an associated vector bundle to $\tilde{P}^s$ with respect to the representation of the group Spin(n) which is the restriction to Spin(n) of the irreducible representation of Clifford algebra $\mathbf{Cl_n}$.

The classical Dirac operator D^s associated to $s \in \mathbf{S}(M, g)$ is defined using Levi-Civita covariant derivative ∇^g on M. The covariant derivative ∇^g induces connection on $\tilde{P}^s$ and also covariant derivative ∇^s on S_s.

The operator

$$D^s : \Gamma(M, S_s) \to \Gamma(M, S_s)$$

is composition of maps:

$$\Gamma(M, S_s) \xrightarrow{\nabla^s} \Gamma(M, S_s \otimes T^*) \xrightarrow{\hat{g}} \Gamma(M, S_s \otimes T) \xrightarrow{\mu} \Gamma(M, S_s)$$

where $\hat{g} : S_s \otimes T^* \to S_s \otimes T$ is identity times duality induced from riemannian metric g, μ is the Clifford multiplication.

A local expression for D^s using a local orthonormal frame $\{e_1, \ldots, e_n\}$ is

$$D^s = \sum_{i=1}^{n} e_i . \nabla^s_{e_i} .$$

The space of harmonic spinors for the spin structure s is

$$\mathbf{H}^s = \{\xi \in \Gamma(M, S_s); D^s \xi = 0\}.$$

Moreover in the even-dimensional case n = 2m the spinor bundle S_s decomposes with respect of the action of Spin(n) into direct sum of two (called halfspinor) bundles (corresponding to irreducible representations of Spin(n))

$$S_s = S_s^+ \oplus S_s^- .$$

The restriction of the Dirac operator D^s to the half-spinor bundles gives us two operators (called again Dirac operators) D^{s+} and D^{s-}

$$D^{s+} : \Gamma(M, S_s^+) \rightarrow \Gamma(M, S_s^-)$$

and

$$D^{s-} : \Gamma(M, S_s^-) \rightarrow \Gamma(M, S_s^+).$$

We have also two following spaces of harmonic spinors called the space of positive harmonic spinors and the space of negative harmonic spinors:

$$\mathbf{H}^{s+} = \mathrm{Ker} D^{s+}, \mathbf{H}^{s-} = \mathrm{Ker} D^{s-}.$$

2.3 Harmonic spinors.

DEFINITION 2.2 *An oriented manifold M is called spin manifold if M admits spin structures (it is equivalent with $w_2(M) = 0$).*

For any $g \in \mathrm{M}(M)$ and $s \in \mathrm{S}\ (M, g)$ we shall consider
a) the corresponding spinor bundle S_g^s ,
b) the corresponding Dirac operator D_g^s and its space of harmonic spinors $\mathbf{H}_g^s = Ker D_g^s$
and denote

$$h_g^s = \dim \mathbf{H}_g^s$$

Moreover we define $h(M) \doteq sup\{h_g^s; g \in \mathrm{M}(M), s \in \mathrm{S}\ (M, g)\}$. (h(M) $\in$ $\mathbf{Z} \cup \{\infty\}$).

Fix some (let say standard) riemannian metric g_0 on M, denote S $_M$ = S (M, g_0). Then for any riemannian metric $g \in$ M(M) we have from the Lemma 2.2 unique spin structure in S (M, g) which we denote again s.

We shall say that M is of finite spin-type iff h(M) is finite, of null spin- type iff h(M) = 0 .

Moreover we shall say that dimension of harmonic spinors on M depend on the metric iff there exist $s \in$ S $_M$ such that h^s_g depends on g.

If two riemanniam metrics $g, g' \in$ M(M) are conformally equivalent, then for any spin structure s the corresponding spinor bundles are isomorphic, the corresponding Dirac operators are related aas well as the spaces of harmonic spinors. Then we have a theorem:

THEOREM 2.2 *The dimension of harmonic spinors is conformal invariant (depends only on the conformal class of metric).*

For any even dimension we have also the decomposition of spinor spaces into two subspaces given above and the spaces of positive and negative harmonic spinors.

