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ISOSPECTRAL, NON-ISOMETRIC RIEMANNIAN MANIFOLDS

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By Dorothee Schueth*

ABSTRACT: First we give a survey of the history of isospectral manifolds that are not isometric, especially the first example by Milnor, the examples by Marie-France Vignéras, Sunada's method of constructing isospectral manifolds and its generalization by DeTurck and Gordon (Chapter 1). Then we describe some applications of these methods: the construction of continuous isospectral deformations (as introduced by Gordon, Wilson, DeTurck et al.), including some new examples (Chapter 2), and on the other hand the construction of isospectral plane domains due to Gordon, Webb and Wolpert (Chapter 3).

CHAPTER 1: Introduction, First Examples, General Methods

1.1 Introduction

Let (M,g) be a compact Riemannian manifold with metric g, possibly with boundary. Let $\Delta = -\operatorname{div}\operatorname{grad}$ be the Laplace operator associated to g, acting on functions. Consider the eigenvalue problem

$$\Delta f = \lambda \cdot f$$

with either Dirichlet or Neumann boundary conditions on ∂M (resp. empty boundary condition if ∂M is empty).

It is well known that in each of the three cases the occurring eigenvalues form a discrete series

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \rightarrow \infty$$

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in which each λ_j occurs with finite multiplicity; the sum of all eigenspaces is dense in $L^2(M,v_g)$, the eigenspaces corresponding to different eigenvalues are L^2-orthogonal, and all eigenfunctions are actually C^∞ . Zero is an eigenvalue (with multiplicity 1) if and only if $\partial M=\emptyset$. (Basic references for these facts are e.g. [BGM] or [Bé1].)

Denote the eigenvalue series, counting multiplicities, by

$$\mathrm{spec}(M,g) \quad \text{resp.} \quad \mathrm{spec}_D(M,g) \quad \text{resp.} \quad \mathrm{spec}_N(M,g)$$

for empty resp. Dirichlet resp. Neumann boundary conditions.

A natural question arising in this context is the following: To which extent does the spectrum of (M,g) determine its geometry? For the special case of a bounded domain in $\mathbb{R}^2$ this was formulated by M. Kac ([Ka]) in 1966 as "Can one hear the shape of a drum?"

Examples of partial answers to this question in the "positive" sense are the following well-known results:

a) The spectrum of a manifold (M,g) determines its dimension, its volume, and its total scalar curvature (hence also its Euler characteristic if M is 2-dimensional without boundary). This follows from the asymptotic expansion of the heat kernel by Minakshisundaram-Pleijel. (See e.g. [BGM].)

b) Some examples of Riemannian manifolds which are distinguished (i.e. uniquely determined) by their spectrum are the standard spheres in dimension ≤ 6, the standard real projective spaces in dimension ≤ 6, and compact 3-dimensional manifolds with constant positive sectional curvature (see [BGM], [Ta], [Tn]).

c) There exists at most a finite number of flat n-dimensional tori isospectral to a given flat torus (this is a result due to Kneser; see also [Pe1] for an explicit bound for the number of such tori); the analogous result is true for compact hyperbolic surfaces ([McK]).

d) Compact closed manifolds with negative sectional curvature are infinitesimally spectrally rigid: There are no nontrivial *continuous* isospectral deformations of them ([GK1], [GK2], [MO]).

e) On Riemann surfaces the Laplace spectrum and the length spectrum (i.e. the collection of lengths of shortest closed geodesics in different free homotopy classes) determine each other mutually ([Hu]); for similar results in more general, but generic situations see [CV].

1.2 First Examples

In spite of the above and other "positive" results, the answer to the question whether the spectrum of a Riemannian manifold determines its geometry *completely* is negative: In 1964, J. Milnor gave the first example of isospectral, non-isometric Riemannian manifolds, namely, a pair of two 16-dimensional flat tori.

1.2.1 The Milnor Example ([Mi])

A flat torus is a quotient $\Gamma\backslash\mathbb{R}^n$ of $\mathbb{R}^n$ by a lattice $\Gamma \subseteq \mathbb{R}^n$ of rank n where the quotient is endowed with the metric induced by the standard euclidean metric on $\mathbb{R}^n$.
The Γ-invariant eigenfunctions of Δ on $\mathbb{R}^n$ are precisely those of the form

$$f_\tau = \exp\,(2\pi i \langle\tau\;,\;.\rangle)$$

where τ ranges over the "dual lattice"

$$\Gamma^* := \{\tau\in\mathbb{R}^n \mid \langle\tau,\gamma\rangle \in \mathbb{Z} \quad \forall\gamma\in\Gamma\}\;.$$

The eigenvalue corresponding to f_τ is $4\pi^2\|\tau\|^2$.
Thus isospectrality of $\Gamma_1\backslash\mathbb{R}^n$ and $\Gamma_2\backslash\mathbb{R}^n$ is equivalent to the condition

$$\forall\; t > 0 : \;\# \{\tau\in\Gamma_1^* \mid \|\tau\| = t\} \;=\; \# \{\tau\in\Gamma_2^* \mid \|\tau\| = t\}\;.$$

This is in turn equivalent to

$$(*) \qquad \forall\; t > 0 : \;\# \{\gamma\in\Gamma_1 \mid \|\gamma\| = t\} \;=\; \# \{\gamma\in\Gamma_2 \mid \|\gamma\| = t\}$$

by the Poisson summation formula.
On the other hand, $\Gamma_1\backslash\mathbb{R}^n$ and $\Gamma_2\backslash\mathbb{R}^n$ are isometric if and only if there exists $T\in O_n$ such that $\Gamma_2 = T\Gamma_1$.
Thus if Γ_1 and Γ_2 satisfy condition (*) (i.e. have the same length spectrum), but are not congruent, then $\Gamma_1\backslash\mathbb{R}^n$ and $\Gamma_2\backslash\mathbb{R}^n$ provide an example of two isospectral, non-isometric Riemannian manifolds.
The two tori found by Milnor are quotients of $\mathbb{R}^{16}$ by two lattices with just this property.

Remark: Today there is also an example of two flat isospectral, non-isometric tori in dimension four (see [CS]).

1.2.2 Isospectral hyperbolic surfaces (Marie-France Vignéras 1980; [Vi])

After the example of Milnor in 1964, fifteen years passed until, at the end of the seventies, new examples of isospectral, non-isometric manifolds were found. An important one of these examples is the following:

Let $G := PSL(2,\mathbb{R})$ be the group of positively oriented isometries of the hyperbolic plane H^2, i.e. the upper half plane endowed with the metric $(dx^2+dy^2)/y^2$.
Let Γ be a discrete subgroup of G such that $\Gamma\backslash H^2$ is a compact Riemannian manifold. In particular, $\Gamma-\{id\}$ can contain no elliptic or parabolic but only hyperbolic isometries. Let $\Gamma\backslash H^2$ be endowed with the metric induced from H^2.
By a theorem of Huber ([Hu]), the length spectrum of $\Gamma\backslash H^2$ determines its Laplace spectrum.
The length spectrum can be described as follows:

The directed closed geodesics of $\Gamma\backslash H^2$ are in one-to-one correspondence to the Γ-conjugacy classes in Γ; the closed geodesic which corresponds to the conjugacy class $[\gamma]_\Gamma \in [\Gamma]$ has length

$$s_\gamma := \min \{\mathrm{dist}(z,\gamma z) \mid z \in H^2\} .$$

The number s_γ can be computed as $s_\gamma = \ln(|\lambda|^2)$ where λ is the bigger (with respect to absolute values) eigenvalue of a representant of γ in $SL(2,\mathbb{R})$. Hence s_γ is determined by the number

$$\mathrm{trace}(\gamma) := |\lambda| + |\lambda|^{-1} > 2 .$$

Because of these facts, isospectrality of $\Gamma_1\backslash H^2$ and $\Gamma_2\backslash H^2$ is equivalent to

$$(**) \quad \forall\, t > 0 : \quad \# \{[\gamma]_{\Gamma_1} \in [\Gamma_1] \mid \mathrm{trace}(\gamma) = t\} = \# \{[\gamma]_{\Gamma_2} \in [\Gamma_2] \mid \mathrm{trace}(\gamma) = t\}$$

On the other hand, $\Gamma_1\backslash H^2$ and $\Gamma_2\backslash H^2$ are isometric if and only if Γ_1 and Γ_2 are conjugate in $PGL(2,\mathbb{R})$.

