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POISSON GEOMETRY OF CERTAIN MODULI SPACES

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1. Introduction

The study of certain moduli spaces leads to concepts which are otherwise known in mathematical physics, in particular to that of a Poisson structure. Let me remind you of its definition: On $\mathbf{R}^{2n}$ with its usual coordinates $\{q_1, \ldots, q_n, p_1, \ldots, p_n\}$, the formula

$$\{f, h\} = \sum \left(\frac{\partial f}{\partial p_j} \frac{\partial h}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial h}{\partial p_j} \right)$$

yields a bracket $\{\cdot, \cdot\}$ on the algebra of smooth functions. This bracket was introduced by POISSON around 1809 and he observed that, given three functions f, g, h with $\{f, g\} = 0$ and $\{f, h\} = 0$, one also has $\{f, \{g, h\}\} = 0$. This means that if g and h are integrals of motion for (the hamiltonian vector field of) f, so is $\{g, h\}$. See e. g. [1 p. 196], [4 p. 216]. Poisson's collegues at Paris would not trust him, and only thirty years later *Jacobi* discovered what is nowadays called the *Jacobi identity* which of course immediately explains Poisson's observation. We now have almost completely reproduced the definition of a Poisson algebra: a Poisson algebra is a commutative algebra with the additional structure of a Lie bracket which behaves as a derivation in each variable with respect to the algebra structure.

After their discovery, Poisson structures have been explored by S. LIE [37], E. CARTAN [10], P. DIRAC [11], and others. They were the basic tool for Lie's work and provided for example an appropriate language for the proof of Lie's third theorem. For many discoveries of modern symplectic geometry, there are precedents in Lie's work which could not have been spelled out without the concept of a Poisson structure. Dirac made the fundamental observation that Poisson brackets provide the right framework in which classical mechanics is seen as an approximation of quantum mechanics. He also noticed their importance for classical constrained

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systems and developed the miraculous notion of Dirac bracket. Poisson brackets are nowadays in a period of intense development, cf. e. g. [53, 54]. Among others they are used to study collective hamiltonian systems, see e. g. [12]. They have been useful even in engineering mathematics. Some references and discussion in this direction may be found in [39].

The geometry of certain moduli spaces may be described in similar ways as that of collective hamiltonian systems. Poisson structures then yield geometric insight which goes beyond what can be obtained from standard symplectic geometry. These moduli spaces include those of central Yang-Mills connections for a principal bundle on a Riemann surface, in particular moduli spaces of semi stable holomorphic vector bundles, certain representation spaces, moduli spaces of parabolic bundles, and related ones. Work in progress applies these ideas to certain moduli spaces of Einstein-Hermite vector bundles. We now explain a special case:

Let Σ be a closed (real) surface, G a compact Lie group, with Lie algebra g, and $\xi\colon P \to \Sigma$ a principal G-bundle. Further, pick a Riemannian metric on Σ and an *orthogonal structure* on g, that is, an adjoint action invariant scalar product. These data then give rise to a Yang-Mills theory which has been studied in great detail by ATIYAH-BOTT [7]. They showed in particular that, in a sense, it suffices to study the geometry of the moduli space $N(\xi)$ of gauge equivalence classes of *central* Yang-Mills connections. Here a connection A is said to be *central* provided the values of its curvature lie in the Lie algebra of the centre of G. When the bundle ξ is flat the central Yang-Mills connections are precisely the flat ones. In particular, for $G = \mathrm{SU}(2)$, the moduli space $N(\xi)$ is that of flat connections coming into play in Chern-Simons gauge theory. Another important special case is that of $G = \mathrm{U}(n)$, the unitary group; the bundle ξ is then topologically classified by its Chern class (say) k, and the space $N(\xi)$ is homeomorphic to the NARASIMHAN-SESHADRI moduli space $N(n,k)$ of semi stable holomorphic rank n and degree k vector bundles over Σ (with reference to a choice of holomorphic structure). In general, these moduli spaces still carry additional geometric structures such as e. g. symplectic or Kähler ones which are at first well defined only away from certain singularities, though. Our program is aimed at extending such structures, suitably generalized, over the whole space, its singularities included. Here is the first result.

Theorem 1. *The decomposition of the space $N(\xi)$ into connected components of orbit types of central Yang-Mills connections is a (Whitney) stratification in such a way that each stratum, being a smooth manifold, inherits a symplectic structure, in fact, a Kähler structure, from the given data. Moreover the data determine an algebra $C^\infty(N(\xi))$ of continuous functions on $N(\xi)$ together with a Poisson structure $\{\cdot,\cdot\}$ which, on each stratum, restricts to the corresponding symplectic Poisson structure.*

A comment might be in order: On each stratum, the algebra $C^\infty(N(\xi))$ restricts to the compactly supported functions; obviously we cannot get all smooth functions. However the compactly supported functions certainly suffice to recover the symplectic Poisson structure on a stratum. In general, on a stratified space, an algebra of continuous functions which on each stratum restricts to the compactly supported functions is referred to as a *smooth structure*. A stratified space can support different smooth structures. An example will be given after Theorem 2 below.

A space with a structure of the kind spelled out in the theorem has been christened *stratified symplectic space* by SJAMAAR-LERMAN [51].

REMARK 1. A formal consequence of the theorem is the existence of a smooth open connected and dense stratum, referred to as *top stratum.* In the vector bundle case, it consists of the *stable* points. Its existence in the symplectic context is not as obvious as one might believe since a priori we do not have a structure of a complex analytic variety at our disposal.

REMARK 2. It has been known for a while, cf. NARASIMHAN-SESHADRI [46], ATIYAH-BOTT [7], that an orthogonal structure on the Lie algebra gives rise to a symplectic structure on the top stratum. In particular, in the vector bundle case, for n and k relatively prime, there is only a single stratum, the top stratum, and a result of NARASIMHAN-SESHADRI [46] says that, the moduli space inherits a structure of a compact Kähler manifold. Our theorem extends the symplectic part thereof to the general case. In fact, our Poisson structure is defined also at the *singular* points; it encapsulates the *mutual positions* of the symplectic structures on the strata. Borrowing some language from algebraic geometry one could say that it describes what happens to the symplectic structure under "specialisation". A structure of this kind *cannot* even be spelled out in ordinary symplectic geometry. Furthermore, it is known that, for general n and k, the spaces $N(n,k)$ have a structure of a normal projective variety SESHADRI [48], [49]; however, this does not shed too much light on the singular behaviour of the symplectic or Poisson structures in general; in fact it may happen that the symplectic structure is singular whereas the complex analytic one is not. An example will be explained in Theorem 2 below.

