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An Application of Principal Bundles to Coloring of Graphs and Hypergraphs

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§1. Introduction

An interesting connection between the chromatic number of a graph G and the connectivity of an associated simplicial complex N(G), its "neighborhood complex", was found by Lovász in 1978 (cf. [4], or [2], p.260). In 1986 a generalization to the chromatic number of a k–uniform hypergraph H, for k an odd prime, using an associated simplicial complex C(H), was found ([1], Prop. 2.1).

It was already noted in the above mentioned papers that there is an action of Z/2 on N(G), and of Z/k on C(H), for any graph G and any k–uniform hypergraph H, $k \geq 2$ (a 2–uniform hypergraph is just a graph). In this note we take advantage of this action to construct an associated principal (Z/k)–bundle ξ, and state theorems relating the chromatic number of the graph or hypergraph to the classifying map of ξ into B(Z/k).

In §2 the necessary definitions are given and the main theorems (Theorem 2.1 and Theorem 2.2) are stated. In §3 these theorems are compared with the previous ones based on connectivity. A spectral sequence argument shows they imply the previous ones, and examples are given where the previous theorems give no information whereas the ones in this paper give sharp lower bounds for the chromatic number. In §4 the proofs of Theorems 2.1, 2.2, which in fact are surprisingly elementary, are given.

§2. Definitions and Statements of the Main Theorems

For convenience we recall several definitions that appear in [1], [2] and [4], as well as several results proved there. First we consider a graph G (finite, undirected, with no isolated points or loops). Its neighborhood complex $N = N(G)$ has the same vertex set V_G as G, and a set $(v_0,\dots,v_r)$ of vertices is a simplex iff $v_0,\dots,v_r$ have a common neighbor in G. Write $Y = |N(G)|$ for the underlying polyhedron. For any vertex v of G, let $\gamma(v) \subseteq V_G$ be the (non-empty) set of all neighbors of v, then N(G) can also be thought of as the collection of $\gamma(v)$'s and all their faces. Notice $v \notin \gamma(v)$ for any $v \in V_G$.

A vertex of N′(G), the barycentric subdivision of N(G), will be written $b(v_0,\dots,v_r)$ where $(v_0,\dots,v_r)$ is a simplex of N. Setting $\Gamma b(v_0,\dots,v_r) = b(\cap\{\gamma(v_i): 0 \le i \le r\})$, it is shown in [4] that Γ defines a simplicial map $\Gamma: N' \to N'$ with $\Gamma^3 = \Gamma$. Thus, setting $M = \operatorname{Im}\Gamma \subseteq N'$ and $X = |M| \subseteq Y$, it was noted in [4] that X is a retract of Y and Γ an involution of X. Here we show further that X is a strong deformation retract of Y and Γ is fixed point free (Prop. 4.3 and Prop. 4.2 respectively). Letting W be the orbit space $X/(Z/2)$, we have a principal Z/2 bundle ξ: $Z/2 \hookrightarrow X \twoheadrightarrow W$. Such a bundle has a classifying map $c = c(\xi): W \to B(Z/2) = \mathbb{R}P^\infty$, unique up to homotopy ([3], Ch.4).

2.1 Theorem: If G can be colored with n + 2 colors, then c compresses to $\mathbb{R}P^n$, i.e., there is a map $c_1: W \to \mathbb{R}P^n$ with c homotopic to the composition $W \xrightarrow{c_1} \mathbb{R}P^n \xhookrightarrow{i} \mathbb{R}P^\infty$.

Now consider a k–uniform hypergraph (H,E). Following [1], the simplicial complex $C = C(H)$ is defined as follows. Its vertices are all ordered k–tuples $[v_1,\dots,v_k]$ where $\{v_1,\dots,v_k\}$ is a k–edge of H, i.e. $\{v_1,\dots,v_k\} \in E$. A set of vertices

$[v_1^i,\ldots,v_k^i]$, $0 \leq i \leq r$, forms an r–simplex of C if the subsets $V_j = \{v_j^0,v_j^1,\ldots,v_j^r\}$ are mutually disjoint and form a complete k–partite subgraph of H.

Let $X = |C|$, and Z/k acts freely in the obvious way on both C and X by cyclic rotation $\alpha[v_1,\ldots,v_k] = [v_k,v_1,\ldots,v_{k-1}]$. Again, we take $W = X/(Z/k)$ as the orbit space giving a principal (Z/k)–bundle ξ, with classifying map c: $W \to B(Z/k)$. To state the theorem, a specific model for B(Z/k) will be useful. Take $S^{(t-1)(k-1)-1}$ to be the set of all $t \times k$ real matrices (a_{ij}) with every row and column sum zero, and $\Sigma a_{ij}^2 = 1$. Cyclic permutation of columns induces a Z/k action on $S^{(t-1)(k-1)-1}$, which for fixed k is compatible with the inclusion $S^{(t-1)(k-1)-1} \subseteq S^{t(k-1)-1}$. Letting $S^{(t-1)(k-1)-1}/(Z/k) = Q_t$, we thus have $Q_1 = \emptyset \subseteq Q_2 \subseteq Q_3 \subseteq \ldots \subseteq Q = \cup\{Q_t\colon t \geq 1\}$, where Q is given the weak topology. For k prime the action of Z/k on $S^{(t-1)(k-1)-1}$ is also free, and Q is then a model for B(Z/k).