Let us denote h^{s+}_g (h^{s-}_g) the dimension of the space of corresponding positive (negative) harmonic spinors.

In some more concrete cases the notions defined in this and previous section will be shortened in a natural way.

3 The complex spin structure.

3.1 The $Spin^c$ structure.

Let (M,J,$\tilde{g}$) be a complex hermitian manifold which is also spin manifold. Let us consider spin structures on M which correspond to the riemannian metric g related to the hermitian metric $\tilde{g}$.

There is the following question: Are there relations between properties of M as a complex hermitian manifold (the operators $\partial, \bar{\partial}, \bar{partial}^{(}*)$, cohomology etc) and properties of M as a spin manifold (Dirac operator, harmonic spinors etc).

To get an answer (which is strictly positive in the Kaehlerian case) we shall introduce a notion of $Spin^c(n)$-structure on manifold which is in some sense unification of both notions.

Let us denote

$$Spin^c(n) = Spin(n) \times_{\mathbf{Z_2}} U(1)$$

where the action of $\mathbf{Z_2}$on $Spin(n) \times U(1)$ is given by:

$$-1(\lambda, t) = (-\lambda, -t); \ -1 \in \mathbf{Z_2}.$$

There is the the exact sequence og groups

$$0 \to \mathbf{Z_2} \to Spin^c(n) \overset{\xi_0}{\to} SO(n) \times U(1) \to 0$$

where the map ξ_0 is given by the projection, and it has a kernel $\mathbf{Z_2} = \{[1,1],[1,-1]\}$.

The group $Spin^c(n)$ can be identified with a subgroup of $\mathbf{Cl}^{\mathbf{C}}(n)$, namely because Spin(n)$\subset$ $\mathbf{Cl}(n)$ and $U(1) \subset \mathbf{C}$, after "tensoring" we get

$$Spin^c(n) \subset \mathbf{Cl}(\mathrm{n}) \otimes \mathbf{C} = \mathbf{Cl}^{\mathbf{C}}(n).$$

We have the following interesting commutative diagram of groups and homomorphisms.

$$\begin{array}{lcl} U(n) & \searrow^{\iota \times det} & \\ \lambda\downarrow & & \\ Spin^c(2n) & \xrightarrow{\xi_0} & SO(2n) \times U(1) \end{array}$$

where the map λ is defined in the following way (see e.g.[2]): let $g \in U(n)$ let $\{e_1, \ldots, e_n\}$ be the unitary basis where g is diagonalized namely if

$$g = diag(e^{\mathrm{i}\theta_1}, \ldots, e^{\mathrm{i}\theta_n})$$

then

$$\lambda(g) = \prod_{k=1}^{k=n} (cos\frac{1}{2}\theta_k + sin\frac{1}{2}\theta_k.e_kJe_k) \times e^{1/2\mathrm{i}\sum_k \theta_k}.$$

in $Spin^c(2n)$. Moreover we have also the following commutative diagram:

$$\begin{array}{lcl} Spin(2n) \times U(1) & \searrow^{\pi} & \\ \mu\downarrow & & \\ SO(2n) \times U(1) & \xleftarrow{\xi_0} & Spin^c(2n) \end{array}$$

Let (M,g) be an oriented Riemannian manifold, P_g be the principal fibre bundle of oriented orthonormal frames of (M,g).

DEFINITION 3.1 *A $Spin^c(n)$-structure on (M,g) consists of a principal U(1) bundle $\mathcal{P}_{U(1)}$ and a principal $Spin^c(n)$-bundle $\mathcal{P}_{Spin\mathbf{C}(n)}$ with a $Spin^c(n)$- equivariant bundle map*

$$\mathcal{P}_{Spin\mathbf{C}(n)} \xrightarrow{\xi} P_g \times \mathcal{P}_{U(1)}.$$

i.e. $\xi(r.h = \xi(r).\xi_0(h)$ *for any* $r \in \mathcal{P}_{Spin\mathbf{C}(n)}$ *and* $h \in Spin^c(n)$.

An oriented Riemannian manifold with a $Spin^c(n)$- structure is called a $Spin^c(n)$-manifold.