The Poisson structure gives rise to some interesting Poisson geometry. A special case worthwhile studying is that where $G = \mathrm{SU}(2)$. In another guise, cf. what was said above, after a choice of compatible holomorphic structure on Σ has been made, the moduli space $N(\xi)$ is then that of semi stable holomorphic vector bundles on Σ of rank 2, degree 0, and trivial determinant. This space and related ones have been studied extensively in the literature [43 – 49]. In particular, for genus $\ell \geq 2$, NARASIMHAN-RAMANAN proved that the complement $\mathcal{K}$ of the top stratum is the *Kummer* variety of Σ associated with its Jacobien $\mathcal{J}$ and the canonical involution thereupon [43]. This has the following consequence, established in [25].

Theorem 2. *For $G = \mathrm{SU}(2)$, when Σ has genus $\ell \geq 2$, the Poisson algebra $(C^\infty(N(\xi)), \{\cdot,\cdot\})$ detects the Kummer variety $\mathcal{K}$ in $N(\xi)$ together with its $2^{2\ell}$ double points. More precisely, $\mathcal{K}$ consists of the points where the rank of the Poisson structure is not maximal, the double points being those where the rank is zero.*

For genus ≥ 3, the Kummer variety $\mathcal{K}$ is precisely the (complex analytic) singular locus of $N(\xi)$, a result due to NARASIMHAN-RAMANAN [43]. This has been reproved in [25] within our framework. When Σ has genus two the space $N(\xi)$ equals complex projective 3-space and $\mathcal{K}$ is the Kummer surface associated with the Jacobien of Σ. In the literature, this case has been considered somewhat special since as a *space* $N(\xi)$ is then actually smooth. More precisely, its algebra of smooth functions in the usual sense is a smooth structure in the above sense. However, from our point of view there is *no* exception. Our smooth structure is *not* the standard one, and as a stratified symplectic space, $N(\xi)$ still has singularities, that is to say, the Poisson algebra $(C^\infty(N(\xi)), \{\cdot,\cdot\})$ detects a Kummer surface together with its 16 singularities and hence the underlying algebra of functions can plainly *not* be that of smooth functions in the ordinary sense; in particular, the symplectic structure on the top stratum does *not* extend to the whole space. It is interesting to observe that the stratification mentioned in Theorem 1 is *finer* than the standard complex analytic one on complex projective 3-space.

One way of proving these results is by means of suitable local models. These local models yield insight interesting in their own right. We explain this briefly for the special case where $G = \mathrm{SU}(2)$. Let $Z \subseteq G$ denote the centre and $T \subseteq G$ a maximal torus. The space $N = N(\xi)$ has three strata

$$N = N_G \cup N_{(T)} \cup N_Z$$

where $N_{(K)}$ denotes the points of orbit type (K). The top stratum is N_Z.

Theorem 3. *Near a point of $N_{(K)}$, N and $(C^\infty(N), \{\cdot,\cdot\})$ may be described in the following way:*
$K = Z$: the space $\mathbf{C}^{3(\ell-1)}$ with its standard symplectic Poisson structure;
$K = T$: a product of $\mathbf{C}^\ell$ with its standard symplectic Poisson structure and of the reduced space and reduced Poisson algebra of a system of $\ell - 1$ particles in the plane with total angular momentum zero;
$K = G$: the reduced space and reduced Poisson algebra of a system of ℓ particles in 3-space with total angular momentum zero.

For illustration, consider the special case where $\ell = 2$, so that, by virtue of the Narasimhan-Ramanan result cited above, the space N is complex projective 3-space. By Theorem 3, near a point of $N_{(T)}$, the space N looks like the product of a copy of $\mathbf{C}^2$ with the reduced system of a single particle in the plane $\mathbf{R}^2$. It turns out that the latter is indeed $\mathbf{R}^2$, with reduced Poisson algebra generated by the coordinate functions x_1, x_2 and, which is *crucial* here, an additional function r which is the radius function, so that the usual relation

$$x_1^2 + x_2^2 = r^2$$

holds. Notice r is *not* smooth in the ordinary sense. The Poisson structure $\{\cdot,\cdot\}$ is given by the formulas

$$\{x_1, x_2\} = 2r, \quad \{x_1, r\} = 2x_2, \quad \{x_2, r\} = -2x_1.$$

Thus we see that the algebra $C^\infty(N)$ contains the usual smooth functions but is strictly larger than that of smooth functions on complex projective 3-space in the ordinary sense.

The present paper is intended as a leisurely introduction to the Poisson geometry of the mentioned moduli spaces which has been developped in our papers [20 – 28]. In Section 2 below we briefly explain the idea of a stratified symplectic space while in Section 3, after a very short description of Yang-Mills theory over a surface which follows the paper [7] by Atiyah and Bott, we give the construction of suitable local models. In Section 4 we explain the resulting (local) Poisson geometry whereas in Section 5 a finite dimensional approach is presented. Moduli spaces of parabolic bundles [41] are not touched in this paper, cf. [16], [28].

What remains to be done? Well, a compatible complex analytic structure on a symplectic manifold can be described in terms of a polarization. To extend this relationship to stratified symplectic spaces, in [29] we introduced the concept of a *stratified Kähler space*. This is a stratified symplectic space whose strata are endowed with structures of Kähler manifolds, together with an additional piece of structure which describes the mutual positions of the Kähler structures, generalizing what was said above about the mutual positions of the symplectic structures on a stratified symplectic space. The description of this additional piece of structure is achieved by means of results in our paper [18], see also [19]. We plan to prove that a moduli space of central Yang-Mills connections inherits a structure of a complex analytic space together with that of a compatible stratified Kähler space which is integral in an appropriate sense. We note that, for a singular Kähler space in the sense of GRAUERT [15], a stratified Kähler structure in our sense amounts to an additional requirement which the local Kähler potentials have to satisfy in the singular points. Thereafter we plan to demonstrate that an integral stratified Kähler space inherits a structure of projective variety. While it is known from geometric invariant theory constructions that a moduli space of central Yang-Mills connections inherits a structure of a projective variety, the link of the latter to the symplectic or more generally Poisson geometry presented here is not completely understood. Our program is aimed at yielding a purely analytical construction of the projective variety structure, thereby providing a better understanding of how this structure is related with the Poisson geometry. We then plan to study *holomorphic quantization* over the resulting stratified Kähler space. Also the closures of the strata of the stratification in Theorem 1 are worth further investigation. These are presumably interesting projective varieties; they are detected by the Poisson structure and generalize the Kummer varieties mentioned before.