What Vignéras constructed were examples of Γ_1 and Γ_2 which are *not* conjugate, but nevertheless satisfy (**).

She obtained Γ_1 and Γ_2 as the images of certain subgroups of a quaternional algebra over an algebraic number field under its embedding into $M(2,\mathbb{R})$; this number theoretical part was the difficult part of her construction.

Remarks:

(i) Vignéras also constructed isospectral, non-isometric examples of compact hyperbolic manifolds in higher dimensions. In these examples, not even the fundamental groups are the same (by the Mostow Rigidity Theorem) which shows that the spectrum does not determine the topology of a manifold.

(ii) Note that in spite of Vignéras' examples, spectral *rigidity* holds for compact hyperbolic manifolds (i.e. there are no *continuous* isospectral deformations of them) by the results of Guillemin/Kazhdan ([GK1], [GK2], see also [MO]) which hold for arbitrary compact Riemannian manifolds with negative sectional curvature. Moreover, MacKean ([McK]) showed that the number of hyperbolic surfaces isospectral to a given one is always finite.

(iii) Note a certain community between the Milnor and the Vignéras examples:
There exists a *bijection*

$$\Phi : \Gamma_1 \to \Gamma_2 \qquad \text{(in the Milnor example), resp.}$$
$$\Phi : [\Gamma_1] \to [\Gamma_2] \qquad \text{(in the Vignéras example)}$$

such that

$\Phi(\gamma_1)$ is conjugate to γ_1 in $\mathrm{Iso}(\mathbb{R}^n) = O_n \ltimes \mathbb{R}^n$ (in the Milnor example), resp.
$\Phi([\gamma_1]_{\Gamma_1}) \in [\Gamma_2]$ and $[\gamma_1]_{\Gamma_1} \in [\Gamma_1]$ are subsets of the same conjugacy class in $PGL(2,\mathbb{R})$ (in the Vignéras example),

although Γ_1 and Γ_2 are *not* conjugate in $\mathrm{Iso}(\mathbb{R}^n)$ resp. $PGL(2,\mathbb{R})$.

1.3 Systematical Methods

1.3.1 The Sunada Theorem

The first systematical method of constructing isospectral manifolds was found by T. Sunada in 1985 (see [Su]) and is based on the same kind of principle as in the above Remark (iii) in 1.2.2:

> THEOREM (Sunada):
> Let G be a finite group acting by isometries on a compact Riemannian manifold (M,g) without boundary; let Γ_1 and Γ_2 be subgroups of G acting freely on M and satisfying
>
> (S) $\forall\ h \in G :\ \#\,(\Gamma_1 \cap [h]_G) = \#\,(\Gamma_2 \cap [h]_G)$.
>
> Then $(\Gamma_1\backslash M,g)$ and $(\Gamma_2\backslash M,g)$ (where g denotes the induced metrics again by abuse of notation) are isospectral.

In 1991, P. Bérard generalized this theorem to the cases M with boundary and Γ_1, Γ_2 not necessarily acting without fixed points (see [Bé3] and Theorem 3.1 below).
We will give a sketch of Bérard's proof in Chapter 3 where we will also describe how Bérard's version of Sunada's Theorem was used by Gordon, Webb and Wolpert for the first construction of isospectral bounded plane domains ([GWW1], [GWW2]).

For examples of isospectral, non-isometric Riemannian manifolds that can be constructed by using Sunada's Theorem (and many of which can be realized as paper models using scissors and paste) see [Bu1] and [Bu2] by P. Buser, also [Br] and [BT] by R. Brooks and R. Tse.
One of Buser's examples will play an important role in Chapter 3.

1.3.2 A more general Theorem by DeTurck and Gordon

In 1987 Dennis DeTurck and Carolyn Gordon proved a strong generalization (see [DG3]) of Sunada's Theorem which recovers also the previous examples described in 1.2 above.

> THEOREM (DeTurck/Gordon):
> Let G be an arbitrary Lie group acting by isometries on a Riemannian manifold (M,g) without boundary. Let Γ_1 and Γ_2 be discrete cocompact subgroups of G that act freely and properly discontinuously on M such that $\Gamma_1\backslash M$ and $\Gamma_2\backslash M$ are compact.
> Then if the following condition (DG) is satisfied, $(\Gamma_1\backslash M,g)$ and $(\Gamma_2\backslash M,g)$ are isospectral (where g denotes the induced metrics again by abuse of notation):

> (DG) $\forall\ h \in G :\ r(G,\Gamma_1;h) = r(G,\Gamma_2;h)$
>
> where (for $h \in G$ and a cocompact discrete subgroup Γ of G)

$$r(G,\Gamma;h) \;:=\; \sum_{[\gamma]_\Gamma \subseteq [h]_G} \rho^{[h]}(\Gamma_\gamma\backslash G_\gamma) \quad ;$$

here, G_γ and Γ_γ denote the centralizers of $\gamma\in\Gamma$ in G and Γ respectively, and the $\rho^{[h]}$ are biinvariant Haar measures on every G_b ($b\in[h]_G$) which are chosen in such a way that the maps

$$I_a : G_b \to G_{aba^{-1}}$$

are measure preserving (compatibility condition).
$r(G,\Gamma;h) := 0$ if $[h]_G$ does not meet Γ at all.

Note that $\Gamma_\gamma\backslash G_\gamma$ is compact if $\Gamma\backslash G$ is compact, thus the G_γ are unimodular and there exist indeed biinvariant Haar measures on these groups.
For the proof of this theorem see [DG3]; it makes use of a trace formula for the heat kernel on the manifolds $(\Gamma_1\backslash M,g)$ and $(\Gamma_2\backslash M,g)$.

Remarks:

(i) G is not assumed to be connected. G is also allowed to be 0-dimensional (discrete).

(ii) The above theorem can be used to construct *continuous* isospectral deformations. See Chapter 2 for this.

(iii) We shortly sketch how the Theorem of DeTurck and Gordon can be applied in order to recover the examples of Milnor and Vignéras and the Sunada Theorem:

(a) Tori: $G := O_n \ltimes \mathbb{R}^n$, $\Gamma \subseteq \mathbb{R}^n \subseteq G$, $\Gamma_\gamma = \Gamma$, $G_\gamma = \mathrm{Stab}_{O_n}(\gamma) \ltimes \mathbb{R}^n$.
Choose $\rho^{[h]}$ such that $\rho^{[h]}(\Gamma_\gamma\backslash G_\gamma) = |O_{n-1}|\cdot|\Gamma\backslash\mathbb{R}^n|$ (these $\rho^{[h]}$ satisfy the compatibility condition required in the theorem). Then

$$r(G,\Gamma;h) = |O_{n-1}|\cdot|\Gamma\backslash\mathbb{R}^n|\cdot \#(\Gamma\cap[h]_G) = |O_{n-1}|\cdot|\Gamma\backslash\mathbb{R}^n|\cdot \#\{\gamma\in\Gamma \mid \|\gamma\| = \|h\|\}$$

for $h\in\mathbb{R}^n\subseteq G$ ($r(G,\Gamma;h) = 0$ for $h\in G-\mathbb{R}^n$) .
For all $h\in G$ these numbers are equal for $\Gamma:=\Gamma_1$ and $\Gamma:=\Gamma_2$ if Γ_1 and Γ_2 satisfy condition (*) from 1.2.1. Thus condition (DG) is satisfied.