DEFINITION 3.2 *Let M be a $Spin^c(n)$ manifold with a $Spin^c(n)$-structure s. By a $Spin^c(n)$- spinor bundle with respect to s for M we mean a vector bundle S_s associated to a representation of $Spin^c(n)$ by Clifford multiplication*

$$S_s = P^s_{Spin^c(n)} \times_\nu V$$

where V is an irreducible complex $\mathrm{Cl}^{\mathbf{C}}(n)$*-module and* ν *is a restriction of the* $\mathrm{Cl(n)}$*-representation to* $Spin^c(n) \subset \mathrm{Cl}^{\mathbf{C}}(n)$.

The Dirac operator on a $Spin^c(n)$-manifold (with the $Spin^c(n)$-structure s) is a differential operator of the first order

$$D_s : \Gamma(M, S_s) \to \Gamma(M, S_s)$$

given by a composition

$$\Gamma(M, S_s) \xrightarrow{\nabla^g} \Gamma(M, S_s \otimes T^*_C M) \xrightarrow{id\otimes\hat{g}} \Gamma(M, S_s \otimes T_C M) \xrightarrow{m} \Gamma(M, S_s)$$

where ∇^g is a covariant derivative induced from the Levi-Civite connection of g, $\hat{g}$ is an isomorphism between $T_C M$ and $T^*_C M$ given by hermitian metric h related to g and m is the Clifford multiplication restricted to $T_C M$.

THEOREM 3.1 *An orientable manifold M carries a $Spin^c(n)$-structure if and only if the second Stiefel-Whitney class $w_2(M)$ is the mod 2 redution of an integral class. In this case $Spin^c(n)$ -structures on M are parametrized by the elements of*

$$2H^2(M; Z) \oplus H^1(M; \mathbf{Z}_2).$$

Any orientable Riemannian manifold of dimension ≤ 3 admits spin-structure, any orientable Riemannian manifold of dimension ≤ 4 admits $Spin^c(n)$-structure.

EXAMPLE 3.1 *Any spin-structure s on (M,g) determines a canonically defined $Spin^c(n)$-structure $\tilde{s}$. Namely, let $\tilde{P}_g$ be a spin-structure on M, put*

$$\mathcal{P}_{U(1)} \equiv P^0_{U(1)} = M \times U(1)$$

the trivial bundle and

$$\mathcal{P}^s{}_{Spin\mathbf{C}(n)} = \mathcal{P}^s_{Spin(n)} \times_{\mathbf{Z}_2} U(1)$$

EXAMPLE 3.2 *Any complex structure J on (M,g) compatible with the metric g carries canonical $Spin^c(2n)$-structure $\tilde{J}$ in the following way.*

Let us recall, that we have in a canonical way defined principal fibre bundle $\mathcal{P}_{U(n)}$on M consistiong of oriented unitary frames on M. Take

$$\mathcal{P}^J_{Spin\mathbf{C}(2n)} = \mathcal{P}_{U(n)} \times_\lambda Spin^c(2n).$$

and let $\mathcal{P}_{U(1)}$be the determinant bundle of M (i.e. the principal fibre bundle of $\Lambda^{2n}(TM)$).

Then $(\mathcal{P}^J_{Spin\mathbf{C}(2n)}, \mathcal{P}_{U(1)})$ is $Spin^c(2n)$ structure on M.

REMARK 3.1 *We have seen that spin-manifolds and complex manifolds are all $Spin^c(n)$-manifolds. One of manifolds which are $Spin^c(n)$ but neither spin nor complex is the real projective space $\mathbf{P}^{4n+1}(R)$.(see [9]). An example of a manifold which is not $Spin^c(n)$ is SU(3)/SO(3).See([9]).*

For a complex manifold M with its canonical $Spin^c(2n)$-structure, the $Spin^c(2n)$-spinor bundle is

$$S_{\tilde{J}} = \Lambda^*_{\mathbf{C}}TM = \Lambda^{0,*}.$$

If M is kaehlerian, then the corresponding Dirac operator D^J is possible to identify with the operator

$$D^J = \bar{\partial} + \bar{\partial}^*$$

where the operator $\bar{\partial}^*$ is adjoint to $\bar{\partial}$ with respect to the hermitian metric on M.