We presented a possible construction of the Poisson structure for the first time at the AMS-meeting on classical field theory in Seattle (USA) in the summer of 1991 and shortly thereafter in a Berkeley seminar talk. The research program has been presented thereafter at various meetings; a brief account will appear in [30]. I am much indebted to A. Weinstein for his encouragement to carry out this program; in fact, at the Berkeley MSRI workshop on quantization in the spring of 1989, he suggested to me to relate the results of my paper [18], see also [19], to the moduli spaces arising in Chern-Simons gauge theory and related ones. I am indebted to M. S. Narasimhan for having pointed out to me the need for an entirely analytical construction of the projective variety structure on such moduli spaces. It is a pleasure to thank the organizers of the 14th Winter School GEOMETRY AND PHYSICS for the opportunity to deliver this series of lectures.

2. Symplectic reduction and stratified symplectic spaces

Symplectic reduction of Hamiltonian spaces is a rich source of symplectic manifolds. It has been widely applied to the study of Hamiltonian systems with symmetries since the days of Jacobi. A general framework for symplectic reduction at regular values of a momentum mapping has been set up by MEYER [42] and MARSDEN-WEINSTEIN [40]. However, in many of the applications one would like to carry out reduction at a singular value of a momentum mapping. It is therefore of interest to devise a reduction scheme for singular levels of a momentum mapping and to study the singularities arising from it.

Let M be a symplectic manifold, with symplectic structure σ, and G a Lie group acting on M in a Hamiltonian way, with momentum mapping μ from M to g^*. To explain what this means, for X in g, we denote by μ^X the smooth function on M which is the composite of μ and X, viewed as a linear function on g^*, and by X_M the vector field on M determined by X and the G-action. Now the action to be hamiltonian with momentum mapping μ means that

— μ is G-equivariant, and that
— for every $X \in g$, we have

$$d(\mu^X) = \sigma(X_M, \cdot).$$

The G-action reflects a certain symmetry of the system. For example, consider a system of n particles in $\mathbf{R}^3$, moving with constant angular momentum. Here $G = \mathrm{O}(3)$, its Lie algebra g equals $\mathbf{R}^3$, with standard inner product, so that we can identify g with its dual g^*, and the manifold M is the corresponding phase space, that is, the total space $(\mathrm{T}^*\mathbf{R}^3)^{\times n}$ of the cotangent bundle on $(\mathbf{R}^3)^{\times n}$, which amounts to $(\mathbf{R}^6)^{\times n}$, acted upon by G in the usual way; furthermore, with the usual notation $(q_1, p_1, \ldots, q_n, p_n) \in (\mathbf{R}^6)^{\times n}$, the momentum mapping μ from $(\mathbf{R}^6)^{\times n}$ to g is given by the association

$$(q_1, p_1, \ldots, q_n, p_n) \mapsto q_1 \wedge p_1 + \cdots + q_n \wedge p_n.$$

Picking a value of the momentum mapping amounts to fixing a constant total angular momentum.

Let μ be a momentum mapping as above, and let o be a point of g^*, fixed under the coadjoint action. The space

$$M_o = \mu^{-1}(o)/G$$

is called the *Marsden-Weinstein* reduced space at o. By suitably modifying μ we may assume that o is the origin. This is the so-called "shifting trick", which allows one to talk exclusively about reduction at zero. It is common to write

$$M_{\mathrm{red}} = M//G = \mu^{-1}(0)/G$$

for this space. What is the significance of this construction? For any G-invariant function h and any element X of g, we have

$$\{h, \mu^X\} = X_M h = 0.$$

In other words, the functions μ^X are preserved by the hamiltonian flows of the function h. Thus for hamiltonian actions symmetry implies conservation laws. Using the momentum map one can cut down the number of degrees of freedom of a symmetric hamiltonian system. The reduced space M_{red} is the one underlying the new system. The function h passes to a function on M_{red}.

There is a problem here swept under the rug: How do we know whether M_{red} is a *smooth* manifold at all? Suppose that (i) 0 is a regular value so that the zero locus $\mu^{-1}(0)$ is a smooth submanifold of M (not necessarily connected), and that (ii) the G-action on the zero locus is free. Then M_{red} will obviously be a smooth manifold. Moreover, it will inherit a symplectic structure. This is the MARSDEN-WEINSTEIN reduction [40]. We explain briefly some of the details: Let (V, ω) be a symplectic vector space, and let W be a subspace of V. Recall that the *annihilator* W^ω (or *skew complement*) of W is the subspace of V consisting of all v so that $\omega(v, w)$ is zero for every w in W. Now W is called *isotropic* if it lies in its annihilator and *coisotropic* if it contains its annihilator. Likewise a smooth submanifold of a symplectic manifold is *(co)isotropic* provided for every point its tangent space is (co)isotropic in the tangent space of the ambient manifold. To see that, under the present circumstances, M_{red} inherits a symplectic structure, let $p \in \mu^{-1}(0)$. Recall at first that the tangent space $\mathrm{T}_p\mu^{-1}(0)$ equals the kernel of the tangent map $d\mu_p$ from T_pM to g^*. In view of the momentum mapping property, the tangent space $\mathrm{T}_p\mu^{-1}(0)$ thus consists of all vectors v in T_pM so that $\sigma_p(X_M(p), v)$ is zero for *every* $X \in g$. In other words: The tangent space $\mathrm{T}_p\mu^{-1}(0)$ is the annihilator of the tangent space $\mathrm{T}_p(Gp)$ to the G-orbit Gp through p. Moreover, the latter is isotropic in M: In fact, for every X, Y in g we have

$$\sigma_p(X_M(p), Y_M(p)) = d\mu_p^X(Y_M(p)).$$

But $\mu(p)$ is zero and hence μ is zero on all of the orbit Gp by equivariance whence $d\mu_p(Y_M(p))$ is zero for every $Y \in g$. Consequently the G-orbit Gp through p is isotropic in T_pM. Since the tangent space $\mathrm{T}_p\mu^{-1}(0)$ is the annihilator of T_pGp we conclude that $\mathrm{T}_p\mu^{-1}(0)$ is coisotropic in T_pM in such a way that the restriction of the form σ_p to $\mathrm{T}_p\mu^{-1}(0)$ vanishes exactly in the vertical directions. Since p was an arbitrary point of the zero locus, we see that, under the present circumstances, the zero locus $\mu^{-1}(0)$ is a coisotropic submanifold of M, the canonical projection map from the zero locus to M_{red} is the projection map π of a principal G-bundle, and the restriction $\sigma|\mu^{-1}(0)$ of the form descends to a symplectic form σ_{red} so that

$$\pi^*(\sigma_{\text{red}}) = \sigma\big|\mu^{-1}(0).$$

The form σ_{red} is the symplectic structure on the reduced space we are aiming at.

