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ON COTANGENT BUNDLES OF SOME NATURAL BUNDLES

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Abstract. We first explain how natural operators transforming vector fields on manifolds into vector fields on a natural bundle F can be used for constructing natural operators transforming vector fields on manifolds into functions on the cotangent bundle of F. Then we characterize some natural bundles with the property that all operators of the latter type can be constructed in such a way. As a special case we determine all natural functions on the cotangent bundle of the bundle of one-dimensional velocities of arbitrary order.

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All manifolds and maps are assumed to be infinitely differentiable.

1. Let $\mathcal{M}f_m$ be the category of m-dimensional manifolds and their local diffeomorphisms. Consider a natural bundle F over m-manifolds, [9], [5].

Definition 1. A natural function g on F is a system of functions $g_M : FM \to \mathbb{R}$ for every m-manifold M satisfying $g_M = g_N \circ Ff$ for all $f : M \to N$ from $\mathcal{M}f_m$.

The simpliest example of a natural function is the Liouville form of the contangent bundle interpreted as a map $\lambda_M : TT^*M \to \mathbb{R}$. We remark that the results of Section 26 in [5] imply that all natural functions on TT^* are of the form $h \circ \lambda$, where $h \in C^\infty(\mathbb{R}, \mathbb{R})$ is an arbitrary smooth function of one variable.

Some natural functions on the cotangent bundle $T^*FM = T^*(FM)$ can be constructed by means of the natural vector fields on the natural budle F.

Definition 2. A natural vector field ξ on F is a system of vector fields $\xi_M : FM \to TFM$ for every m-manifold M satisfying $TFf \circ \xi_M = \xi_N \circ Ff$ for all $f : M \to N$ from $\mathcal{M}f_m$.

In general, every section s of a vector bundle $E \to X$ defines a function $\tilde{s}$ on the dual vector bundle $q : E^* \to X$ by

$$\tilde{s}(w) = \langle s(qw), w \rangle, \qquad w \in E^*.$$

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Clearly, for every natural vector field ξ on F, the maps $\tilde{\xi}_M\colon T^*FM \to \mathbf{R}$ form a natural function on T^*F. Moreover, if we have k natural vector fields $\xi_1, \dots, \xi_k$ on F and a smooth function $h\colon \mathbf{R}^k \to \mathbf{R}$, then $h(\tilde{\xi}_1, \dots, \tilde{\xi}_k)$ also is a natural function on T^*F.

A natural vector field on the tangent bundle is the Liouville vector field L_M generated by the homotheties in the individual fibers of TM. One verifies easily that $\tilde{L}_M\colon T^*TM \to \mathbf{R}$ is identified with the Liouville function $\lambda_M\colon TT^*M \to \mathbf{R}$ by the canonical isomorphism $TT^*M \to T^*TM$, [8], [5].

2. Let $C^\infty TM$ denote the set of all smooth sections of a tangent bundle $TM \to M$. In [4] we have clarified that the natural vector fields on F can be interpreted as the so-called absolute (or constant) natural operators $C^\infty TM \to C^\infty TFM = C^\infty(T(FM))$ transforming vector fields on M into vector fields on FM. Now we are going to deduce that under certain assumptions on F all natural operators $C^\infty TM \to C^\infty(T^*FM, \mathbf{R})$ transforming vector fields on M into functions on T^*FM can be constructed from the natural operators $C^\infty TM \to C^\infty TFM$. Analogously to [4], the natural functions on T^*F correspond to the constant operators.

The set N_F of all natural operators $C^\infty TM \to C^\infty TFM$ is a vector space, provided we define

$$(A+B)_M(X) = A_M X + B_M X, \quad (kA)_M(X) = k(A_M X)$$

$A, B \in N_F$, $k \in \mathbf{R}$, $X \in C^\infty TM$. Our first assumption is

I. The dimension of N_F is finite.

By [4] and [6], this is true for all Weil bundles and for the bundles of higher order tangent vectors.