(b) Hyperbolic surfaces: $G := PSL(2,\mathbb{R})$, $\Gamma \subseteq G$ such that $\Gamma\backslash H^2$ and hence also $\Gamma\backslash G$ is compact; G_γ corresponds to the transvections along the axis of the isometry γ .
Choose $\rho^{[h]}$ such that $\rho^{[h]}(\Gamma_\gamma\backslash G_\gamma) = s_\gamma$ as in 1.2.2 (these $\rho^{[h]}$ satisfy the compatibility condition). Then, since $s_\gamma = s_h$ for all $\gamma\in[h]_G$:

$$r(G,\Gamma;h) = s_h \cdot \# \{[\gamma]_\Gamma \mid [\gamma]_\Gamma \subseteq [h]_G\} = s_h \cdot \# \{[\gamma]_\Gamma \mid \mathrm{trace}(\gamma) = \mathrm{trace}(h)\}$$

for $h\in G$ hyperbolic ($r(G,\Gamma;h) = 0$ for the others since Γ contains only hyperbolic isometries).
For all $h\in G$ these numbers are equal for $\Gamma:=\Gamma_1$ and $\Gamma:=\Gamma_2$ if Γ_1 and Γ_2 satisfy condition (**) from 1.2.2. Thus condition (DG) is satisfied.

(c) Sunada's Theorem: G finite; choose $\rho^{[h]} := \#$ (the counting measure). Then

$$\rho^{[h]}(\Gamma_\gamma\backslash G_\gamma) = (\# G_\gamma)/(\#\Gamma_\gamma) = (\# G \cdot \#[\gamma]_\Gamma) \;/\; (\#[\gamma]_G \cdot \#\Gamma)$$

and hence

$$r(G,\Gamma;h) = ((\# G) \;/\; (\#[h]_G \cdot \#\Gamma)) \cdot \sum_{[\gamma]_\Gamma \subseteq [h]_G} \#[\gamma]_\Gamma$$

$$= ((\# G) \;/\; (\#[h]_G \cdot \#\Gamma)) \cdot \#(\Gamma \cap [h]_G)\,.$$

For all $h \in G$ these numbers are equal for $\Gamma := \Gamma_1$ and $\Gamma := \Gamma_2$ if Γ_1 and Γ_2 satisfy the Sunada condition (S) from 1.3.1. Thus again condition (DG) is satisfied.

CHAPTER 2: Continuous Isospectral Deformations

As already mentioned above, the Theorem of DeTurck and Gordon (see 1.3.2) is a tool for the construction of continuous isospectral deformations. The first examples of such deformations were given already in 1984 by Gordon and Wilson (see [GW1]) where a different isospectrality argument (using Kirillov Theory of unitary representations) was applied; in [DG3], DeTurck and Gordon proved the above Theorem and applied it to obtain a generalization of the first construction of continuous isospectral deformations.

In this context, the notion of "almost inner automorphisms" ([DG1], [Go2], [DG3]), first introduced in [GW1] in a slightly different way, plays a central role.

2.1 Definition

Let G be a Lie group and Γ a subgroup of G. Then

$$AIA(G;\Gamma) := \{\Phi \in Aut(G) \mid \forall\, \gamma \in \Gamma\ \exists\, a \in G : \Phi(\gamma) = a\gamma a^{-1}\}$$

is called the group of almost inner automorphisms of G relative to Γ .

2.2 Proposition ([DG3])

Let G be a Lie group, g a left invariant metric on G, Γ a cocompact discrete subgroup of G, and let $\Phi \in AIA(G;\Gamma)$ be such that the following additional condition is satisfied:

(∗) Whenever $\gamma \in \Gamma$, and $a \in G$ is chosen such that $\Phi(\gamma) = I_a(\gamma)$, then

$$\det\, ((I_a^{-1} \circ \Phi)_{*e}|_{T_e G_\gamma}) = 1\,.$$

Then $(\Phi(\Gamma)\backslash G,g)$ and $(\Gamma\backslash G,g)$ are isospectral.

The proof of this proposition is an application of the Theorem of DeTurck and Gordon (1.3.2): In order to establish condition (DG), it only remains to show that the diffeomorphism $\Phi : G_\gamma \to G_{\Phi(\gamma)}$ which descends to the quotients $\Gamma_\gamma\backslash G_\gamma$ and

$\Phi(\Gamma)_{\Phi(\gamma)}\backslash G_{\Phi(\gamma)}$ is measure preserving with respect to the measures ρ. But this is the case because $I_a^{-1} : G_{\Phi(\gamma)} \to G_\gamma$ is measure preserving by the choice of the measures (compatibility condition) and $I_a^{-1} \circ \Phi : G_\gamma \to G_\gamma$ is also measure preserving by condition (*) in the proposition.

2.3 Observation ([DG3])

If G is simply connected and nilpotent, Γ a cocompact discrete subgroup of G and $\Phi \in AIA(G;\Gamma)$, then condition (*) of 2.2 is automatically satisfied.

(All the $(I_a^{-1} \circ \Phi)_{*e}$ as in (*) can be shown to be unipotent automorphisms of T_eG in this case. Hence they have determinant 1 on every invariant subspace.)

Remarks:

(i) If the automorphism Φ is not only almost inner but inner, then the manifolds $(\Gamma\backslash G,g)$ and $(\Phi(\Gamma)\backslash G,g)$ are isometric, and their isospectrality is trivial.
But in general AIA(G;Γ) is bigger than Inn(G), the group of inner automorphisms, and for $\Phi \in AIA(G;\Gamma)-Inn(G)$ the two manifolds are not isometric in general.

(ii) In some Lie groups G one can find continuous families Φ_t in AIA(G;Γ), not contained in Inn(G), with Φ_0=id and all Φ_t satisfying condition (*) of 2.2. By such families, examples of continuous isospectral deformations can be constructed (see below). Note that obviously, $(\Phi_t(\Gamma)\backslash G,g)$ is isometric to $(\Gamma\backslash G,\Phi_t^*g)$; so the deformation can indeed be interpreted as a deformation of the metric on a fixed manifold.

2.4 Examples

We now describe three explicit examples and in this context discuss also some approaches to the description of the geometrical changes occurring during the deformations.

2.4.1 The "standard" example ([GW1])

In this first explicit example of a continuous isospectral deformation ever given (in [GW1], studied also in [DG1], [DG2], [DGGW1]) the Lie group G is two-step nilpotent. Such groups provide the "simplest" kind of groups in which almost inner but non-inner automorphisms can occur.

Let G be the simply connected Lie group corresponding to the Lie algebra $\mathfrak{g}$ which is spanned by $\{X_1, Y_1, X_2, Y_2, Z_1, Z_2\}$ and whose nontrivial Lie brackets are given by $[X_i, Y_i] = Z_1$ (i = 1, 2) and $[X_1, Y_2] = Z_2$.

Let Γ be the subgroup of G which is generated by $\{\exp X_1, \ldots , \exp Z_2\}$, where exp : $\mathfrak{g} \to G$ denotes the exponential mapping. Γ is a cocompact discrete subgroup of G.

Define $\Phi_t \in Aut(G)$ by requiring that $(\Phi_t)_{*e}$ maps Y_1 to Y_1+tZ_1 and equals the identity on span$\{X_1, X_2, Y_2, Z_1, Z_2\}$. One can check that $\Phi_t \in AIA(G;\Gamma)$ (indeed Φ_t

$\in$ AIA(G;G)). Moreover, by the above Observation 2.3, all Φ_t satisfy condition (*) of Proposition 2.2.

Let g be the left invariant metric on G which makes the left invariant vectorfields X_1, Y_1, ... , Z_2 orthonormal, and let $g_t := \Phi_t^* g$. Then $\{X_1, Y_1 - tZ_1, X_2, Y_2, Z_1, Z_2\}$ is an orthonormal frame for g_t .

Now by Proposition 2.2,

$$(\Gamma\backslash G, g_t)$$

is a continuous isospectral deformation.