In most interesting cases, zero will *not* be a regular value. This happens for example for the system of n particles in $\mathbf{R}^3$ with total angular momentum zero. Under suitable circumstances, a way out is provided by the following result.

Theorem 2.1 (SJAMAAR-LERMAN [51]). *Suppose G compact and μ proper. Then the orbit type decomposition of M_{red} is a stratification, and the data determine a structure of stratified symplectic space, that is to say, a Poisson algebra $(C^\infty(M_{\text{red}}), \{\cdot,\cdot\}_{\text{red}})$*

of continuous functions which, on each stratum, restricts to a smooth symplectic Poisson algebra in the usual sense.

As before, on each stratum, the algebra $C^\infty(N(\xi))$ restricts to the compactly supported functions, which suffices to describe the symplectic Poisson structure on that stratum.

We now explain briefly the orbit type decomposition. For each closed subgroup K of G let $M_{(K)}$ denote the subspace of points having symmetry type conjugate to K, and let

$$(M_{\text{red}})_{(K)} = (\mu^{-1}(0) \cap M_{(K)})/G.$$

Then

$$M_{\text{red}} = \cup (M_{\text{red}})_{(K)}.$$

Each connected component of $(M_{\text{red}})_{(K)}$, if non-empty, inherits a structure of a smooth symplectic manifold. More precisely: For K so that $(M_{\text{red}})_{(K)}$ is non-empty, write M_K for the smooth submanifold of $(M)_{(K)}$ of points with symmetry K. Then the quotient $L = N_G(K)/K$ of the normalizer $N_G(K)$ of K in G by K acts on M_K in a hamiltonian way so that 0 is a regular value of the momentum mapping, and the corresponding reduced space amounts to the space $(M_{\text{red}})_{(K)}$. Furthermore, SJAMAAR AND LERMAN prove that the decomposition $M_{\text{red}} = \cup (M_{\text{red}})_{(K)}$ is a stratification. This means that

— $(M_{\text{red}})_{(K)}$ lies in the closure of $(M_{\text{red}})_{(H)}$ if and only if H is a subgroup of some conjugate gKg^{-1} of K, and that

— for every point of M_{red}, a certain precise local cone condition (not spelled out here) holds describing a neighborhood of that point in M_{red}.

The algebra $C^\infty(M_{\text{red}})$ is taken to be the algebra $(C^\infty(M))^G/I^G$ of G-invariant functions $C^\infty(M)^G$ on M modulo the ideal I^G of smooth G-invariant functions that vanish on the zero locus $\mu^{-1}(0)$. The symplectic Poisson bracket $\{\cdot,\cdot\}$ on M passes to a Poisson bracket $\{\cdot,\cdot\}_{\text{red}}$ on $C^\infty(M_{\text{red}})$. This is not at all obvious. It relies on the following

Lemma 2.2 (ARMS-CUSHMAN-GOTAY [2]). *Let f and h be smooth G-invariant functions on M and suppose that h vanishes on the zero locus $\mu^{-1}(0)$. Then $\{f,h\}$ also vanishes on the zero locus.*

We include a proof, for the sake of illustration.

Proof. Since μ is a momentum mapping and since f is G-invariant, we have

$$\{\mu^X, f\} = X_M f = 0$$

for every $X \in g$. Consequently for every $X \in g$ the function μ^X is constant ong the integral curves of the hamiltonian vector field X_f for f, that is, the val of μ is constant along the integral curves of X_f. Hence for every $m \in \mu^{-1}(($ the integral curve ϕ^f_m of X_f through m lies in the zero locus $\mu^{-1}(0)$. Since h vanishes on the latter, the real function $h \circ \phi^f_m$ is identically zero. Differentiation yields

$$\{f,h\}(m) = 0.$$

This completes the proof. □

Under circumstances of the present kind, we shall refer to the Poisson algebra $(C^\infty(M_{\text{red}}), \{\cdot,\cdot\}_{\text{red}})$ as the *Arms-Cushman-Gotay* algebra.

There is a way to make sense of a smooth curve in M_{red}, cf. [51]. Given a Hamiltonian $h \in C^\infty(M_{\text{red}})$, an *integral curve* of h through a point m_0 is a smooth curve γ with $\gamma(0) = m_0$ such that, for all functions $f \in C^\infty(M_{\text{red}})$,

$$\frac{d}{dt} f(\gamma(t)) = \{f, h\}(\gamma(t)).$$

Theorem 2.3 (SJAMAAR-LERMAN [51]). *Given m_0, there is a unique integral curve of h passing through m_0. Moreover hamiltonian flows exist and are unique and preserve the pieces of the decomposition of M_{red}, and the restriction of a hamiltonian flow to a piece equals the corresponding hamiltonian flow on the piece.*

3. Yang-Mills theory over a surface and local models

We return to the situation of the Introduction. Thus $\xi\colon P \to \Sigma$ is a principal G-bundle, having structure group a compact Lie group G. Moreover, we suppose Σ oriented and endowed with a Riemannian metric, and we write vol for the unique volume form of length one in the chosen orientation. Moreover, we pick an orthogonal structure on g, that is, an adjoint action invariant scalar product, which we denote by $\langle\cdot,\cdot\rangle$. We write $\Omega^* = \Omega^*(\Sigma, \mathrm{ad}(\xi))$ for the forms on Σ with values in the adjoint bundle

$$\mathrm{ad}(\xi)\colon P \times_G g \to \Sigma.$$

Abusing notation somewhat, we write

$$\langle\cdot,\cdot\rangle\colon \Omega^* \otimes \Omega^* \to \mathrm{C}^\infty(\Sigma)$$

for the induced Riemannian structure on each Ω^*. The scalar product on g also induces a graded pairing

$$\wedge\colon \Omega^j \otimes \Omega^k \to \Omega^{j+k}(\Sigma, \mathbf{R}).$$

Now the data determine a star operator $*$ on Ω^* which sends an $\mathrm{ad}(\xi)$-valued j-form to an $\mathrm{ad}(\xi)$-valued $(2-j)$-form; this operator is characterized by

$$\alpha \wedge *\beta = \langle\alpha, \beta\rangle \mathrm{vol}.$$

Consider the space $\mathcal{A}$ of connections on ξ. For a connection A we write d_A for its operator of covariant derivative and K_A for its curvature; the latter is an element of Ω^2. The data give rise to a Yang-Mills theory studied extensively by ATIYAH-BOTT [7]. The resulting Yang-Mills equations amount to

$$d_A * K_A = 0.$$

Recall that a connection is said to be *central* provided the values of its curvature lie in the Lie algebra z of the centre Z of G. Notice when G is not connected z may be smaller than the centre of the Lie algebra g. ATIYAH-BOTT [7] showed that, for connected structure group G, in a sense, it suffices to study central Yang-Mills connections; moreover they showed that, for a central Yang-Mills connection A, its curvature looks like

$$K_A = X_\xi \otimes \mathrm{vol},$$

for a constant element X_ξ in z determined by the topology of the bundle. For example, for $G = U(n)$, the unitary group, the bundle ξ is classified by its topological characteristic class in the infinite cyclic group $\mathrm{H}^2(\Sigma, \pi_1(U(n))$ which amounts to an integer k, also called *degree* (in the vector bundle picture), and the element X_ξ looks like

$$X_\xi = 2\pi i \frac{k}{n} \mathrm{diag}(1, \ldots, 1) \in \mathrm{u}(n).$$

The moduli space of central Yang-Mills connections N is now the quotient $\mathcal{N}/\mathcal{G}$ of the subspace $\mathcal{N}$ of central Yang-Mills connections in $\mathcal{A}$ divided out by the group $\mathcal{G}$ of gauge transformations. This is the space commented on already in the Introduction.