2.2 <u>Theorem</u>: If H is t–colorable and k prime then the classifying map c: $W \to Q$ compresses into Q_t.

§3. <u>Comparison With The Connectivity Theorems</u>

We assume Theorems 2.1, 2.2 for the remainder of this section. Our first result is precisely Prop. 2.1 of [1].

3.1 <u>Proposition</u> For any k–uniform hypergraph H, k prime, if x is ((t–1)(k–1)–1)–connected then H cannot be colored with t colors.

Proof: Using standard homotopy techniques the map c may be replaced by a fibration, so up to homotopy type one has a fibration $X \hookrightarrow W \xrightarrow{c} Q = B(Z/k)$.

We consider the Serre spectral sequence of this fibration in cohomology, with coefficients Z/k. Recall $H^i(Q;Z/k) \approx H^i(K(Z/k,1); Z/k)$, which in turn is the group cohomology $H^i(Z/k; Z/k) \approx Z/k$, $i \geq 0$ (cf. [5], p.122). This spectral sequence in general has E_2 term with twisted coefficients, however the assumption that the fibre X is $((t-1)(k-1)-1)$–connected implies in any case that $Z/k = H^i(Q;Z/k)$ cannot be in the image of any differential for $i \leq (t-1)(k-1)$ and thus survives to E_∞. In particular $0 \neq c^*: H^{(t-1)(k-1)}(Q;Z/k) \to H^{(t-1)(k-1)}(W;Z/k)$.

However, if c compressed to $c_1: W \to Q_t$ (a CW–complex having dimension $(t-1)(k-1)-1$), then obviously $c_1^*: H^{(t-1)(k-1)}(Q_t;Z/k) = 0 \to H^{(t-1)(k-1)}(W;Z/k)$ satisfies $c_1^* = 0$, which is impossible since this would imply $c^* = 0$ in the same dimension. Hence c does not compress to Q_t, and by Theorem 2.2 H is not t–colorable.

Similarly we obtain Theorem 2 of [4], which we now state.

3.2 <u>PROPOSITION</u> If $Y = |N(G)|$ is $(t-2)$–connected for a graph G, then G is not t–colorable.

The proof is quite similar to that of the previous proposition, noting first that if Y is $(t-2)$–connected then so is its retract X, then that $H^{t-1}(\mathbb{R}P^\infty;Z/2) \to H^{t-1}(W;Z/2)$ is non–zero so c cannot be compressed into $\mathbb{R}P^{t-2}$, and finally applying Theorem 2.1. We omit the details.

It was shown as a corollary in [4] that $Y = |N(G)|$ is never contractible. From our Theorem 2.1, we can easily improve this as follows.

3.3 <u>COROLLARY</u> (of Theorem 2.1): If a finite polyhedron Y equals N(G) for some G, then its Euler characteristic $\chi(H)$ is even.

PROOF: From the principal $(Z/2)$–bundle, $\chi(X) = 2\chi(W)$. But $\chi(X) = \chi(Y)$ since they have the same homotopy type (Proposition 4.3).

3.4 EXAMPLE: To any vertex "v" of the complete graph K_n, attach a "tail" {v,a},{a,b},{b,c},{c,d},{c,e},{d,e}. The resulting neighborhood complex of the new graph G will only be 0–connected, so the Lovász theorem (Proposition 3.2 above) implies only chr(G) $\geq$ 3 for its chromatic number. On the other hand the classifying map for the graph K_n is the inclusion $\mathbb{R}P^{n-2} \hookrightarrow \mathbb{R}P^{\infty}$, which does not compress to $\mathbb{R}P^{n-3}$, and this will remain the case for the classifying map of G. So Theorem 2.2 gives chromatic number (G) $\geq$ n, which is sharp. Similar examples can be given for hypergraphs.

§4. PROOFS OF THE MAIN THEOREMS

4.1 PROOF OF THEOREM 2.2: To compress the classifying map into Q_t it suffices (cf. [3], Ch.4, Th. 12.2) to find a (Z/k)–equivariant map $X=|C(H)|\to S^{(t-1)(k-1)-1}$, with the action of Z/k as given in §2. But Lemma 3.3 of [1] shows precisely the existence of such an equivariant map g: $X \to \mathbb{R}^{(t-1)(k-1)}-\{0\} \to S^{(t-1)(k-1)-1}$ (which is (Z/k)–equivariant) completes the proof.

