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ON SOME RATIONAL FIBRATIONS WITH NON-VANISHING MASSEY PRODUCTS OVER HOMOGENEOUS SPACES

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It is well known that formality or non-formality (in the sense of Sullivan) of a manifold implies strong geometric consequences. For example, if a manifold carries a Kählerian structure, it must be formal [3]. In fact it is not easy to determine whether the given topological space is formal or not (see, e.g. [5]). It is known that if a topological space has non-zero Massey products, it is not formal. Therefore, it is an interesting question to describe various classes of topological spaces with non-vanishing Massey products. The first example of that kind in the case of compact homogeneous nilmanifold was obtained by L. Cordero, M. Fernandez and A. Gray in [2]. In [11] the author considered the "opposite" case of homogeneous spaces of compact simple Lie group. In geometric applications also the natural question of constructing a fibration with the given base and fiber and with the total space of the fibration possessing non-vanishing Massey products arises.

In the present note we desribe a certain method of constructing such fibrations.

THEOREM. *There exists a rational fibration*

$$S^2 \to E \to M \approx CP^3$$

with the fiber being 2-dimensional homotopy sphere and the base of the cohomology type of 3-dimensional complex projective space such that E has non-vanishing Massey products.

[1] This paper is in final form and no version of it will be submitted for publication elsewhere.

REMARK. In fact it is possible to obtain more general results, using our methods, but we present here the simplest version.

Introduce appropriate notations. To keep this note reasonably short, the reader is supposed to be familiar with the standard facts of cohomology theory of homogeneous spaces and rational homotopy theory. We shall use all the notations and facts in accordance with [1, 5, 6, 7] without additional explanations.

Let $M = G/H$ be a homogeneous space of compact simply connected Lie group G. Choose maximal tori T and T' of Lie groups G and H respectively in such a way that $T' \subset T$. Denote by $W(G,T)$ and $W(H,T')$ the Weyl groups of G and H with respect to their maximal tori T and T'. For any vector space V, $k[V]$ denotes the polynomial algebra over V and $k[V]^{\Gamma}$ denotes the ring of Γ - invariants of the discrete group $\Gamma \leq GL(V)$. A free graded commutative algebra over a graded vector space V is denoted by $\Lambda(V)$ and the same notation is used for the exterior algebra over V (but the meaning of the notation will always be clear from the context). Lie algebra of a Lie group G is denoted by $L(G)$. We also use the notation $k[X_1, \ldots, X_n]$ for the polynomial ring of variables $X_1, \ldots, X_n$ over the field k.

Proof of the theorem

Everywhere we consider the category DGA of commutative graded differential algebras $(A, d) \in DGA$. We use the notions of minimal models, DGA - models, formality etc. (see [6,7,9], for details). The symbol $V^{\vee}$ is used for the vector space which is dual to V. Recall Chevalley isomorphisms

$$\begin{aligned} Q[L(T)]^{W(G,T)} &\cong Q[f_1, \ldots, f_n], \quad n = \mathrm{rank}(G) \\ Q[L(T')]^{W(G,T')} &\cong Q[u_1, \ldots, u_s], \quad s = \mathrm{rank}(H) \end{aligned} \tag{1}$$

and the definition of a Cartan algebra of a homogeneous space G/H. By definition $(C, \delta) \in DGA$, defined by the formula

$$\begin{aligned} &(C, \delta) = (Q[u_1, \ldots, u_s] \otimes \Lambda(y_1, \ldots, y_n),\ \delta) \\ &\delta u_i = 0 \ \ (i = 1, \ \ldots, s),\ \delta y_j = \tilde{f}_j(u_1, \ldots, u_s) = f_j \mid_{L(T')} \end{aligned} \tag{2}$$

is called a *Cartan algebra* of $M = G/H$ (here u_i, f_j are defined by (1), u_i have even and y_j have odd degrees). In addition, $\deg(y_j) = \deg(f_j) - 1$. Here and everywhere below the notation $\deg(u)$ is used for the degree of the element u.