How does symplectic geometry come into the picture? The data yield a bilinear pairing $(\cdot,\cdot)$ on Ω^* given by

$$(\alpha, \beta) = \int_\Sigma \langle \alpha, \beta \rangle.$$

This pairing is weakly non-degenerate, meaning that its adjoint is injective. In particular, we get a weakly symplectic structure σ on Ω^1, and we can view Ω^2 as the dual of Ω^0; the latter is in fact the Lie algebra of the group of gauge transformations, whence Ω^2 appears as the dual of the Lie algebra of infinitesimal gauge transformations. The 2-form σ is translation invariant and hence yields a 2-form on the space $\mathcal{A}$ of connections, in fact, a weakly symplectic structure. A crucial observation of ATIYAH-BOTT [7] is that the assignment to a connection A of its curvature K_A is a momentum mapping μ for the action of $\mathcal{G}$ on $\mathcal{A}$, and that, with the notation of the previous Section, the space N arises as the Marsden-Weinstein reduced space

$$\mathcal{A}_{X_\xi} = \mu^{-1}(X_\xi \otimes \mathrm{vol})/\mathcal{G}$$

at $X_\xi \otimes \mathrm{vol}$. Notice this makes sense since $X_\xi \otimes \mathrm{vol}$ remains fixed under the induced $\mathcal{G}$-action on Ω^2, viewed as the dual of Ω^0. In particular, the weakly symplectic structure on $\mathcal{A}$ descends to a symplectic structure on a certain smooth submanifold of N. There are a great deal of technical difficulties to be overcome here; in particular, $X_\xi \otimes \mathrm{vol}$ will in general *not* be a regular value of the momentum mapping. Further, to make sense of the above heuristic approach technically, one has to invoke Sobolev spaces, regularity results of elliptic differential equations, Hodge theory, slice theorems, and the like.

Obviously, the SJAMAAR-LERMAN result (2.1) cited in the previous Section is *not* applicable here. Theorem 1 spelled out in the Introduction says that the *statement* of the SJAMAAR-LERMAN result is still true, though. Thus the space N is stratified by orbit types, and the data determine a Poisson algebra $(C^\infty(N), \{\cdot,\cdot\})$ of continuous functions which, on each stratum, restricts to a symplectic Poisson structure on that stratum. This result has been established in a series of papers [20 – 24].

A crucial step in the construction of the stratified symplectic structure is provided by suitable local models. These local models contain geometric information interesting in its own right. We now explain this.

Let A be a central Yang-Mills connection. Its operator d_A of covariant derivative is manifestly a differential on Ω^*, that is to say, (Ω^*, d_A) is a complex. Hence we can talk about its cohomology

$$\mathrm{H}^*_A = \mathrm{H}^*(\Omega^*, d_A).$$

Since the holonomies of any central connection yield a representation of the fundamental group π of Σ in the quotient G/Z of G by its centre Z, the cohomology H^*_A may be identified with the Eilenberg-Mac Lane group cohomology $\mathrm{H}^*(\pi, g_\phi)$ where ϕ denotes the representation determined by A and g_ϕ the Lie algebra g, made into a π-module via ϕ. In particular, the cohomology $\mathrm{H}^*(\pi, g_\phi)$ is finite dimensional and hence so is H^*_A. Thus we see that, while the complex (Ω^*, d_A) is certainly elliptic, finite dimensionality of H^*_A may be established without appealing to the index theorem.

The orthogonal structure on g induces a symplectic structure σ_A on the (finite dimensional) vector space H^1_A and the Lie bracket on g induces a graded Lie bracket $[\cdot,\cdot]_A$ on H^*_A which, for degree reasons, is *symmetric* on H^1_A. Let $Z_A \subseteq \mathcal{G}$ be the stabilizer of A. After a choice of base point has been made, the group $\mathcal{G}^Q$ of based gauge transformations acts freely on the space of connections whence restriction to the point Q identifies Z_A with a closed subgroup of G. Consequently Z_A is a compact Lie group which acts canonically on H^*_A preserving σ_A and $[\cdot,\cdot]_A$, and its Lie algebra z_A equals H^0_A. The orthogonal structure on g induces a canonical isomorphism between H^2_A and the dual z^*_A of z_A preserving the Z_A-actions. Furthermore the assignment to $\eta \in \mathrm{H}^1_A$ of $\Theta_A(\eta) = \frac{1}{2}[\eta,\eta]_A$ yields a momentum mapping Θ_A for the Z_A-action on the symplectic vector space H^1_A. MARSDEN-WEINSTEIN reduction then yields the space $\mathrm{H}_A = \Theta_A^{-1}(0)/Z_A$.

Theorem 3.1 [20]. *The symplectic quotient* H_A *is a local model for* N *near* $[A]$. *More precisely, a suitable Kuranishi map identifies a neighborhood of* $[A]$ *in* N *with a neighborhood of the class of zero in* H_A.

We shall say that a point $[A]$ of N is *non-singular* provided Z_A acts trivially on H^1_A. Theorem 3.1 entails that N is *smooth* near a non-singular point $[A]$, and that the data yield a symplectic structure on the subspace of non-singular points of N. This is the symplectic structure constructed by ATIYAH-BOTT [7] by symplectic reduction in infinite dimensions.

A formal consequence of (3.1) is the following.