Note that topologically, $\Gamma\backslash G$ is a nontrivial T^2-bundle over T^4 (T^2 and T^4 denote the two- resp. four-dimensional torus). The fibers are integral manifolds of the left invariant distribution span$\{Z_1, Z_2\}$ and are totally geodesic for each g_t .

Why is the above deformation nontrivial?

The manifolds occurring during the deformation have many communities, for example all of them have the same volume (see 1.1 a)), they are pairwise locally isometric because $(\Gamma\backslash G, g_t)$ is isometric to $(\Phi_t(\Gamma)\backslash G, g)$ which is locally isometric to $(\Gamma\backslash G, g)$, and they all have the same length spectrum by a theorem of Carolyn Gordon ([Go2]) which holds for a wide class of examples constructed by using Theorem 1.3.2.

Nevertheless the deformation is nontrivial, as guaranteed for example by a result of Gordon and Wilson (see [GW1]) about the isometry classes of the metrics $\{\Phi^* g \mid \Phi \in \mathrm{Aut}(G)\}$ where G is a simply connected solvable group with only real roots and g is a left invariant metric on G.
Their result implies in particular that if G and g are as just mentioned, if Γ is a cocompact discrete subgroup and Φ_t a continuous family of automorphisms of G starting from id, then a deformation of the form $(\Gamma\backslash G, \Phi_t^* g)$ can not be trivial if not all of the Φ_t are inner automorphisms.
But this is easily seen not to be the case in the above example, so the deformation must be nontrivial.

This approach is rather abstract and does not really shed light into *how* the geometry changes during the deformation.
In [DGGW1] this question was studied thoroughly for the above example. It turned out that the geometrical change during the deformation can be detected by considering the *relative position* of shortest closed geodesics in different free homotopy classes.

More explicitly, in this example the situation is the following:
The g_t-shortest geodesic loops in the two free homotopy classes corresponding to the Γ-conjugacy classes $[\exp Y_1]_\Gamma$ and $[\exp Y_2]_\Gamma$ foliate two submanifolds $M_1(t)$ resp. $M_2(t)$ of $\Gamma\backslash G$ whose g_t-distance equals dist$(t,\mathbb{Z}) + $ const.
Now by the countability of the set of all free homotopy classes of $\Gamma\backslash G$ the deformation must be nontrivial. (DeTurck/Gluck/Gordon/Webb in [DGGW1])

Note that the two-dimensional direction span$\{Y_1, Y_2\}$ which plays an important role in this geometrical argument is just the two-dimensional space on which the automorphisms $(\Phi_t)_{*e}$ reveal their non-innerness: Indeed, an inner automorphism which maps Y_1 to Y_1+tZ_1 would necessarily map Y_2 to Y_2+tZ_2 .

There is a series of papers ([DGGW1] - [DGGW6]) by the same authors where they study the geometry of numerous examples of continuous isospectral deformations using the above and other similar approaches.

A natural question which was *not* considered in those papers is the following:

> How long is the interval $[0, t_0)$ during which all the $(\Gamma\backslash G,g_t)$ are pairwise non-isometric?

Or more strongly:

> What is the parameter of the isometry classes of the manifolds $(\Gamma\backslash G,g_t)$?

We will now exhibit a method of determining a "strong isometry invariant" for the above example, i.e. a geometrically defined number, depending on t, which distinguishes different isometry classes.
This number is obtained by an appropriately chosen geometrical algorithm. A similar algorithm, serving the mentioned purpose, can be formulated in all examples where G is two-step nilpotent and also in many other examples.

For the example considered above, we define a number d_t, depending on t, as follows:

Step 1: Determine the *Killing vectorfields* of $(\Gamma\backslash G,g_t)$. These are precisely those induced by the *central* left invariant vectorfields of G.

Step 2: *Divide* $(\Gamma\backslash G,g_t)$ by the flow of the Killing vectorfields. The basis manifold is the four-dimensional torus T^4. The only metric on T^4 for which the projection becomes a *Riemannian submersion* is the flat standard metric. So we have obtained a geometrically defined Riemannian submersion from $(\Gamma\backslash G,g_t)$ onto (T^4, standard).

Step 3: Consider *all shortest closed geodesics* of T^4 and determine those *g_t-horizontal lifts* of them to $(\Gamma\backslash G,g_t)$ which remain *closed* (with the original length). Observe that the closed geodesics in $(\Gamma\backslash G,g_t)$ obtained in this way belong to 4 free homotopy classes ($[\exp X_1]_\Gamma$, $[\exp Y_1]_\Gamma$, $[\exp X_2]_\Gamma$, $[\exp Y_2]_\Gamma$) and that each of these 4 subfamilies of geodesics foliates a *submanifold $M_i(t)$ (i = 1, 2, 3, 4)* of $(\Gamma\backslash G,g_t)$.

Step 4: Determine the six *g_t-distances* between the $M_i(t)$. They turn out to be 0, 0, 0, 0, 0, and $d_t := \mathrm{dist}(t,\mathbb{Z})$.

Since the number d_t is thus geometrically defined, we can conclude for t, t' $\in \mathbb{R}$:

| If $d_t \neq d_{t'}$ then $(\Gamma\backslash G, g_t)$ is not isometric to $(\Gamma\backslash G, g_{t'})$.

On the other hand, it is not very difficult to write down an isometry between $(\Gamma\backslash G, g_t)$ and $(\Gamma\backslash G, g_{t'})$ if $d_t = d_{t'}$ (i.e. isometries between g_t and g_{t+1} and between g_t and g_{-t}).
Thus d_t is exactly the parameter of the deformation in the example considered, and $[0, 1/2)$ is the biggest interval during which the manifolds $(\Gamma\backslash G, g_t)$ are pairwise non-isometric. (See also [Sch1] for details.)

2.4.2 An example with G not nilpotent but exponentially solvable

Introducing Remark:
The first continuous isospectral deformations, constructed by Gordon and Wilson, date back to 1984, as mentioned already at the beginning of this chapter.
They were of the form $(\Gamma\backslash G, \Phi_t^* g)$ with G simply connected and exponentially solvable and with an almost innerness condition for the automorphisms Φ_t on the dual of the Lie algebra.
However, in all explicit examples that were given G was actually nilpotent, except in one example ([GW1], example 2.4(iv)) where G was solvable; but nevertheless the deformation that was constructed in this example and the corresponding family of almost inner automorphisms were interesting again only on the nilradical of G .
The following is an example in the solvable case which can *not* be reduced – as the first solvable example just mentioned – to a nilpotent situation.

Let G be the simply connected Lie group corresponding to the Lie algebra $\mathfrak{g}$ which is spanned by $\{X_1, Y_1, X_2, Y_2, Z\}$ and whose nontrivial Lie brackets are given by $[X_i, Y_i] = Z$ $(i = 1, 2)$ and $[X_1, X_2] = X_2$, $[X_1, Y_2] = -Y_2$. Note that G is unimodular (otherwise there would be no hope to construct a cocompact discrete subgroup).

Let Γ be the subgroup of G which is generated by $\{\exp(t_0X_1), \exp(Y_1/t_0), \exp Q, \exp U, \exp(Z/2)\}$, where t_0 is chosen such that $\exp(\mathrm{ad}(t_0X_1))$ is similar to $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$ on $\mathrm{span}\{X_2, Y_2\}$ and $\{Q, U\}$ is a basis of $\mathrm{span}\{X_2, Y_2\}$ with respect to which $\exp(\mathrm{ad}(t_0X_1))$ has this form. Then Γ is indeed a cocompact discrete subgroup of G. (This construction of Γ is analogous to the one in the solvable example 2.4(iv) in [GW1] which was mentioned above.

Define $\Phi_t \in \mathrm{Aut}(G)$ by requiring that $(\Phi_t)_{*e}$ maps Y_1 to Y_1+tZ and equals the identity on $\mathrm{span}\{X_1, X_2, Y_2, Z\}$. One can check that $\Phi_t \in \mathrm{AIA}(G;\Gamma)$ and all Φ_t satisfy condition (*) of Proposition 2.2 (note that here, this last fact does *not* follow from Observation 2.3).