The Clifford multiplication on $S_{\tilde{J}}$ is given by the relation:

$$\mu(v)(\phi) = v \wedge \phi - v^* \lrcorner \phi$$

for $v \in T_x(M), \phi \in S_{\tilde{J}} = \Lambda^*_{\mathbf{C}}TM$. where v^* is the element of $T^*_x(M)$ which corresponds to v in the duality given by metric on M.

Let us suppose that M is a complex hermitian spin-manifold and denote by K the canonical holomorphic line bundle on M ($K = \Lambda^{n,0}$). From the equality

$$c_1(K)\text{mod}2 \equiv c_1(M)\text{mod}2 = -w_2(M) = 0$$

it follows that $c_1(K)$ is an even number, and there exist holomorphic line bundles L on M with $L^2 = L \otimes L = K$ (so called square roots of K.) The number of square roots L of K is card$(H^1(M, \mathbf{Z}_2)$.

THEOREM 3.2 *Let (M,g,J) be a complex hermitian spin manifold. There is natural 1-1 correspondence between spin structures on (M,g) and square roots L of the canonical line K.*

Proof: The proof follows from the commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{P}^s_{Spin(2n)} \times_{\mathbf{Z}_2} \mathcal{P}_{U(1)}((L^s)^{-1}) & \searrow & \\ \downarrow \cong & & \\ \mathcal{P}^J_{Spin\mathbf{C}(2n)} & \longrightarrow & \mathcal{P}_{SO(2n)} \times \mathcal{P}_{U(1)}((L^s)^{-2} \end{array}$$

where $(L^s)^{-2} = K^{-1}$.

3.2 Spin-structures on complex manifold.

Let on a riemannian manifold (M,g) a complex structure J compatible with the riemannian metric g and a spin-structure s be given.

We try to compare two $Spin^c(2n)$ structures on M, namely $\tilde{J}$ and $\tilde{s}$, their spinor bundles and Dirac operators.

For any spin-manifold , the bundle $S_{\tilde{s}}$ is the usual complex spinor bundle.

THEOREM 3.3 *The relation between the spinor bundle S_J for M as a complex manifold and the spinor bundle S_s for M as a spin manifold is*

$$S_J = S_s \otimes (L^s)^{-1}$$

and

$$S_s = S_J \otimes L^s$$

where L^s is the square root which corresponds to the spin structure s.

If moreover (M,g) is spin-manifold and s spin structure on it with the corresponding line bundle L^s (holomorphic square root of K) then the associated Dirac operator can be identified with

$$D_{\tilde{s}} = \bar{\partial} + \bar{\partial}^* : \Lambda^{0,*} \otimes L_s \to \Lambda^{0,*} \otimes L^s$$

It is the standard operator on (0,*)-forms with values in the holomorphic line bundle L^s.

The splitting of the spinor bundle

$$S_J = S_J^+ \oplus S_J^-$$

corresponds to the decomposition to the even and odd forms

$$\Lambda^{0,*} = \Lambda^{0,even} \oplus \Lambda^{0,odd}$$

and the Dirac operators defined above can be (after restriction) taken as operators from even (odd) forms into odd (even) forms.

The solutions of the equation $D_s\phi = 0$ are called harmonic spinors (with respect to the $Spin^c(n)$-structure s) on M. Let us denote:

$$\mathbf{H} = \{\phi \in \Gamma(M, S_s), D_s\phi = 0\}$$

$$\mathbf{H}^+ = \{\phi \in \Gamma(M, S_s^+), D_s\phi = 0\}$$

$$\mathbf{H}^- = \{\phi \in \Gamma(M, S_s^-), D_s\phi = 0\}.$$

Let us demote by **O** (L) the sheaf of germs of holomorphic sections of the holomorphic bundle L.