Corollary 3.2 [21]. *The decomposition of* N *according to orbit types is a (Whitney) stratification in such a way that each stratum inherits a symplectic structure from the data.*

Locally, the Arms-Cushman-Gotay Poisson algebra on the models yields a Poisson structure near every point of the moduli space N. A single global Poisson structure on N which encompasses all these has been constructed in our paper [24]. It may be described in the following way: In [22] we constructed a homeomorphism $\rho_\flat$, referred to as *Wilson loop mapping*, from $N(\xi)$ onto a certain representation space $\mathrm{Rep}_\xi(\Gamma, G)$ for the universal central extension Γ of the fundamental group π of Σ. While the space $N(\xi)$ *depends* on the choices of Riemannian metric on Σ the space $\mathrm{Rep}_\xi(\Gamma, G)$ does not. In [23] we constructed smooth structures $C^\infty(N(\xi))$ and $C^\infty(\mathrm{Rep}_\xi(\Gamma, G))$ on these spaces, and we have shown that $\rho_\flat$ is a diffeomorphism with respect to these structures. By construction, the space $\mathrm{Rep}_\xi(\Gamma, G)$ is the quotient $\mathrm{Hom}_\xi(\Gamma, G)/G$ of a certain space of homomorphisms $\mathrm{Hom}_\xi(\Gamma, G)$ of Γ into G determined by ξ. For $\phi \in \mathrm{Hom}_\xi(\Gamma, G)$, we denote by g^*_ϕ the dual g^* of the Lie algebra g, made into a π-module via ϕ and the coadjoint action. For every $[\phi] \in \mathrm{Rep}_\xi(\Gamma, G)$, a choice of representative $\phi \in \mathrm{Hom}_\xi(\Gamma, G)$ induces a linear map λ^*_ϕ from the real vector space $\Omega_{[\phi]}\mathrm{Rep}_\xi(\Gamma, G)$ of differentials at $[\phi]$, with reference to $C^\infty(\mathrm{Rep}_\xi(\Gamma, G))$, into the first homology group $\mathrm{H}_1(\pi, g^*_\phi)$ of π with coefficients in g^*_ϕ [24 (1.16)], and λ^*_ϕ is an isomorphism if and only if $[\phi]$ is a non-singular point of $\mathrm{Rep}_\xi(\Gamma, G)$. In view of [24 (1.17)], λ^*_ϕ is independent of the choice of ϕ in the sense that, given $x \in G$, the corresponding linear map $\lambda^*_{x\phi}$ from $\Omega_{[\phi]}\mathrm{Rep}_\xi(\Gamma, G)$ to $\mathrm{H}_1(\pi, g^*_{x\phi})$ equals the composite of λ^*_ϕ with the isomorphism $\mathrm{Ad}^*(x)_\flat$ induced by x. The induced coadjoint action invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ on g^* then gives rise to, for every $\phi \in \mathrm{Hom}_\xi(\Gamma, G)$, an intersection pairing $\langle\cdot,\cdot\rangle_\phi$ on $\mathrm{H}_1(\pi, g^*_\phi)$, and the corresponding Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(\mathrm{Rep}_\xi(\Gamma, G))$ will then satisfy the formula

$$\{f,h\}[\phi] = \langle \lambda^*_\phi(df[\phi]), \lambda^*_\phi(dh[\phi])\rangle_\phi \tag{3.3}$$

where $f, h \in C^\infty(\mathrm{Rep}_\xi(\Gamma, G))$ and where $\phi \in \mathrm{Hom}_\xi(\Gamma, G)$ is a representative of the point $[\phi] \in \mathrm{Rep}_\xi(\Gamma, G)$. As a set function, the bracket $\{\cdot,\cdot\}$ is determined by (3.3). This formula is *intrinsic* in the sense that it does *not* involve choices except that of the representative ϕ which has been taken care of by the discussion of the dependence of λ^*_ϕ on the choice of ϕ. The details of this construction are given in our paper [24]. Theorem 3.1 now has the following extension, established in [24 (4.3)].

Theorem 3.4. *For every central Yang-Mills connection A, near $[A] \in N(\xi)$, the Arms-Cushman-Gotay Poisson algebra $(C^\infty(\mathrm{H}_A), \{\cdot,\cdot\}_A)$ yields a local model of $N(\xi)$ with its Poisson structure and likewise, near the point $\rho_\flat[A]$, a local model of $\mathrm{Rep}_\xi(\Gamma, G)$ with its Poisson structure, where $\rho_\flat$ refers to the Wilson loop mapping from $N(\xi)$ to $\mathrm{Rep}(\Gamma, G)$. More precisely, the choice of A (in its class $[A]$) induces a Poisson diffeomorphism (of smooth spaces) of an open neighborhood W_A of $[0] \in \mathrm{H}_A$ onto an open neighborhood U_A of $[A] \in N(\xi)$, where W_A and U_A are endowed with the induced smooth and Poisson structures, and a similar statement holds for $\mathrm{Rep}_\xi(\Gamma, G)$ near the point $\rho_\flat[A]$.*

The intrinsic description (3.3) of the Poisson structure has the following consequence, cf. [24 (Theorem 2)].

Theorem 3.5. *The induced action of the mapping class group of Σ respects the Poisson structure. More precisely, its subgroup of orientation preserving elements preserves the Poisson bracket on $\mathrm{Rep}_\xi(\Gamma, G)$ whereas the orientation reversing elements yield Poisson bracket preserving diffeomorphisms from $\mathrm{Rep}_\xi(\Gamma, G)$ to $\mathrm{Rep}_{-\xi}(\Gamma, G)$*

where $-\xi$ *refers to the (topologically) "opposite" bundle (which may coincide with* ξ*).*

Another construction of the Poisson structure will come out of the finite dimensional approach explained in Section 5 below. The above construction is more elementary but less elegant than the finite dimensional one. The advantage of the above construction is its invariance properties which entail invariance under the mapping class group as just explained.

The globally defined Poisson structure has another important consequence: Let f be a smooth invariant real function on G, and let C be a closed curve in Σ, having starting point Q. The homotopy class of C induces a homomorphism $[C]$ from $\mathbf{Z}$ to the fundamental group π of Σ, and the assignment to ϕ of the real number $f(\phi([C](1)))$ induces a real valued function f^C on $\operatorname{Rep}(\pi, G) = \operatorname{Hom}(\pi, G)/G$. Let ξ be a flat G-bundle over Σ. Since $[C]$ lifts to a homomorphism from $\mathbf{Z}$ to the free group F on the chosen generators, f^C yields a function in $C^\infty(\operatorname{Rep}_\xi(\pi, G))$, and hence $\{f^C, \cdot\}$ is a derivation of $C^\infty(\operatorname{Rep}_\xi(\pi, G))$. On each stratum it amounts of course to a smooth vector field. The corresponding flow on the *non-singular* part of $\operatorname{Rep}_\xi(\pi, G)$ has been studied by GOLDMAN in [14], referred to as a *twist flow*. However, the derivation $\{f^C, \cdot\}$ in fact integrates to a "twist flow" on the whole space $\operatorname{Rep}_\xi(\pi, G)$, that is, an action of the real line on this space preserving the smooth structure. Theorem 2.3 applied to the local model spelled out in (3.4) above shows that this is locally so, and, using the partition of unity established in our paper [23], we can conclude that these twist flows in fact exist globally and are unique.