Let $Nop(T, T^*F \times \mathbf{R})$ denote the set of all natural operators $C^\infty TM \to C^\infty(T^*FM, \mathbf{R})$. For every smooth function $h\colon N_F^* \to \mathbf{R}$ we construct a natural operator $Dh \in Nop(T, T^*F \times \mathbf{R})$. Since the intrinsic definition of Dh is somewhat abstract, we start with a "coordinate" description of Dh. Fix a basis $A_1, \dots, A_n$ of N_F, which identifies N_F^* with $\mathbf{R}^n$. Then every $h \in C^\infty(\mathbf{R}^n, \mathbf{R})$ defines $Dh \in Nop(T, T^*F \times \mathbf{R})$ by

$$(Dh)_M X = h(\widetilde{A_{1M} X}, \dots, \widetilde{A_{nM} X})\colon T^*FM \to \mathbf{R}, \tag{1}$$

$X \in C^\infty TM$. To describe the same construction in an intrinsic way, we have to take into account that every $X \in C^\infty TM$ and every $w \in T^*FM$ define a linear map $\varphi(X, w)\colon N_F \to \mathbf{R}$ by

$$\varphi(X, w)(A) = \widetilde{A_M X}(w)$$

This is an element of N_F^* and (1) can be rewritten as

$$(Dh)_M X(w) = h(\varphi(X, w)) \tag{2}$$

with $h \in C^\infty(N_F^*, \mathbf{R})$. Thus we obtain a map $D\colon C^\infty(N_F^*, \mathbf{R}) \to Nop(T, T^*F \times \mathbf{R})$, $h \mapsto Dh$.

3. Write ∂_1 for the vector field $\partial/\partial x^1$ on $\mathbb{R}^m$ and $\tilde{A}(\partial_1)$ for $\widetilde{A_{\mathbb{R}^m}(\partial_1)}$. To reconstruct a function $h\colon N_F^* \to \mathbb{R}$ from a natural operator $A \in Nop(T, T^*F \times \mathbb{R})$, we assume F has the following property.

II. There exists a smooth map $j\colon N_F^* \to (T^*F)_0\mathbb{R}^m$ such that

$$\langle A, u\rangle = \tilde{A}(\partial_1)(ju), \qquad A \in N_F,\ u \in N_F^*. \tag{3}$$

Then we define a map $S\colon Nop(T, T^*F \times \mathbb{R}) \to C^\infty(N_F^*, \mathbb{R})$ by

$$S(A) = \tilde{A}(\partial_1) \circ j \tag{4}$$

Lemma 1. *It holds* $S \circ D = \mathrm{id}$.

Proof. If we use a basis $A_1, \dots, A_n$ of N_F, we obtain by (4), (1) and (3)

$$S(Dh)(u) = \widetilde{Dh}(\partial_1)(ju) = h(\tilde{A}_1(\partial_1)(ju), \dots, \tilde{A}_n(\partial_1)(ju)) = h(u_1, \dots, u_n). \quad \square$$

4. Let $\mathrm{Diff}_0^1\mathbb{R}^m \subset \mathrm{Diff}\mathbb{R}^m$ be the subgroup of all diffeomorphisms of $\mathbb{R}^m$ preserving the origin and the vector field ∂_1. To deduce the converse relation $D \circ S = \mathrm{id}$, we need another assumption.

III. The orbit of $j(N_F^*)$ with respect to $\mathrm{Diff}_0^1\mathbb{R}^m$ is dense in $(T^*F)_0\mathbb{R}^m$.

Proposition 1. *If I, II and III hold, then all natural operators* $C^\infty TM \to C^\infty(T^*FM, \mathbb{R})$ *are of the form*

$$Dh \qquad \text{for all } h \in C^\infty(N_F^*, \mathbb{R}).$$

Proof. It is well known that every $X \in C^\infty TM$ nonvanishing at $x \in M$ can be transformed into ∂_1 by a local diffeomorphism. This implies that if $A_1, A_2 \in Nop(T, F^*T \times \mathbb{R})$ satisfy $A_1(\partial_1)|T^*F_0\mathbb{R}^m = A_2(\partial_1)|T^*F_0\mathbb{R}^m$, then $A_1 = A_2$, [5], [6]. By Lemma 1 we have $(S \circ D \circ S)(A) = S(A)$, i.e.