It is well known that the projective space CP^n can be represented in the form of the homogeneous space

$$CP^n = SU(n+1)/S(U(n) \times U(1)) \tag{3}$$

LEMMA. *A Cartan algebra of $M = CP^3$ represented in the form (3) is determined by the formulae*

$$\begin{aligned}
&(C,\delta) = (Q[u_1,u_2,z]\otimes\Lambda(y_1,y_2,y_3),\delta)\\
&\quad \delta u_1 = \delta u_2 = \delta z = 0\\
&\quad \delta y_1 = u_1 + z^2,\ \delta y_2 = u_2,\ \delta y_3 = u_1^2 + z^4\\
&\deg(u_1) = 4,\ \deg(u_2) = 6,\ \deg(y_1) = 3,\ \deg(y_2) = 5,\\
&\deg(y_3) = 7.
\end{aligned} \tag{4}$$

Proof. Taking into consideration (3) one can use the explicit expressions for f_j and u_i in Chavalley isomorphisms (1) (see, for example, [4]). One easily obtains

$$\begin{aligned}
&u_1 = x_1^2 + x_2^2 + x_3^2,\ u_2 = x_1^3 + x_2^3 + x_3^3,\ x_1 + x_2 + x_3 = 0\\
&f_1 = x_1^2 + \ldots + x_4^2,\ f_2 = x_1^3 + \ldots + x_4^3,\\
&f_3 = x_1^4 + \ldots + x_4^4,\ x_1 + \ldots + x_4 = 0,
\end{aligned}$$

where u_i are $W(A_2)$ - invariants and f_j are $W(A_3)$ - invariants (A_n is a standard notation for the type of a simple Lie group). Now, the expressions for $\tilde{f}_1,\ \tilde{f}_2,\ \tilde{f}_3$ can be easily found and (2) implies (4). Lemma is proved.

Following [11] we shall consider topological fibrations (= fibrations) with the fiber F, that is maps $F \to E \xrightarrow{p} M$ with p satisfying the homotopy lifting property and with $p^{-1}(*) = F$.

It is known [9], [10] that for any fibration

$$F \to E \to M \tag{5}$$

there exists a corresponding DGA-model

$$(\mathcal{M}_F, d_F) \to (\mathcal{M}_F \otimes_\tau A^*(M), d) \to (A^*(M), d_M) \tag{6}$$

where $A^*(M)$ is a DGA - model of M and d is determined by formulae

$$\begin{aligned}
&d(1\otimes b) = 1\otimes d_M(b),\ b\in A^*(M),\\
&d(x\otimes 1) = d_F(x)\otimes 1 + \sum_{i\geq 0}(-1)^{(i+1)\deg(x)}\sum_{\nu\geq 1}\Phi_i^\nu(x)\otimes b_{i+1}^\nu,\ x\in\mathcal{M}_F.
\end{aligned} \tag{7}$$

Here the following notations are used: $\Phi^\nu{}_i$ is a derivation of $\mathcal{M}_F$ decreasing degree by i, that is

$$\Phi_i^\nu(xy) = \Phi_i^\nu(x)y + (-1)^{i\deg(x)}x\Phi_i^\nu(y)$$

and $\{b^{\nu}{}_{i+1},\ \nu = 1,2,\ldots\}$ constitute a basis of $A^{i+1}(M)$. One can see that the "twisted" tensor product $(\mathcal{M}_F \otimes_{\tau} A^*(M),\ d)$ is determined by (7), that is, by the choice of $\Phi^{\nu}{}_i$. To stress the latter observation, let us use the notation τ_{Φ}.

Let $D_i(F)$ be a set of Q - derivations of $\mathcal{M}_F$, decreasing degree by i for $i > 0$, $D_0(F) = \{d_F\Phi - \Phi d_F,\ \Phi \in D_1(F)\}$. Then

$$D_*(F) = \oplus_{i \geq 0} D_i(F)$$

forms a graded differential Lie algebra by the rule

1) $[\Phi_1, \Phi_2] = \Phi_1\Phi_2 - (-1)^{ij}\Phi_2\Phi_1, \Phi_1 \in D_i(F),\ \Phi_2 \in D_j(F)$
2) $\Delta\Phi = d_F\Phi - (-1)^{\deg(\Phi)}\Phi_F$

Δ denotes the appropriate derivation. Let

$$(\Lambda(sD_*(F)^{\vee}), \Theta)$$

be the Koszul cochain complex of graded differential Lie algebra $D_*(F)$, that is, the free graded differential algebra on the dual vector space $D_*(F)^{\vee}$ with the degree being shifted up by one, with the differential defined as the dual of Δ + the dual of the bracket. We have the following family of linear maps

$$\alpha : A^{i+1}(M)^{\vee} \to D_i(F),\ \alpha(b^{\vee}) = \sum_{i \geq 0}(-1)^{i+1} \sum_{\nu \geq 1} \Phi^{\nu} i < b_{i+1}, b^{\vee} > \tag{8}$$

Consider now the dual $\alpha^{\vee}$ of α. Observe that α is determined by Φ_i^{ν}. To stress this observation denote α by α_{Φ}.

To continue the proof we need the following result of Stasheff-Schlessinger [8], which we formulate in the form [10], which is a bit more convenient in our calculations.