Then we have a theorem which gives a summary of preceding results:

THEOREM 3.4 *Let (M,J,g) be a Kaehler spin-manifold ,then*

- *Spin structures on (M,g) are in one-to-one correspondence with holomorphic line bundles L on (M,J) which are square-roots of the canonical line bundle K_M. (i.e L satisties $L \otimes L = K$.)*
- *For the corresponding spaces of (odd and even) harmonic spinors there are isomorphisms*

$$\mathbf{H}^+ \simeq H^{even}(M, \mathbf{O}(L))$$

$$\mathbf{H}^- \simeq H^{odd}(M, \mathbf{O}(L))$$

4 Spin structures on Riemann surfaces.

A Riemann surface M is a one dimensional complex manifold. We restrict ourselves only on compact Riemann surfaces, so in the following text we use the notion Riemann surface only for compact Riemann surface.

Let g be a Riemannian metric on M compactible with complex structure J on M. Then (M, J, g) is Kaehler manifold. Every two metrics compatible with J are conformally equivalent.

There is one-to-one correspondence between the set of complex structures on a two dimensional manifold M and the set C (M) of conformal classes of metrics on M.

Every oriented 2-dimensional manifold M is a spin manifold since $w_2(M) = 0$, so that any Riemannian surface is also spin manifold.

REMARK 4.1 : *If (M, g) is an oriented Riemannian manifold, then there exists a uniquely defined complex structure J on M such that (M, J) is Riemann surface and g is its compactible metric.*

LEMMA 4.1 : *Let M be of genus* **g**. *There are*

$$N(\mathbf{g}) = card H^1(M, \mathbf{Z_2}) = 2^{2\mathbf{g}}$$

different spin structures on M.

Let K be a canonical line bundle of Riemann surface M. From the Theorem3.4 it follows:

THEOREM 4.1 :

(a) Spin structures on M are in one-to-one correspondence with holomorphic line bundles L on M such that $L^2 = K$.

(b) There are isomorphisms

$$\mathbf{H}_L^+ \simeq H^0(M, O(L))$$

$$\mathbf{H}_L^- \simeq H^1(M, O(L))$$

Let us denote $h^i = dim H^i(M, \mathrm{O}(L))$, then from the Serre duality follows that

$$H^0(M, \mathrm{O}(L)) \simeq H^1(M, \mathrm{O}(L))$$

and $h^0 = h^1$.

Fix for any Riemannian surface M a riemannian metric and spin structure. Then we can associate to every element $\alpha \in H^1(M, \mathbf{Z_2})$

a) spin structure s_α on (M, g) (a principal Spin(2)-bundle P_α which is 2-1 covering of B_g.

b) holomorphic line bundle L_α on (M,g) satisfying $L_\alpha^2 := L\alpha \otimes L_\alpha = K_M$.

The element $\alpha \in H^1(M_{\mathbf{g}}, \mathbf{Z_2})$ then determines spin structure for any metric g' on M.

Let us denote $h^0_{\alpha,g} = \dim H^0(M, O(L_\alpha)$, then

$$h_g^{s(\alpha)} = 2.h^0_{\alpha,g}$$

THEOREM 4.2 *For any metric g and $\alpha \in H^1(M, \mathbf{Z_2})$ on a surface of genus* **g** *we have an equality:*

$$h^0_{\alpha,g} \leq [\frac{1}{2}(\mathrm{g}+1)]$$

THEOREM 4.3 ([1]) *Let M be a Riemann surface of genus* **g**. *Then for any metric g there is precisely $2^{\mathrm{g}-1}(2^{\mathrm{g}}+1)$ spin structures α on M for which $h^0_{\alpha,g}$ is even number. We call these spin structures even spin structures on M.*

For the other $2^{\mathrm{g}-1}(2^{\mathrm{g}}-1)$ spin structures α on M $h^0_{\alpha,g}$ is odd number. These spin structures are called odd spin structures.

From these two theorems immediately follows :

THEOREM 4.4 ([8]) *If the genus* **g** *of M is less then 3 then the dimension of the space of harmonic spinors on M is independent on the metric.*

Proof: We shall study the situation for different these genera separately.