$$A(\partial_1)(ju) = (D \circ S)(A)(\partial_1)(ju)$$

By naturality, it holds

$$A(\partial_1)|W = (D \circ S)(A)(\partial_1)|W \tag{5}$$

for the whole orbit W of $j(N_F^*)$ in $T^*F_0\mathbb{R}^m$. Since W is dense in $T^*F_0\mathbb{R}^m$ by III, the restrictions of both sides of (5) to $T^*F_0\mathbb{R}^m$ coincide. Hence $(D \circ S)(A) = A$. $\square$

5. We are going to apply Proposition 1 to the bundle $T_1^r M = J_0^r(\mathbb{R}, M)$ of one-dimensional velocities of order r. First of all we determine all natural functions on $T^*T_1^r$. We have the generalized Liouville vector field L_M on $T_1^r M$ induced by the reparametrization $x(t) \mapsto x(kt)$, $0 \neq k \in \mathbb{R}$, of a curve $x\colon \mathbb{R} \to M$ and a natural linear morphism $Q_M\colon TT_1^r M \to TT_1^r M$ introduced by de León and Rodrigues, [1]. According to [4], all natural vector fields on T_1^r form an r-parameter family linearly generated by

$$L_1 = L,\ L_2 = Q \circ L, \dots, L_r = Q^{r-1} \circ L \tag{6}$$

Proposition 2. *All natural functions on $T^*T_1^r$ are of the form*

$$h(\tilde{L}_1, \dots, \tilde{L}_r) \qquad \text{for all } h \in C^\infty(\mathbb{R}^r, \mathbb{R}).$$

Proof. If x^i are the canonical coordinates on $\mathbb{R}^m$, the r-th order Taylor expansions of a curve $x^i(t)$ determine the induced coordinates $y_1^i, \dots, y_r^i$ on $T_1^r\mathbb{R}^m$. The coordinate form of Q is $Q_{\mathbb{R}^m}(dx^i, dy_1^i, \dots, dy_r^i) = (0, dx^i, \dots, dy_{r-1}^i)$ while the coordinate expression of $L_{\mathbb{R}^m}$ is $dx^i = 0$, $dy_s^i = s y_s^i$, $s = 1, \dots, r$, [5]. If we introduce the additional coordinates on $T^*T_1^r\mathbb{R}^m$ by

$$q_i dx^i + p_i^1 dy_1^i + \dots + p_i^r dy_r^i \tag{7}$$

then the coordinate form of the natural functions $\tilde{L}_1, \tilde{L}_2, \dots, \tilde{L}_r$ on $T^*T_1^r\mathbb{R}^m$ is

$$\begin{aligned} &p_i^1 y_1^i + \dots + r p_i^r y_r^i \\ &p_i^2 y_1^i + \dots + (r-1) p_i^r y_{r-1}^i \\ &\quad\vdots \\ &p_i^r y_1^i \end{aligned} \tag{8}$$

Denote by B_1^r the vector space of all natural vector fields on T_1^r. The basis (6) of B_1^r induces some coordinates $a_1, \dots, a_r$ on B_1^{r*}. Define a map $j\colon B_1^{r*} \to (T^*T_1^r)_0\mathbb{R}^m$ by

$$y_1^1 = 1,\ p_1^1 = a_1, \dots p_1^r = a_r \text{ and zero at all other places.} \tag{9}$$

Using (8) one verifies directly

$$h(\tilde{L}_{1\mathbb{R}^m}, \dots \tilde{L}_{r\mathbb{R}^m}) \circ j = h \qquad \text{for all } h \in C^\infty(\mathbb{R}^r, \mathbb{R}). \tag{10}$$