Let g be the left invariant metric on G which makes the left invariant vectorfields $X_1, Y_1, \ldots , Z$ orthonormal, and let $g_t := \Phi_t^* g$. So $\{X_1, Y_1-tZ, X_2, Y_2, Z\}$ is an orthonormal frame for g_t .

By Proposition 2.2,

$$(\Gamma\backslash G, g_t)$$

is a continuous isospectral deformation.

Topologically, $\Gamma\backslash G$ is a T^2-bundle (with fibers that are integral manifolds of the left invariant distribution span$\{Y_1, Z\}$) over a T^2-bundle (with fibers that are integral manifolds of the left invariant distribution span$\{X_2, Y_2\}$ in the basis manifold of the first bundle) over a circle. In both bundles, the two-dimensional fibers are *not* totally geodesic with respect to the g_t resp. with respect to the metric associated to the g_t on the basis manifold of the first bundle such that the submersion becomes a Riemannian submersion (for every t).
Note that the 3-dimensional Lie-group $\bar{G} := G / \exp(\text{span}\{Y_1, Z\})$ is just the group E(1, 1) of rigid motions of the plane with respect to the pseudo-metric with signature (1,1).

Nontriviality of the deformation in this example:

One can show, for example, that the g_t-shortest geodesic loops in the two free homotopy classes corresponding to $[\exp(Y_1/t_0)]_\Gamma$ and $[\exp Q]_\Gamma$ foliate two submanifolds $M_1(t)$ and $M_2(t)$ of $\Gamma\backslash G$ which have non-constant g_t-distance dist(t, $t_0\mathbb{Z}$). Thus the deformation is nontrivial by the same countability argument as in the previous example 2.4.1.
But here it turns out that the number dist(t, $t_0\mathbb{Z}$) is *not yet* the parameter of this deformation; instead $d_t :=$ dist(t, $t_0/2 \cdot \mathbb{Z}$) is the parameter. The reason for this is that the g_t-shortest geodesic loops in $[\exp Q]_\Gamma$ can not be distinguished geometrically from the g_t-shortest geodesic loops in $[\exp U]_\Gamma$; only the *union* of these two families of geodesic loops can be defined geometrically.
A geometrically defined algorithm that yields the number d_t = dist(t, $t_0/2 \cdot \mathbb{Z}$) and thus shows it to be a strong isometry invariant is the following:

- *Divide* $(\Gamma\backslash G, g_t)$ by the flow of its *Killing fields* (these are the *central* fields again) and consider the *closed g_t-horizontal lifts* of the *integral curves* of the *Killing fields of the quotient* which is endowed with the unique metric that makes the projection a Riemannian submersion. (These lifts are closed integral curves of the left invariant vectorfield $Y_1 - tZ_1$.)
  Observe that they foliate a submanifold $M_1(t)$ of $\Gamma\backslash G$.

- *Divide* $(\Gamma\backslash G, g_t)$ *twice* by the flow of the respective *Killing fields* ("first Z, then Y_1") and consider the *closed g_t-horizontal lifts of the (globally) shortest geodesic loops* of the three-dimensional quotient. (These lifts are closed integral curves of the left invariant vectorfields Q and U .)
  Observe that they foliate a submanifold $M_2(t)$ of $(\Gamma\backslash G, g_t)$.

- The *g_t-distance* between $M_1(t)$ and $M_2(t)$ is just d_t = dist(t, $t_0/2 \cdot \mathbb{Z}$) .

Thus d_t is indeed a strong isometry invariant.
Since on the other hand it is not difficult to write down explicitly an isometry

between $(\Gamma\backslash G, g_t)$ and $(\Gamma\backslash G, g_{t'})$ if $d_t = d_{t'}$, we conclude that d_t is exactly the parameter of the deformation.
For more details about this example see [Sch3].

2.4.3 An example of an isospectral deformation on a manifold that is diffeomorphic to a Heisenberg manifold

Introducing Remark:

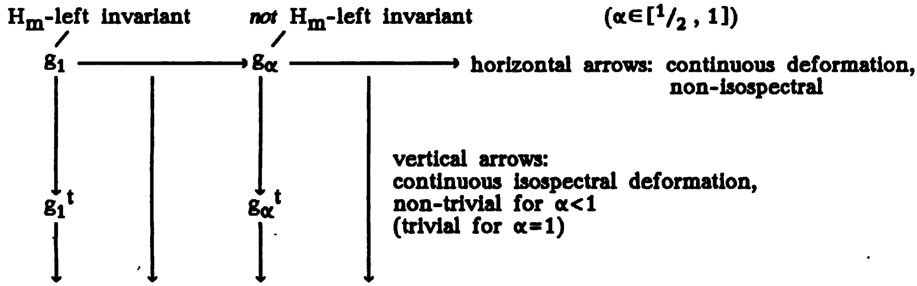
In 1991, He Ouyang and Hubert Pesce proved independently:

> If G is 2-step nilpotent, then *all* continuous isospectral deformations of the type $(\Gamma\backslash G, g_t)$ with left invariant metrics g_t are obtained by the method of almost inner automorphisms, i.e. there is a continuous family of automorphisms $\Phi_t \in AIA(G;\Gamma)$ with $\Phi_0 = id$ and $g_t = \Phi_t^* g_0$. (See [Ou], [Pe1], [Pe2], or [OP].)

This implies in particular that if G is 2-step nilpotent and does not admit nontrivial almost inner automorphisms, i.e. if $AIA(G;\Gamma) = Inn(G)$, then for every left invariant metric g, $(\Gamma\backslash G, g)$ is *infinitesimally spectrally rigid* within the family of left invariant metrics.
This is the case, for example, for the classical Heisenberg groups H_m which have Lie algebra $\mathfrak{h}_m$ generated by $\{X_1, Y_1, \ldots , X_m, Y_m, Z\}$ with nontrivial Lie brackets given by $[X_i, Y_i] = Z$ $(i = 1, 2, \ldots m)$.
Thus every compact Riemannian Heisenberg manifold of the type $(\Gamma\backslash H_m, g)$ (where g is a left invariant metric on H_m) is infinitesimally spectrally rigid within the family of left invariant metrics.

But the following example will show that nevertheless there are nontrivially isospectrally deformable Riemannian manifolds diffeomorphic to and arbitrarily "close" to certain Heisenberg manifolds.
More precisely, we will exhibit a (non-isospectral) continuous deformation of a certain Heisenberg manifold $(\Gamma\backslash H_m, g)$ such that every manifold occurring in this deformation, except the initial manifold, *is* nontrivially isospectrally deformable:

The underlying manifold will be $\Gamma\backslash H_m$, and we will construct a continuous two-parameter family of metrics on it with the following properties:



All of the metrics ${g_\alpha}^t$ will be left invariant with respect to some other (not nilpotent, but solvable) group structure on the manifold H_m, but only $g_1 \cong {g_1}^t$ will be H_m-left invariant.
We give the construction explicitly for m=2:

Let $\mathfrak{h} := \mathfrak{h}_2$ be the five-dimensional Heisenberg algebra which is spanned by $\{X_1, Y_1, X_2, Y_2, Z\}$ and whose nontrivial Lie brackets are given by $[X_i, Y_i] = Z$ ($i = 1, 2$).
Let $\mathfrak{g}$ be the five-dimensional Lie algebra which is spanned by $\{X_1, Y_1, X_2, Y_2, Z\}$ and whose nontrivial Lie brackets are given by $[X_i, Y_i] = Z$ ($i = 1, 2$) and moreover $[X_1, X_2] = Y_2$, $[X_1, Y_2] = -X_2$.