0) For **g**= 0 , $M = P^1(\mathbf{C}) = S^2$ which is simple connected. There exists just one spin structure on M which does not admits nonzero harmonic spinors. So $h^0 = h^1 = 0$.

1) For **g** = 1, there exist altogether 4 spin-structures on M, namely 3 even spin-structures which does not admit nonzero harmonic spinors ($h^0 = 0$) one odd spin structure (the trivial one) has space of positive harmonic spinors one dimensional.

REMARK 4.2 *Riemann surfaces of genus* **g**= *1 can be represented by regular cubic curves in $P^2(\mathbf{C})$, they are also known as the elliptic curves.*

2) For **g** = 2 there exist altogether 16 spin structures on M, there are 10 even spin structures with no nonzero harmonic spinors ($h^0 = 0$) and also 6 odd spin-structures with one-dimensional space of positive harmonic spinors ($h^0 = h^1 = 1$).

REMARK 4.3 *Riemann surfaces of genus* **g**= *2 are all hyperelliptic.*

For the case of genus $\mathbf{g} \geq 3$ we have the following results.

THEOREM 4.5 ([8]) *The dimension of the space of harmonic spinors on a Riemann surface of genus* $\mathbf{g} \geq 3$ *varies with the choice of metric.*

THEOREM 4.6 ([8]) *If (M,J) is hyperelliptic, there exists spin structure* α *on (M,J) such that* $h^0_{\alpha,g} = [\frac{1}{2}(\mathbf{g}+1)]$. *Moreover if* $\mathbf{g}$ *is even, there are just 2(*$\mathbf{g}$ *+1) such structures.*

THEOREM 4.7 ([10]) *If there exist on Riemann surface spin structures* α *such that* $h^0_{\alpha,g} = [\frac{1}{2}(\mathbf{g}+1)]$ *then (M,J) is one of the following types*

(a) hyperelliptic

(b) $\mathbf{g} = 4$

(c) $\mathbf{g} = 6$

In the nonhyperelliptic case of $\mathbf{g} = 4$ *and 6 , there is at most one spin structure having* $h^0_{\alpha,g} = [\frac{1}{2}(\mathbf{g}+1)]$.*[[3]]*

REMARK 4.4 *From the theory of divisors, Jacobi manifolds and* θ*-functions for Riemann surface hold*

a)Harmonic spinors are in 1-1 correspondence with holomorphic sections of line bundles associated to the divisors D of order $\mathbf{g}$*- 1 and satisfying the equality* $2.D = K \equiv$ *meromorphic functions on M having divisors greater or equal to D)*

b)Spin-structures are in 1-1 correspondence with θ*-characteristics*

REMARK 4.5 *a) Compact Riemann surfaces are possible to study also as complex projective algebraic curves.*

b) Compact Riemann surfaces (algebraic curves) are divided into two classes, the first class consists of so called hyperelliptic surfaces, the others are called nonhyperelliptic (see next section). There exist as hyperelliptic as well as nonhyperelliptic surfaces for all genera $\mathbf{g}$*with* $\mathbf{g} \geq 3$.

4.1 Hyperelliptic Riemann surfaces.

4.1.1 General theory.

From the results of a complex projective algebraic geometry we can get the complete description of spin-structures and harmonic spinors for all hyperelliptic surfaces (see e.g. [10]).

We shall use in the following text the correspondences betweem Riemannian surfaces and complex algebraic projective curves and other results from [5] .

DEFINITION 4.1 : *A hyperelliptic curve C is a complex projective curve, which admits a rational surjective map π onto the projective line $P^1(\mathbf{C})$ which is 2-1 up to finite set of point, which are called branching points of π.*