Analogously to Proposition 1 it suffices to deduce that the orbit of $j(B_1^{r*})$ with respect to the subgroup $\mathrm{Diff}_0\mathbb{R}^m \subset \mathrm{Diff}\mathbb{R}^m$ of all origin preserving diffeomorphisms is dense in $(T^*T_1^r)_0\mathbb{R}^m$. Since $T^*T_1^r$ is a natural bundle of the order $r+1$, the action of

$\mathrm{Diff}_0\mathbb{R}^m$ on $(T^*T_1^r)_0\mathbb{R}^m$ factorizes through the $(r+1)$-th order jet group G_m^{r+1}, [5]. One deduces easily that the transformation laws of $y_1^i, \dots, y_r^i$ are

$$\begin{aligned} \bar{y}_1^i &= a_j^i y_1^j \\ &\vdots \\ \bar{y}_r^i &= a_{j_1\dots j_r}^i y_1^{j_1} \dots y_1^{j_r} + \dots + a_j^i y_r^j \end{aligned} \tag{11}$$

where the dots in the last row denote a polynomial expression we shall not indicate explicitely.

Consider first the case $m = 1$. If $y_1^1 \neq 0$, then $y = (y_1^1, \dots, y_r^1)$ is r-jet of a local diffeomorphism $\mathbb{R} \to \mathbb{R}$. Hence we can have $y = (1, 0, \dots, 0)$ in a suitable coordinate system. From (7) and (11) we deduce the following transformation law of q_1 on the kernel of the jet projection $G_1^{r+1} \to G_1^r$

$$\bar{q}_1 = q_1 - a p_1^r \tag{12}$$

where $a \in \mathbb{R}$ is the only coordinate on $\mathrm{Ker}(G_1^{r+1} \to G_1^r)$. If $p_1^r \neq 0$, we can obtain $\bar{q}_1 = 0$ by a suitable choice of a. This proves the denseness of $j(B_1^{r*})$.

For $m \geq 2$, let $y = (y_1^i, \dots, y_r^i)$ be the r-jet of an immersion $\mathbb{R} \to \mathbb{R}^m$. Then we have

$$y_1^1 = 1 \text{ and all other } y\text{'s vanishing} \tag{13}$$

in a suitable coordinate system. By (11), the subgroup of G_m^{r+1} preserving (13) is characterized by

$$a_1^1 = 1,\ a_1^t = 0,\ a_{11}^i = 0, \dots, a^i_{\underbrace{1\dots1}_{r\text{-times}}} = 0, \qquad t = 2, \dots, m \tag{14}$$

It suffices to show that we can transform each element from a dense subset of $(T^*T_1^r)_0\mathbb{R}$ into (13) and

$$p_t^1 = 0, \dots, p_t^r = 0, q^i = 0, \qquad t = 2, \dots, m \tag{15}$$

by means of a suitable element of G_m^{r+1}. First of all, from (7) and (8) we deduce

$$\bar{p}_i^r = \tilde{a}_i^j p_j^r$$

where $(\tilde{a}_j^i)$ is the inverse matrix to (a_j^i). Hence $p^r \in \mathbb{R}^{m*}$ and for $p_1^r \neq 0$ we can select a basis in $\mathbb{R}^m$ such that $y_1 = (1, 0, \dots, 0)$ and $p^r = (p_1^r, 0, \dots, 0)$.

Assume by induction we have (13) and

$$p^s = (p_1^s, 0, \dots, 0) \qquad \text{for } s = k+1, \dots, r \tag{16}$$

From (7) and (11) we deduce the following transformation law of p_i^k on the kernel of the jet projection $G_m^{r-k+1} \to G_m^{r-k}$

$$\bar{p}_i^k = p_i^k + c a_{i1\dots1}^1 p_1^r \tag{17}$$

where c is a non-zero integer. For $p_1^r \neq 0$ we can obtain $\bar{p}_t^k = 0$ by means of $a_{t1\dots1}^1$, $t = 2, \dots, m$. In the last step of such a procedure we can obtain $q = (0, \dots, 0)$ by using the kernel of the jet projection $G_m^{r+1} \to G_m^r$. □

We remark that the case $r = 2$ was studied in another setting by Doupovec, [2].