Let H_2 resp. G be the simply connected Lie groups corresponding to these Lie algebras.
Note that G is solvable but not exponentially solvable (unlike the group in example 2.4.2), i.e. $\exp : \mathfrak{g} \to G$ is not bijective, since $\mathrm{ad}X_1$ has nontrivial purely imaginary eigenvalues. Note furthermore that the 3-dimensional group $\bar{G} := G / \exp(\mathrm{span}\{Y_1, Z\})$ is just the group $E(2)$ of rigid motions of the plane endowed with the standard euclidean metric.

Let h resp. g_1 be the left invariant metrics on H_2 resp. G that make the left invariant vectorfields $X_1, Y_1, \ldots, Z$ orthonormal in both cases.

Let $\tilde{\Gamma} \subseteq H_2$ resp. $\Gamma \subseteq G$ be the subgroups generated by $\{\exp(2\pi X_1), \exp(Y_1/2\pi), \exp X_2, \exp Y_2, \exp(Z/2)\}$. These groups can be shown to be indeed discrete and cocompact in H_2 resp. G.

It is not difficult to see that there is an isometry F from (G,g) to (H_2,h) which equals the "identity" on the tangent spaces $T_eG = \mathfrak{g}$ and $T_eH_2 = \mathfrak{h}$ with respect to the names of the basis elements; moreover, this F satisfies $F(\Gamma) = \tilde{\Gamma}$ and $F(\gamma \cdot x) = F(\gamma) \cdot F(x)$ for all $\gamma \in \Gamma$ and $x \in G$. Thus F descends to an isometry between the compact quotients $(\Gamma\backslash G, g_1)$ and $(\tilde{\Gamma}\backslash H_2, h)$.

Note that $\Gamma \cong \tilde{\Gamma}$ is nilpotent although it is cocompact in the non-nilpotent group G. Note furthermore that topologically, $\Gamma\backslash G$ is – just like $\tilde{\Gamma}\backslash H_2$ – a nontrivial circle bundle over T^4.

Let Φ_t be the automorphism of G defined by requiring that $(\Phi_t)_{*e}$ maps Y_1 to Y_1+tZ and equals the identity on $\mathrm{span}\{X_1, X_2, Y_2, Z\}$. One can check by somewhat tedious calculations that $\Phi_t \in AIA(G;\Gamma)$ and Φ_t satisfies condition $(*)$ of Proposition 2.2. Thus

$$(\Gamma\backslash G, {\Phi_t}^* g)$$

is an isospectral deformation for every metric on $\Gamma\backslash G$ that is induced by a G-left invariant metric.

– Now if we start with $g := g_1 \cong h$, then the above deformation turns out to be trivial .

– But if we start with the slightly different metric $g := g_\alpha$ $(0<\alpha<1)$ which makes $\{X_1, Y_1, X_2, \alpha Y_2, Z\}$ orthonormal and set $g_\alpha{}^t := \Phi_t{}^* g_\alpha$, then the deformation

$$(\Gamma\backslash G, g_\alpha{}^t) \quad \text{with } \alpha \text{ fix and } t \text{ varying}$$

is nontrivial at least for $(0.157... \approx)\ (4\pi^2+1)^{-1/2} < \alpha < 1$.

This can be shown by geometrical arguments similar to those in Examples 2.4.1 and 2.4.2; the parameter of each of the deformations (for α in the open interval written down three lines above) turns out to be $d_t :=$ dist$(t, \pi\mathbb{Z})$.

Remarks:

(i) For more details about this example see [Sch3].

(ii) The question if there exists a nontrivial isospectral deformation of $(\tilde{\Gamma}\backslash H_2, h)$ *itself* – then necessarily with metrics that are not H_2-left invariant – remains open.

(iii) Another open question is whether the analogon of the above result of Ouyang and Pesce (which is valid for the 2-step nilpotent case) holds also in the n-step nilpotent or in the solvable case. About this there are no results except the following which is a weaker one:

2.5 Proposition:
Let G be simply connected and nilpotent of arbitrary step or solvable with only real roots. If Γ_t is a continuous family of cocompact discrete subgroups of G such that the *quasi-regular unitary representations* ρ_{Γ_t} of G on $L^2(\Gamma_t\backslash G)$ (defined by $((\rho_{\Gamma_t}(h))(f))(x) = f(x \cdot h^{-1})$) are pairwise unitarily equivalent, then there exists a continuous family of automorphisms $\Phi_t \in AIA(G;\Gamma_0)$ with $\Phi_0 = id$ and $\Gamma_t = \Phi_t(\Gamma_0)$.

(For the proof see [Sch2].)

The reason why this result is *weaker* than a possible analogon for isospectral deformations is the fact that equivalence of the quasi-regular representations *implies* isospectrality of the $(\Gamma_t\backslash G, g)$ if g is any fixed left invariant metric. This can easily be seen by expressing the Laplace operator on $(\Gamma_t\backslash G, g)$ in terms of $(\rho_{\Gamma_t})_*$ (see [GW1]).
So Proposition 2.5 is at the same time a result about *some* isospectral deformations of $(\Gamma_0\backslash G, g)$, but a priori these might not be *all*.

On the other hand: All continuous isospectral deformations of the form $(\Gamma_t\backslash G, g)$ that are known until now are of the type $(\Phi_t(\Gamma_0)\backslash G, g)$, where the Φ_t are in $AIA(G;\Gamma)$ and satisfy condition (*) of Proposition 2.2. Thus the discrete subgroups Γ_0 and Γ_t satisfy condition (DG) of Theorem 1.3.2 for all t. But this is in turn equivalent to the unitary equivalence of the corresponding quasi-regular representations, as shown by P. Bérard in [Bé4].

This means that in all *known* examples of continuous isospectral deformations of the form $(\Gamma_t\backslash G, g)$ the quasi-regular representations ρ_{Γ_t} *are* indeed pairwise equivalent.
Thus Proposition 2.5 says also that counterexamples to the result of Ouyang and

Pesce in the higher step nilpotent or in the solvable case – if they exist – must be very "different" from the continuous isospectral deformations known until now.

CHAPTER 3: Isospectral Plane Domains

In this chapter we describe the first example of isospectral, non-congruent bounded plane domains which was found by Carolyn Gordon, David Webb and Scott Wolpert in 1991.
It is based on Bérard's generalized version of the Theorem of Sunada (see 1.3.1):

3.1 Theorem (P. Bérard 1991; [Bé3])
Let G be a finite group acting by isometries on a compact Riemannian manifold (M,g) (possibly with boundary) and let Γ_1 and Γ_2 be subgroups of G satisfying the Sunada condition

(S) $$\forall\, h \in G : \# (\Gamma_1 \cap [h]_G) = \# (\Gamma_2 \cap [h]_G)$$

Then the spectra of eigenvalues belonging to Γ_i-invariant eigenfunctions of Δ_g on M with

a) empty boundary condition if $\partial M = \emptyset$
b) Dirichlet boundary condition
c) Neumann boundary condition

are equal for $i = 1, 2$ in each of the three cases a), b), c).

Remarks:

(i) Note that Γ_1 and Γ_2 are not supposed to act without fixed points.

(ii) In case $\partial M = \emptyset$ (a)) and if Γ_1 and Γ_2 act freely on M, Theorem 3.1 implies the original Theorem of Sunada (1.3.1).

(iii) If $\partial M \neq \emptyset$ and if Γ_1 and Γ_2 act freely on M, then Theorem 3.1 implies that $(\Gamma_1\backslash M, g)$ and $(\Gamma_2\backslash M, g)$ (where g denotes the induced metrics again by abuse of notation) are Dirichlet- and Neumann-isospectral.

(iv) Cases b) and c) will be exploited in a special two-dimensional example below where $(\Gamma_1\backslash M, g)$ and $(\Gamma_2\backslash M, g)$ can be shown to be Dirichlet- and Neumann-isospectral although Γ_1 and Γ_2 act with fixed points and $\Gamma_1\backslash M$ and $\Gamma_2\backslash M$ have "additional" boundary arcs.