4. Local Poisson geometry

Let A be a central Yang-Mills connection. The star operator endows H^1_A with a complex structure compatible with the symplectic structure σ_A. Hence H^1_A inherits a structure of unitary Z_A-representation in such a way that Θ_A is its unique momentum mapping having the value zero at the origin. Now Theorem 3.4 reduces the *local* Poisson geometry of the space N to what is called the "the standard example" (2.4) on p. 52 of ARMS-GOTAY-JENNINGS [3]. This observation is exploited in our paper [25]. We now explain some of the underlying ideas.

Consider a unitary representation V of a compact Lie group K. Associated with it is a unique momentum mapping μ from V to k^* having the value zero at the origin [3]. Later we shall take $V = \mathrm{H}^1_A$, for a central Yang-Mills connection A on ξ and $K = Z_A$, the stabilizer of A in $\mathcal{G}$, and μ will be Θ_A. For ease of exposition, we shall present the situation for arbitrary V and K.

The K-action extends to an action of the complexification $K^{\mathbf{C}}$ of K on V. The idea is to relate the symplectic quotient with the space of orbits of the $K^{\mathbf{C}}$-action. Now the latter is a bad space: There may be orbits which are not closed, and hence the orbit space is not in general Hausdorff. The way out is provided by the affine *categorical* quotient $V//K^{\mathbf{C}}$, cf. e.g. [47]. It is obtained in the following way: Consider the algebra $\mathbf{C}[V]^{K^{\mathbf{C}}}$ of $K^{\mathbf{C}}$-invariant complex polynomials; actually it coincides with that of K-invariant complex polynomials. By Hilbert's theorem, this algebra has a finite set $f_1, \ldots, f_r$ of generators. These yield a $K^{\mathbf{C}}$-invariant algebraic map f from V to $\mathbf{C}^r$ which, by construction, factors through the space of $K^{\mathbf{C}}$-orbits in V. The subspace of *closed* orbits may now be identified with the affine set in $\mathbf{C}^r$ defined by a set of defining relations for the algebra of invariants $\mathbf{C}[V]^{K^{\mathbf{C}}}$. Recall that the reduced space V_{red} is the quotient $\mu^{-1}(0)/K$. The following consequence of an observation of KEMPF-NESS [35], cf. §4 of [47], where the zero locus $\mu^{-1}(0)$ is referred to as a *Kempf-Ness* set, will be crucial for our purposes.

Lemma 4.1. *The canonical map $V_{\text{red}} \to V//K^{\mathbf{C}}$ from the reduced space V_{red} to the (affine) categorical quotient $V//K^{\mathbf{C}}$ induced by the inclusion of $\mu^{-1}(0)$ into V is a homeomorphism.*

Hence as a space, in fact, as a complex affine variety, the reduced space V_{red} looks like the affine categorical quotient $V//K^{\mathbf{C}}$. We shall see below that, as a stratified symplectic space, il looks rather different, though.

To illustrate how this works, consider the special case where $G = \mathrm{SU}(2)$. The following discussion will in particular show what is behind the proof of Theorem 3 above.

The bundle ξ is now necessarily trivial. Let $Z \subseteq G$ denote the centre and $T \subseteq G$ a maximal torus. We already pointed out that the space $N = N(\xi)$ has three strata

$$N = N_G \cup N_{(T)} \cup N_Z$$

where $N_{(K)}$ denotes the points of orbit type (K). The top stratum is N_Z. We can now "see" the strata in the local model in the following way: For a point $[A]$ in the top stratum, H^0_A and H^2_A are zero, and hence near $[A]$, the moduli space looks like a neighborhood of zero in H^1_A, with the symplectic structure σ_A. However, H^1_A amounts to a copy of $\mathbf{C}^{3(\ell-1)}$, with the standard structure.

For a point $[A]$ in the "middle" stratum $N_{(T)}$, up to conjugation, the holonomies of a representative A yield a homomorphism ϕ from π to T, and the cohomology H^*_A amounts to $\mathrm{H}^*(\pi, g_\phi)$. Now g_ϕ decomposes into a direct sum of t and $t^\perp$ where t is the Lie algebra of T which is a copy $\mathbf{R}$ of the reals and $t^\perp$ amounts to $\mathbf{R}^2$, with circle action through the 2-fold covering map onto $\mathrm{SO}(2,\mathbf{R})$. Moreover, $\mathrm{H}^1(\pi, g_\phi)$ decomposes into a direct sum of $(t\otimes\mathbf{C})^\ell$ and $(t^\perp\otimes\mathbf{C})^{\ell-1}$. The action on $(t\otimes\mathbf{C})^\ell$ is trivial - in fact, this summand corresponds to the points *in* the stratum $N_{(T)}$ (locally), while the $\mathrm{SO}(2,\mathbf{R})$-representation on $(t^\perp\otimes\mathbf{C})^{\ell-1}$ is hamiltonian, with momentum mapping Θ_A, restricted to $(t^\perp\otimes\mathbf{C})^{\ell-1}$. The latter boils down to the classical constrained system of $\ell-1$ particles moving in the plane with constant total angular momentum. In particular, reduction at total angular momentum zero yields the reduced space we are looking for (locally).

To be even more specific, let $\ell = 2$ and $V = t^\perp\otimes\mathbf{C} \cong \mathbf{R}^2\times\mathbf{R}^2$. The $\mathrm{SO}(2,\mathbf{R})$-representation is the obvious one, that is $\mathrm{SO}(2,\mathbf{R})$ acts as rotation group on each copy of $\mathbf{R}^2$. With the usual coordinates $(q,p) \in \mathbf{R}^2\times\mathbf{R}^2$, the momentum mapping μ from V to $\mathbf{R}$ is given by the assignment to (q,p) of the determinant $|qp|$. The algebra of (real) invariants in $\mathbf{R}[V]$ is generated by the three scalar products qq, qp, pp, and the determinant $|qp|$. However, on the zero locus $\mu^{-1}(0)$, the determinant vanishes whence the algebra $C^\infty(V_{\text{red}})$ is generated by the three scalar products.