Any hyperelliptic curve C can be describe as follows.
There is a covering

$$C = C_1 \cup C_2$$

where $C_i, i = 1,2$ are regular (affine) complex curves in $\mathbf{C}^2$:
C_1 is given in the plane $\mathbf{C}^2$ with coordinates (t,s) by the equation:

$$s^2 = f(t); f(t) = \prod_{i \in S_p} (t - a_i); a_i \in \mathbf{C}$$

and C_2 is given in the plane $\mathbf{C}^2$ with coordinates (t',s') by the equation:

$$s'^2 = h(t')$$

where

$$h(t') = \prod_{i \in S_p} (1 - a_i.t') \mathrm{if} p = 2k \mathrm{even}$$

or

$$h(t') = \prod_{i \in S_p} t'.(1 - a_i.t') \mathrm{if} p = 2k - 1 \mathrm{odd}$$

with relations on $C_1 \cap C_2$:

$$t' = 1/t, s' = s/t^k; \mathrm{for} t \neq 0, t' \neq 0.$$

The points in infinity of C_1 are:

$$\infty_1 \leftrightarrow (t', s') = (0,1); \infty_2 \leftrightarrow (t', s') = (0,-1)$$

for even p

$$\infty \leftrightarrow (t', s') = (0,0)$$

for odd p.
The map $\pi : C \to P_1(\mathbf{C})$ is given by:

$$(t, s) \mapsto t \mathrm{on} C_1; (t', s') \mapsto t' \mathrm{on} C_2$$

which is 2-1 except of the set of branching points $\mathbf{B} \subset C$.

$$\mathbf{B} := \{P_i, i = 1, \ldots, 2k\}$$

where

$$P_i \leftrightarrow t = a_i \text{for} p = 2k \text{even},$$

or

$$P_i \leftrightarrow t = a_i, P_{2k} = \infty, \text{for} p = 2k - 1 \text{odd}.$$

The genus gof C is

$$\mathbf{g} = \frac{1}{2}(card\mathrm{B}\ ) - 1 = k - 1.$$

There is an involution $\iota : \mathrm{C} \to \mathrm{C}$ with $\pi(P) = \pi(\iota(P)$ for any $P \in \mathrm{C}$, having the set of fixpoints equal to the set of branching points B and divisor class

$$\mathrm{L}_{\mathrm{C}} := \{P + \iota(P); P \in \mathrm{C}\}.$$

Let K_{C} be the canonical divisor class of C (i.e class of divisors of abelian differentials on C).

Let

$$\sum = \{\text{divisorclass} D; 2.D = K_{\mathrm{C}}\}$$

be the set of divisor classes on C , called set of θ- characteristics.

For any divisor class D (or divisor D) we denote by L(D) the line bundle on C which corresponds to the divisor class D (or to the divisor D) and L (D) the space of all holomorphic sections of L(D). For more details see e.g [6, 7].

THEOREM 4.8 *[11]:*

(i) $K_C = (\mathbf{g} - 1)L_C,$

(ii) Any θ-characteristics has a form

$$f_T := \sum_{P \in T} P + \frac{1}{2}(\mathbf{g} - 1 - cardT).L_{cC}$$

where $T \subset B$, with $cardT \equiv \mathbf{g} + 1 \ mod\ 2$

(iii) We have

$$f_{T_1} \equiv f_{T_2} \Leftrightarrow T_1 = T_2 or\ T_1 = \mathbf{c}T_2$$

$$\sum \equiv \{T \subset B; cardT \equiv (\mathbf{g} + 1) mod 2\}/T \sim \mathbf{c}T$$

(iv) For all such T there exist points $\tilde{P}_1, ..., \tilde{P}_{\mathbf{g}-1}$ of C with $f_T \equiv \sum_{i=1}^{i=\mathbf{g}-1} \tilde{P}_i$ iff $cardT \neq \mathbf{g} + 1$ and if $cardT \leq \mathbf{g}$, then

$$dimL(f_T) = \frac{1}{2}(\mathbf{g} + 1 - cardT)$$

(if card $T \geq (\mathbf{g} + 2)$ we change $\mathbf{c}T$ for T itself).

(v) *$L(f_T) \simeq \{$ The set of polynoms in t of degree $\leq \frac{1}{2}(\mathbf{g} - 1 + cardT)$ vanishing in all points $P \in T\}$.*

Following this theorem we get the following facts:

(1) Spin-structures on hyperelliptic curve C correspond in one-to-one way to θ-characteristics. Let $L(f_T)$ be the spin structure corresponding to the θ-characteristics f_T, where T is a subset of B with card T $\leq$ g+ 1 , with an equivalence relation $T \sim \mathbf{c}T$.