Bérard's Theorem follows from

3.2 Proposition ([Bé3]):
Let G be a finite group and Γ_1, Γ_2 two subgroups of G which satisfy the Sunada condition (S). Let V be a Hilbert space on which G acts unitarily. Then there

exists an isometry between the fixed point sets V^{Γ_1} and V^{Γ_2} .

In order to prove the Theorem from the Proposition let in particular

0) $V := L^2(M,v_g)$ with the L^2-metric

a) $V := H^{1,2}(M,v_g)$ with the $H^{1,2}$-metric for $\partial M=\emptyset$

b) $V := H^{1,2}_D(M,v_g) := \{f \in H^{1,2}(M,v_g) \mid f|_{\partial M} = 0\}$ with the $H^{1,2}$-metric

c) $V := H^{1,2}_N(M,v_g) := \{f \in H^{1,2}(M,v_g) \mid \partial_\nu f|_{\partial M} = 0\}$ with the $H^{1,2}$-metric

G acts unitarily on each of these spaces by $f \mapsto f \circ h^{-1}$ (where h^{-1} is interpreted as an isometry of (M,g) on which G acts isometrically).
The isometries between V^{Γ_1} and V^{Γ_2} in the cases a), b), c) can be constructed as the *restrictions* (as maps) of the isometry constructed in case 0). This follows from the proof of the proposition (see a sketch of it below; indeed, once a certain operator T as in step (i) of the proof is chosen, the rest of the construction is "natural", i.e. respects inclusions).
Given this, Theorem 3.1 follows immediately from the *variational characterization* of the eigenvalues of the Laplacian (see e.g. [Bé1]):

The k-th eigenvalue (with multiplicities taken into account) on V, where V is as in a), b), or c), is given by

$$\lambda_k = \inf_{L\in U_k} \sup \{R(f) \mid f \in L-\{0\}\}$$

where U_k is the set of all k-dimensional subspaces of V, and

$$R(f) := \frac{\|f\|^2_{H^{1,2}}}{\|f\|^2_{L^2}} - 1 = \frac{\|df\|^2_{L^2}}{\|f\|^2_{L^2}}$$

is the Rayleigh Quotient.

We now give a <u>sketch of Bérard's proof of Proposition 3.2</u> (see [Bé3]):

Step (i):
Consider the right regular representations ρ_{Γ_i} of G on $L^2(\Gamma_i\backslash G) \cong \mathbb{C}^{\#(\Gamma_i\backslash G)}$ (i = 1, 2). One easily computes $tr(\rho_{\Gamma_i}(h)) = \#(\Gamma_i \cap [h]_G)$ for $h\in G$. Thus by (S), ρ_{Γ_1} and ρ_{Γ_2} are unitarily equivalent, i.e. there exists a unitary operator $T : L^2(\Gamma_1\backslash G) \to L^2(\Gamma_2\backslash G)$ which intertwines ρ_{Γ_1} and ρ_{Γ_2} .

Step (ii):
Denote the action of G on V by ρ . Then G acts on $V\otimes L^2(\Gamma_i\backslash G)$ by $\rho\otimes\rho_{\Gamma_i}$ (i = 1, 2). By purely algebraic methods one can establish a canonical isometry

$$F_i : V^{\Gamma_i} \to (V\otimes L^2(\Gamma_i\backslash G))^G .$$

Step (iii):
By the choice of T, $id\otimes T$ is an isometry from $(V\otimes L^2(\Gamma_1\backslash G))^G$ to $(V\otimes L^2(\Gamma_2\backslash G))^G$.
Thus $F_2^{-1} \circ (id\otimes T) \circ F_1$ is the required isometry from V^{Γ_1} to V^{Γ_2} .

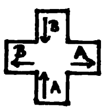
3.3 Application: Construction of an example of two Dirichlet- and Neumann-isospectral plane domains
(Gordon/Webb/Wolpert 1991; see [GWW1], [GWW2])

3.3.1 First we review an example by P. Buser (see [Bu2]) of two isospectral, non-isometric flat (but not planar) 2-dimensional manifolds with boundary, embedded into $\mathbb{R}^3$, which are constructed by applying the Theorem of Sunada/Bérard in its version for manifolds with boundary; see Remark (iii) after Theorem 3.1.

Let G be the group $SL(3,\mathbb{Z}_2)$ and let Γ_1 resp. Γ_2 be the subgroups consisting of all matrices of the form $\begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$ resp. $\begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}$. One can check that Γ_1 and Γ_2 satisfy the Sunada condition (S). Let $A := \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $B := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

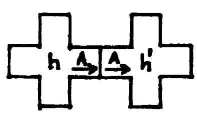
Then the set $\{A, B\}$ generates G. Construct a manifold M as follows:

Take 168 identical flat tiles of the form



labeled with the 168 elements of G, where the arms are labeled with ingoing and outgoing arrows named A and B as shown in the picture. Now put all these tiles together according to the right actions of A and B on G:

Whenever $h' = hA$ ($h, h' \in G$), paste the corresponding tiles together in the following way:

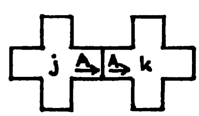


Make the analogous pastings for B and the edges labeled with B.

G acts on the resulting manifold M from the left by isometries in an obvious way and without fixed points. Define $M_1 := \Gamma_1 \backslash M$, $M_2 := \Gamma_2 \backslash M$.

M_1 and M_2 theirselves can be constructed in a similar way as M, by taking tiles labeled with cosets now instead of group elements:
Let $C := ABA^{-1}B^{-1}$. One can compute that $\{C^0, \ldots , C^6\}$ is a representative set for the right cosets of Γ_1 as well as Γ_2 . Now for the construction of M_1 take seven identical tiles as above, labeled from 0 to 6. Put them together according to the right action of A and B on $\Gamma_1 \backslash G$:

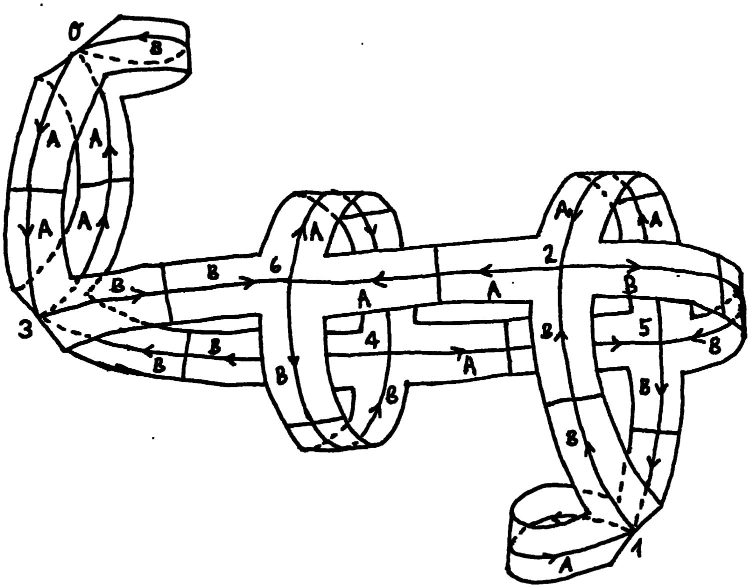
Whenever $\Gamma_1 C^j A = \Gamma_1 C^k$ ($j, k \in \{0, \ldots , 6\}$) then paste the corresponding tiles together in the following way:



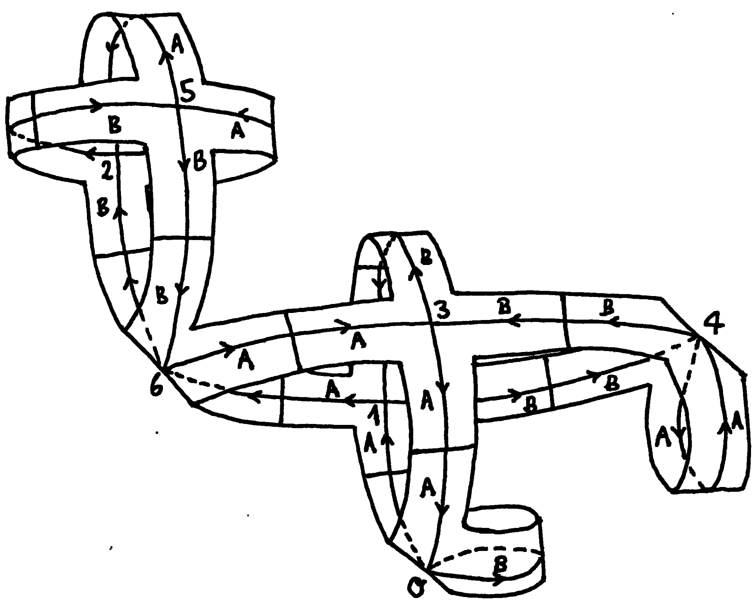
Make the analogous pastings for B and the edges labeled with B.
The construction of M_2 goes analogously with Γ_2 instead of Γ_1 .