STASHEFF - SCHLESSINGER THEOREM. *If $\alpha_{\Phi}^{\vee} : \Lambda(sD_*(F)^{\vee}) \to A^*(M)$ is a DGA-morphism, then the "twisting" τ_{Φ} determines for any DGA-model $A^*(M)$ a DGA-model (6) corresponding to a certain fibration (5).*

REMARK. Not any DGA-model (6) corresponds to a topological fibration (5), the corresponding condition is given by the latter theorem. Now, apply the procedure, described above, to $F = S^2, M = CP^3$ and take

$$(A^*(M), d_M) = (C, \delta)$$

where (C, δ) is determined by lemma. Evidently

$$(\mathcal{M}_F, d_F) = (Q[x] \otimes \Lambda(y), d_F)$$
$$d_F(x) = 0,\ d_F(y) = x^2,\ \deg(x) = 2,\ \deg(y) = 3.$$

Now,take $\Phi_i^\nu \equiv 0$ for all i, ν except $\Phi_3^1 \in D_3(S^2)$,

$$\Phi_3^1(x) = 0,\ \Phi_3^1(y) = 1.$$

Thus the twisted tensor product $(\mathcal{M}_F, d_F) \otimes_{\tau_*} (C, \delta)$ takes the form

$$\begin{aligned}
&(\mathcal{M}_F, d_F) \otimes_{\tau_*} (C, \delta) = (Q[u_1, u_2, z, x] \otimes \Lambda(y_1, y_2, y_3, y), d) \\
&\qquad du_1 = du_2 = dz = dx = 0, \\
&\qquad dy_1 = u_1 + z^2,\ dy_2 = u_2,\ dy_3 = u_1^2 + z^4,\ dy = u_1 + x^2 \\
&\quad \deg(u_1) = 4,\ \deg(u_2) = 6,\ \deg(x) = \deg(z) = 2,\ \deg(y_1) = \deg(y) = 3, \\
&\quad \deg(y_2) = 5,\ \deg(y_3) = 7.
\end{aligned} \tag{9}$$

Now, make the following observation:

$$D_*(F) = \mathrm{Span}(D_1, D_2, D_3)$$

where D_i are the derivations of $\mathcal{M}_F$ whose values on the generators x and y are determined by the equalities

$$D_1(x) = 0,\ D_1(y) = x,\ D_2(x) = 1,\ D_2(y) = 0,\ D_3(x) = 0,\ D_3(y) = 1 \tag{10}$$

The equalities (10) imply

$$[D_1, D_2] = D_3,\ [D_1, D_3] = 0,\ [D_2, D_3] = 0,\ D_3 = \Phi_3^1 \tag{11}$$

Formulae (10) and (11) follow from the compairing of the degrees $\deg(x)$ and $\deg(y)$ and the condition that D_i decreases the degree by i. First of all, we shall show that

$$\alpha^\vee : (\Lambda(sD_*(F)^\vee), \Theta) \to (C, \delta)$$

is a morphism of DGA. In our case

$$\alpha(b^\vee) = b^\vee(u_1)\Phi_3^1 = b^\vee(u_1) D_3 \tag{12}$$

Evidently, α is a homomorphism by definition, therefore the same is valid for $\alpha^\vee$. Therefore it is enough to verify the condition of $\alpha^\vee$ being the DGA-morphism on generators $D_i^\vee$. Evidently

$$\alpha^\vee(D_i^\vee) = 0,\ i = 1,2,\ \alpha^\vee(D_3^\vee) = u_1 \tag{13}$$

The latter equality implies

$$\delta\alpha^\vee(D_i^\vee) = 0 \tag{14}$$

On the other hand (μ denotes the multiplication operator)

$$\alpha^\vee\Theta(D_i^\vee) = \alpha^\vee(\Delta^\vee D_i^\vee + \sum_{k=1}^{3}\mu(D_k^\vee)\Theta(D_k)(D_i^\vee))$$

(see [5]). Obviously,

$$\Delta^\vee D_1^\vee = 2D_2^\vee,\ \Delta^\vee D_2^\vee = \Delta^\vee D_3^\vee = 0$$

and therefore, using (13)

$$\alpha^\vee(\Delta^\vee D_1{}^\vee) = \alpha^\vee(\Delta^\vee D_2{}^\vee) = \alpha^\vee(\Delta^\vee D_3{}^\vee) = 0$$

Further

$$\alpha^\vee(\sum_{k=1}^{3}\mu(D_k^\vee)\Theta(D_k)(D_i^\vee)) = \alpha^\vee(\sum_{k=1}^{3}D_k^\vee\cdot[D_k, D_i]^\vee) =$$
$$= \alpha^\vee(D_1^\vee\cdot D_3^\vee) = \alpha^\vee(D_1^\vee)\cdot\alpha^\vee(D_3^\vee) = 0.$$

Thus

$$\delta\alpha^\vee(D_i^\vee) = \alpha^\vee\Theta(D_i^\vee),\ i = 1,2,3,$$

which shows that $\alpha^\vee$ is a DGA-morphism.