The proof of Theorem 2.1 requires two preliminary lemmas about the simplicial map Γ: $N' \to N'$, in addition to the properties found in [4] (and mentioned in §2 above). It will also be useful to the use the notation V, W, V_i, etc. to represent subsets of V_G which are simplices of N. Notice $V \subseteq V'$ and $\Gamma b(V) = b(W)$, $\Gamma b(V') = b(W')$, implies $W \supseteq W'$, while $\Gamma^2 b(V) = b(V_0)$ with $V \subseteq V_0$.

4.2 LEMMA: Γ: M $\to$ M defines a free Z/2 action on M.

PROOF: For any vertex $w \in V$, let M_w be the subcomplex of N' consisting of all simplices of N' spanned by vertices of the form $b(V)$, where V is a simplex of N with $w \in V$. An arbitrary simplex σ of N' (and hence of M) can be written $\sigma = (b(V_0),...,b(V_s))$, where the V_i form a nested collection of simplices of N, say without loss of generality $V_0 \subset V_1 \subset ... \subset V_s$. If w is any vertex of V_0, then $w \in V_i \geq 0$, and hence σ is a simplex of M_w. On the other hand $\Gamma(M_w) \cap M_w = \emptyset$ since if $w \in V$ then $w \notin \cap \{\gamma(v): v \in V\}$.

4.3 LEMMA: $\Gamma^2 \simeq id_Y$, and thus X is a strong deformation retract of Y.

PROOF: By a standard technique (cf. [6], p.302) it suffices to show, for any $y \in Y$, that y and $\Gamma^2 y$ lie in a common simplex of N. To this end, it suffices to consider the vertices $b(V_0),...,b(V_q)$ of any simplex σ of N' and show there is a simplex W of N such that these vertices together with $\Gamma^2 b(V_0),...,\Gamma^2 b(V_q)$ all lie in W. As before we may assume $V_0 \subset V_1 \subset ... \subset V_q$, and set $\Gamma^2 b(V_i) = bW_i)$. From the observations preceding 4.2,

$$\begin{array}{ccccccc} V_0 & \subset & V_1 & \subset & \cdots & \subset & V_q \\ \cap & & \cap & & & & \cap \\ W_0 & \subseteq & W_1 & \subseteq & \cdots & \subseteq & W_q \end{array}$$

hence taking $W = W_q$ will work.

Finally, $\Gamma^3 = \Gamma$ and $X = \mathrm{Im}\Gamma$ implies $\Gamma^2 | X = id_X$, whence $\mathrm{Im}\Gamma^2 = \mathrm{Im}\Gamma = X_0$ and the above homotopy is constant on X.

4.4 PROOF OF THEOREM 2.1: Given a coloring of G with $n + 2$ colors, we define (as in [2]) $M_i = \cup\{|M_v|: v \text{ has color } i\}$, where M_v is defined as in the proof of 4.2 above, and $X_i = X \cap |M_i|$.. We note (again as in [2])

(a) $X_i \cap \Gamma X_i \neq \emptyset$,

(b) $X = \cup\{X_i: 1 \leq i \leq n + 2\}$.

From (a) and (b) we have,

$$\text{(c)} \quad X = \cup\{X_i \cup \Gamma X_i : 1 \leq i \leq n + 1\}.$$

Since X_i is closed, from (a) there is a Urysohn function $g_i: X \to I$ with $g_i(X_i) = 0$, $g_i(\Gamma X_i) = 1$. Define $g: X \to \mathbb{R}^{n+1}$ by

$$g(x) = (g_1(x) = g_i\Gamma(x).,,,.g_{n+1}(x) - g_{n+1}\Gamma(x)).$$

Clearly $g\Gamma(v) = -g(v)$, and $g(x) \neq 0$ for all $x \in X$ by (c). Composing g with the usual (Z/2–equivariant) retraction $\mathbb{R}^{n+1} - \{0\} \to S^n$ gives a Z/2–equivariant map $X \to S^n$, and this suffices (as mentioned in 4.1 above) to get a classifying map $W \to \mathbb{R}P^n$.

REFERENCES

[1] N. Alon, P. Frankl, and L. Lovász, The chromatic number of Kneser Hypergraphs, Trans. Amer. Math. Soc. V.298, No.1 (1986), 350–370.

[2] B. Bollobás, Extremal Graph Theory, LMS Monographs #11 (1978), Academic Press.

[3] D. Husemoller, Fibre Bundles, Second Edition, Graduate Texts in Mathematics #20 (1974), Springer–Verlag.

[4] L. Lovász, Kneser's conjecture, chromatic number, and homotopy, J. Comb. Theory Ser. A., 25(1978), 319–324.

[5] S. MacLane, Homology, Die Grundlehren der mathematischen Wissenschaften, Band 114(1967), Springer–Verlag.

[6] A. Sieradski, An Introduction to Topology and Homotopy, PWS–KENT(1992).

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