6. According to [4], all natural operators $C^\infty TM \to C^\infty TT_1^r M$ form a $(2r+1)$-parameter family linearly generated by (6) and

$$(18) \qquad \mathcal{T}_1^r,\ V_1 = Q \circ \mathcal{T}_1^r, \dots, V_r = Q^r \circ \mathcal{T}_1^r$$

where $\mathcal{T}_1^r$ denotes the flow operator of T_1^r.

Proposition 3. *For $dimM \geq 2$, all natural operators $C^\infty TM \to C^\infty(T^*T_1^r M, \mathbb{R})$ are of the form*

$$h(\tilde{L}_1, \dots, \tilde{L}_r, \tilde{V}_1, \dots, \tilde{V}_r, \tilde{\mathcal{T}}_1^r) \qquad \textit{for all } h \in C^\infty(\mathbb{R}^{2r+1}, \mathbb{R}).$$

Proof. Write N_1^r for $N_{T_1^r}$. The basis (6) and (18) induces some coordinates $a_1, \dots, a_r, b_1, \dots, b_r, c$ on N_1^{r*}. Define $j\colon N_1^{r*} \to (T^*T_1^r)_0\mathbb{R}^m$ by $y_1^2 = 1$, $p_1^k = b_k$, $p_2^k = a_k$, $q_1 = c$ and zero at all other places, $k = 1, \dots, r$. Consider the subgroup $\mathrm{id}_{\mathbb{R}} \times \mathrm{Diff}_0\mathbb{R}^{m-1} \subset \mathrm{Diff}_0^1\mathbb{R}^m$. Then p_1^k and q_1 remain unchanged, while $p_2^k, \dots, p_r^k$ behave in the same way as in Proposition 2. This implies that the orbit of $j(N_1^{r*})$ is dense. □

In particular, all natural operators $C^\infty TM \to C^\infty(T^*TM, \mathbb{R})$ are of the form $h(\tilde{L}, \tilde{V}, \tilde{\mathcal{T}})$, where L is the classical Liouville vector field on the tangent bundle, V is the operator of vertical lifts, $\mathcal{T}$ is the flow operator of the tangent bundle and $h \in C^\infty(\mathbb{R}^3, \mathbb{R})$. This result was deduced in a quite different setting by Kobak, [3].

References

[1] De León M., Rodrigues P.R., *Almost tangent geometry and higher order mechanical systems*, Differential Geometry and Its Applications, Proceedings, D. Reidel Publishing Company, 1987, pp. 179–195.

[2] Doupovec M., *Natural transformations between $T_1^2T^*M$ and $T^*T_1^2M$*, Ann. Polon. Math. LVI (1991), 67–77.

[3] Kobak P., *Natural liftings of vector fields to tangent bundles of 1-forms*, Mathematica Bohemica 116 (1991), 319–326.

[4] Kolář I., *On the natural operators on vector filds*, Ann. Global Anal. Geometry 6 (1988), 109–117.

[5] Kolář I., Michor P.W., Slovák J., *Natural operations in differential geometry*, Springer-Verlag, 1993.

[6] Mikulski W., *Some natural operations on vector fields*, Rendiconti di Math., Ser VII 12 (1992), 783–803.

[7] Mikulski W., *Natural transformations transforming vector fields to functions on some linear natural bundles*, to appear.

[8] Modugno M, Stefani G., *Some results on second tangent and cotagent spaces*, Quaderni dell'Università di Lecce, 1978.

[9] Nijenhuis A., *Natural bundles and their general properties*, Differential Geometry in honour of K. Yano, Kinokuniya, Tokyo, 1972, pp. 317–334.

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