Now, applying the cited above Schlessinger-Stasheff theorem, one obtains the topological fibration $E \to M$ with a fiber being 2-dimensional homotopy sphere and for which E has a model $(A(E), d) = (\mathcal{M}_F \otimes_\tau A^*(M), d)$ determined by (9). Now we shall calculate the minimal model of E. Use the following general result (lemma V. 8 in [7]).

Let $(A, d) \in DGA$ and $C = \Lambda(W' \otimes W)$ be a contractible DGA-subalgebra. Suppose that there exists a subalgebra $B \subset A$ (not necessarily d-invariant) for which $A = B \otimes C$ in a graded sense. Denote by $\langle C^+\rangle$ the homogeneous ideal generated by the elements of degrees > 0 from C. Then the natural projection $A \to A/\langle C^+\rangle$ is a quasiisomorphism.

Consider $(A(E), d)$ and put $C = \Lambda(u_1+z^2, u_2, u_1+x^2, y_1, y_2, y)$. Then, obviously, C is a contractible algebra with respect to d. Evidently, $A(E) = Q[z,x] \otimes \Lambda(y_3) \otimes C$ and we can apply the above statement. One obtains

$$(A(E)/ < C^+ >, d) = ((Q[z,x]/(z^2 - x^2)) \otimes \Lambda(y_3), d)$$
$$dz = dx = 0,\ dy_3 = z^4.$$

Put

$$\begin{gathered}(\mathcal{M}_E, D) = (Q[z,x] \otimes \Lambda(y_3) \otimes \Lambda(y), D)\\ Dz = Dx = 0,\ Dy = z^2 - x^2,\ Dy = z^4\end{gathered} \tag{15}$$

Observe, that (15) implies the minimality of $(\mathcal{M}_E, D)$.

As far as $(z^2 - x^2)$ is not a zero divisor in $Q[z, x]$, we can apply M. Vigué-Poirrier theorem [12] about the quasiisomorphism of DGA

$$(A \otimes L, d) \overset{c}{\sim} (A \otimes L/(\gamma, d\gamma), \bar{d}),$$

for any DGA of the form $k[V] \otimes \Lambda(W)$ with $k[V]$ being graded by even degrees and $\Lambda(W)$ by odd in the case when γ is not a zero divisor in $k[V]$. In our case it means that (15) is quasiisomorphic to $A(E)/\langle C^+\rangle$ and is minimal. Thus $(\mathcal{M}_E, D)$ is a minimal model for E, because quasiisomorphic algebras have the same minimal models. Now, we shall show that $(\mathcal{M}_E, D)$ *has non-vanishing Massey products.* To show this, consider the elements

$$a = x^2,\ b = z^2,\ c = z^2.$$

It is easy to calculate that

$$ab = D(x^2 y),\ bc = Dy_3$$

and therefore one obtains Massey product $\langle \bar{a}, \bar{b}, \bar{c} \rangle$ which is represented by a cocycle $v = x^2 y_3 + x^2 z^2 y$. The latter element cannot be a coboundary, because the equality $v = Dw$ would imply $w = a y_3 \wedge y$ (otherwise it is impossible to obtain y_3 or y after the derivation). Thus $Dw = a z^4 y + a(z^2 - x^2) y_3$ and therefore $\deg(a) = 0$, but then the equality is impossible. The same argument shows that $\bar{v} \notin \bar{a} H^*(\mathcal{M}_E, D) + \bar{c} H^*(\mathcal{M}_E, D)$, because otherwise $v = x^2 a_1 + z^2 a_2 x^2$ with $\bar{a}_i \in H^*(\mathcal{M}_E, D)$ would imply $\deg(a_1) = 5$, $\deg(a_2) = 3$ and $a_1 = \xi y_3$, $a_2 = \eta y$, $\xi, \eta \in Q$ but y and y_3 are not cocycles. Thus $\mathcal{M}_E$ has non-vanishing Massey products. Theorem is proved.

COROLLARY. *The total space E of the fibration*

$$S^2 \to E \to M \approx CP^3$$

is not formal in the sense of Sullivan.

REMARK. In fact, the possibility of taking (C, δ) as $(A^*(M), d)$ follows from Sullivan's theorem [9] about the spatial realization (that is, about the possibilities of a realization of a graded differential algebra as a model of a CW-complex). Isomorphism $H^*(M) \approx H^*(CP^3)$ follows from the lemma.

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