(2) The dimension of positive harmonic spinors for spin-structure L(f_T) we have the equality :

$$h^0_{L(f_T)} = \frac{1}{2}(\mathbf{g} + 1 - cardT)$$

(3) The set of positive harmonic spinors for a given spin structure L(f_T) is $\mathbf{H}^+_{L(f_T)}$ = The set of all polynoms in t of degree less or equal $\frac{1}{2}(\mathbf{g} - 1 + cardT)$ vanishing in all points $P \in T$.

4.2 Nonhyperelliptic Riemann surfaces.

All Riemann surfaces of genera $g \leq 2$ are hyperelliptic, there exist nonhyperelliptic surfaces of all genera $\mathbf{g} \geq 3$.

Let us present some results on nonhyperelliptic Riemann surfaces and their examples for smaller genera $3 \leq \mathbf{g} \leq 6$.

A. In the case of nonhyperelliptic Riemann surfaces genus g= 3 the situation is simple.

We have N(3) = 64, $N_{odd} = 28, N_{ev} = 36, h^0 \leq 1$. For arbitrary nonhyperelliptic Riemann surface and any odd spin-structure on it we have $h^0 = 1$, and for any even spin-structure on it we have $h^0 = 0$.

Let us remark that nonhyperelliptic Riemann surfaces of genus 3 are just regular quartics in $P^2(\mathbf{C})$.

There is one interesting geometrical relation: The 28 different double tangent to a regular quartic Q are in one-to-one correspondence with odd spin structures on Q.

B. The case of nonhyperelliptic surfaces of genus g= 4 is very interesting. Let us state a theorem

THEOREM 4.9 : *For a nonhyperelliptic Riemann surfaces of genus* g= *4 one of the following conditions is satisfied*

I. There is unique even spin structure with $h^0 = 2$, other 135 even spin structures have $h^0 = 0$.

II. All 136 even spin structures have $h^0 = 0$.

In both cases all odd spin-structures have $h^0 = 1$.

A nonhyperelliptic Riemann surface will be called of type I or II if it satisfies the corresponding condition I or II from the preceding Theorem.

The Theorem follows from the theory of special divisors on a Riemann surfaces, for reference see [4].

Both classes I and II are nonempty as show the following examples:

EXAMPLE 4.1 *Let M be a Riemann surface given as a curve in $P^2(C)$ by the equation in C^2*

$$w^3 = z(z-1)(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)$$

with $\lambda_i, i = 1,2,3$ distinct points in $\mathbf{C} - \{0,1\}$.

The holomorphic line bundle L which corresponds to the divizor $3.Q_1$ where $Q_1 = z^{-1}\infty$ is the spin structure on M with $h^0 = 2$.

EXAMPLE 4.2 *Let M be a Riemann surface given as a curve in $P^2(C)$ by the equation in C^2*

$$w^3 = z(z-1)(z-\lambda_1)^2(z-\lambda_2)^2(z-\lambda_3)^2$$

with $\lambda_i, i = 1,2,3$ distinct points in $\mathbf{C} - \{0,1\}$. Then M has genus 4 and is of type II.

C.The case of nonhyperelliptic Riemann surfaces of genus **g**= 6 is much more complicated.

There are several possible classes of these surfaces with respect to the possible dimensions of the spaces of harmonic spinors for different spin structures.

A complete classification is needed, at present there is the following example:

EXAMPLE 4.3 *There exists a system of nonhyperelliptic curves C of genus 6, namely regular quintics in $P^2(\mathbf{C})$ with the canonical line bundle $K = O(2)/C$, and spin structure $L = O(1)/C$ with $h^0_L = 3$. (O(k) is the k-th power of the dual of tautological bundle on $P^2(\mathbf{C})$.*

Further results will be presented in some of next paper.

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Jarolim BUREŠ
Math. Institute
Charles University
Sokolovská 83
186 00 Praha
Czechoslovakia