The resulting manifolds M_1 and M_2 are Dirichlet- and Neumann-isospectral by Theorem 3.1 (see Remark (iii) after the theorem).

They are shown in the following picture taken from [GWW2] :



M_1



M_2

3.3.2 Gordon, Webb and Wolpert constructed their first example of planar bounded isospectral domains using the above flat isospectral manifolds M_1 and M_2 in the following way:

OBSERVE that M_1 and M_2 each admit an isometric involution τ_1 resp. τ_2 (see the above picture).

One can compute that τ_1 and τ_2 both are induced by a certain isometric involution $\tau : M \to M$ which corresponds to the G-automorphism

$$h \mapsto ShS^{-1} \quad \text{with} \quad S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

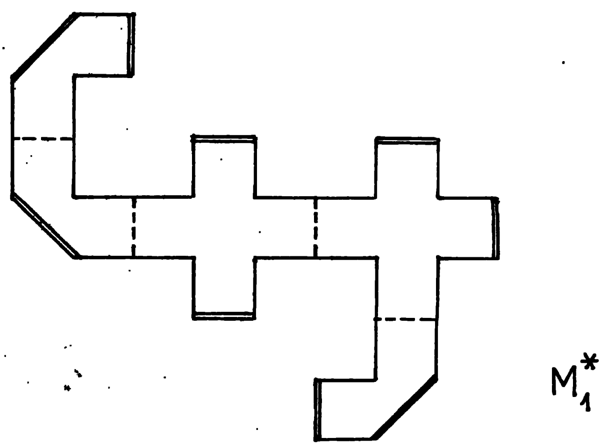
(note that this automorphism of G maps A to A^{-1} and B to B^{-1}, so it inverts the directions of the outgoing arms of the tiles).

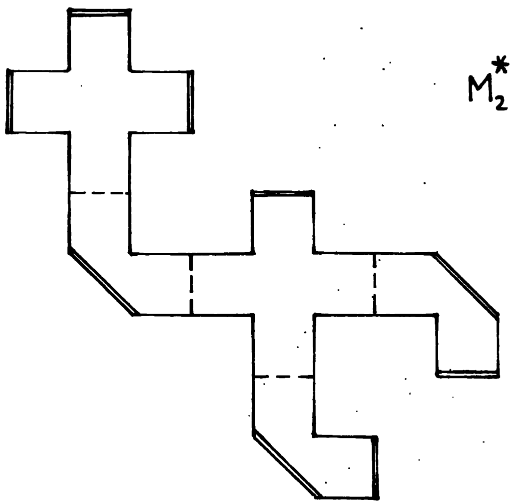
Now define $G^* := \langle\tau\rangle \ltimes G$ where τ acts on G by $h \mapsto ShS^{-1}$, $\Gamma_i^* := \langle\tau\rangle \ltimes \Gamma_i \subseteq G^*$ for $i = 1, 2$ (note that $S \in \Gamma_1 \cap \Gamma_2$).

The subgroups Γ_1^* and Γ_2^* of G^* again satisfy the Sunada condition (S) in G^*: In fact, $G^* \ni \tau^k \cdot h \mapsto (\tau^k, S^k \cdot h) \in \langle\tau\rangle \times G \cong \mathbb{Z}_2 \times G$ is a group isomorphism which carries Γ_i^* to $\mathbb{Z}_2 \times \Gamma_i$ for $i = 1, 2$; but $\mathbb{Z}_2 \times \Gamma_1$ and $\mathbb{Z}_2 \times \Gamma_2$ obviously satisfy (S) in $\mathbb{Z}_2 \times G$ since Γ_1 and Γ_2 do so in G.

Define $M_1^* := \Gamma_1^* \backslash M = \langle\tau_1\rangle \backslash M_1$
and $M_2^* := \Gamma_2^* \backslash M = \langle\tau_2\rangle \backslash M_2$.

These are two bounded flat domains. Some of their boundary arcs are the images of corresponding boundary arcs of M_i under the projection $M_i \to M_i^*$ defined by τ_i ; some other boundary arcs are "additional" and correspond to the fixed point set of the involution τ_i . Denote the first part of the boundary (as just described) by $\partial_1 M_i^*$ and the "new" part of the boundary by $\partial_2 M_i^*$ ($i = 1, 2$).
M_1^* and M_2^* are shown in the following picture, taken again from [GWW2], in which the arcs that belong to $\partial_2 M_i^*$ are depicted by double lines:





By applying Bérard's Theorem (3.1) to M and Γ_1^* , Γ_2^* it follows that the eigenvalue series belonging to Γ_i^*-invariant Dirichlet- (resp. Neumann-) eigenfunctions of Δ on M are the same for i = 1, 2; equivalently: The eigenvalue series belonging to τ_i-invariant Dirichlet- (resp. Neumann-) eigenfunctions on the M_i are the same for i = 1, 2 .

But every τ_i-invariant eigenfunction on M_i descends to an eigenfunction on M_i^* which *automatically* satisfies Neumann boundary conditions on $\partial_2 M_i^*$.
Thus from the equality of the spectra of τ_1-invariant Neumann-eigenfunctions on M_1 and τ_2-invariant Neumann eigenfunctions on M_2 it follows that indeed M_1^* and M_2^* theirselves are Neumann-isospectral:

$$\mathrm{spec}_N(M_1^*) = \mathrm{spec}_N(M_2^*) .$$

And similarly:

(1) $$\mathrm{spec}_{D,N}(M_1^*) = \mathrm{spec}_{D,N}(M_2^*)$$

where $\mathrm{spec}_{D,N}(M_i^*)$ denotes the series of eigennvalues belonging to eigenfunctions that satisfy Dirichlet conditions on $\partial_1 M_i^*$ and Neumann conditions on $\partial_2 M_i^*$.

At last observe that

(2) $$\mathrm{spec}_D(M_i) = \mathrm{spec}_D(M_i^*) \,\dot{\cup}\, \mathrm{spec}_{D,N}(M_i^*)$$

where $\dot{\cup}$ denotes the disjoint union in the sense of series.
To see this, consider the canonical splitting of each Dirichlet eigenfunction on M_i into a τ_i-invariant and a τ_i-antiinvariant Dirichlet eigenfunction. These induce eigenfunctions on M_i^* which satisfy D,N- resp. D,D- boundary conditions on M_i^* – and vice versa. (The inverse mapping is injective since a Dirichlet eigenfuction on M_i which is both τ_i-invariant and τ_i-antiinvariant can easily shown to be zero.)

Now it follows from (1), (2), and the Dirichlet-isospectrality of M_1 and M_2 that

$$spec_D(M_1^*) = spec_D(M_2^*) .$$

Thus M_1^* and M_2^* are also Dirichlet-isospectral and hence yield the first example of two drums which have different shape but the same sound.
(For more details see [GWW2].)

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