To understand V_{red} as a *space*, we observe that the extension of the $\mathrm{SO}(2,\mathbf{R})$-representation to its complexification amounts to the standard $\mathrm{SO}(2,\mathbf{C})$-representation on $\mathbf{C}^2$. This representation has a single invariant, the complex scalar product which, with the notation $w = q + ip$, we write ww, and the algebra of complex invariants is free. Consequently the affine categorical quotient $\mathbf{C}^2//\mathrm{SO}(2,\mathbf{C})$ is a copy of the complex line $\mathbf{C}$. By virtue of Lemma 4.1, the canonical map from V_{red} to $\mathbf{C}$ is a homeomorphism. Under this homeomorphism, with the notation $ww = x_1 + ix_2$, so that x_1 and x_2 are the coordinate functions on $\mathbf{C}$, viewed as the real plane, we have

$$x_1 = qq - pp, \quad x_2 = 2qp, \quad r = qq + pp.$$

It is obvious that the algebra $C^\infty(V_{\mathrm{red}})$ is as well generated by the coordinate functions x_1, x_2 and the radius function r. Moreover, the complex picture tells us that the single obvous relation $x_1^2 + x_2^2 = r^2$ between the coordinate functions and the radius function suffices, that is, is a defining relation for $C^\infty(V_{\mathrm{red}})$. Finally, a straightforward calculation of the Poisson brackets between x_1, x_2, r, viewed as functions on the *original* space V, yields the formulas

$$\{x_1, x_2\} = 2r, \quad \{x_1, r\} = 2x_2, \quad \{x_2, r\} = -2x_1$$

already spelled out in the Introduction. See [25] for details.

A similar reasoning yields a model for a neighborhood of a point in the "bottom" stratum N_G. It consists of $2^{2\ell}$ isolated points and, locally, the Poisson algebra is that of the reduced classical constrained system of ℓ particles in $\mathbf{R}^3$ with total angular momentum zero. Even for $\ell = 2$, the reduced Poisson algebra is already rather complicated: It has ten generators; in fact, a basis of $\mathrm{sp}(2)$ may be taken as coordinate functions on $\mathrm{sp}(2)^*$; the reduced space may be described as the closure of a certain nilpotent orbit in $\mathrm{sp}(2)^*$. This relies on the theory of dual pairs [17] and work done in [36]; see our paper [25] for details.

A general study of the smooth structures needed to describe the Poisson geometry of our moduli spaces has been undertaken in our paper [23]. In particular the problem arises of determining, at an arbitrary point, the Zariski tangent space with reference to the smooth structure. For a general principal bundle ξ over Σ, cf. [23 (7.9)], an arbitrary central Yang-Mills connection A, that is, a choice of representative of the point $[A]$ of the moduli space N of central Yang-Mills connections, induces a linear map λ_A from H^1_A to the Zariski tangent space $\mathrm{T}_{[A]}N$, with reference to the smooth structure $C^\infty(N)$ mentioned earlier. This map is injective on the invariants $(\mathrm{H}^1_A)^{Z_A}$ and has kernel the orthogonal complement of the invariants. Now the subspace $(\mathrm{H}^1_A)^{Z_A}$ of invariants amounts to the tangent space of the stratum in which the point $[A]$ lies. Moreover, for the above example where $G = \mathrm{SU}(2)$ and $\ell = 2$, it is shown in [23] that, for a point in the middle stratum $N_{(T)}$, the Zariski tangent space has dimension 7, while for a point in the bottom stratum N_G, the Zariski tangent space has dimension 10. We have seen above that, for a point $[A]$ in the middle stratum, $(\mathrm{H}^1_A)^{Z_A}$ is of dimension 4, and so is its orthogonal complement; hence the kernel of λ_A is then of dimension 4. Likewise, for a point $[A]$ in the bottom stratum, $(\mathrm{H}^1_A)^{Z_A}$ is trivial and the kernel of λ_A is the whole space H^1_A. The latter has dimension 12 whereas the Zariski tangent space has dimension 10. The same discussion applies to the representation space incarnation of these spaces. Thus we see that, at a singular point $[\phi]$ of a space of the kind $\mathrm{Rep}_\xi(\Gamma, G)$, the Zariski tangent space $\mathrm{T}_{[\phi]}\mathrm{Rep}_\xi(\Gamma, G)$ with respect to the smooth structure $C^\infty(\mathrm{Rep}_\xi(\Gamma, G))$ does *not* boil down to $\mathrm{H}^1(\pi, g_\phi)$. This should be compared to what is said on p. 205 of [13].

It is obvious that the whole description generalizes to get a Poisson model for a neighborhood of an arbitrary point in a general moduli space of central Yang-Mills connections. More examples should be worked out!

5. The finite dimensional construction

Let

$$\mathcal{P} = \langle x_1, y_1, \ldots, x_\ell, y_\ell; r\rangle, \quad r = \Pi[x_j, y_j],$$

be the standard presentation of the fundamental group π of Σ where ℓ denotes the genus. Write F for the free group on the generators and N for the normal closure of the relators, so that $\pi = F/N$. The choice of generators identifies $\mathrm{Hom}(F, G)$ with $G^{2\ell}$ and, furthermore, the space $\mathrm{Hom}(\pi, G)$ with the pre-image of the identity element in G, for the word map

$$r: G^{2\ell} \to G$$

induced by the relator. The idea is now to view r as a momentum mapping. While this might appear a bit strange at first, a minor change will fit the picture into the appropriate framework: Let $O \subseteq g$ be the open G-invariant subset of g where the exponential mapping from g to G is regular; notice that O contains the centre of g. For example, when $G = \mathrm{SU}(2)$, the image of O in G is the 3-sphere with the point -1 removed. Write $\mathcal{H}(\mathcal{P}, G)$ for the space determined by the requirement that a pull back diagram

$$\begin{array}{ccc} \mathcal{H}(\mathcal{P},G) & \xrightarrow{\ \widehat{r}\ } & O \\ {\scriptstyle \eta}\big\downarrow & & \big\downarrow{\scriptstyle \exp} \\ \mathrm{Hom}(F,G) & \xrightarrow[\ r\ ]{} & G \end{array}$$

results, where the induced map from $\mathcal{H}(\mathcal{P}, G)$ to O is still denoted by r. The space $\mathcal{H}(\mathcal{P}, G)$ is a smooth manifold and the induced map η from $\mathcal{H}(\mathcal{P}, G)$ to $\mathrm{Hom}(F, G) = G^{2\ell}$ is a smooth codimension zero immersion whence $\mathcal{H}(\mathcal{P}, G)$ has the same dimension as $G^{2\ell}$; moreover the above injection of $\mathrm{Hom}(\pi, G)$ into $\mathrm{Hom}(F, G)$ induces a canonical injection of $\mathrm{Hom}(\pi, G)$ into $\mathcal{H}(\mathcal{P}, G)$, and in this way $\mathrm{Hom}(\pi, G)$ will be viewed as a subspace of $\mathcal{H}(\mathcal{P}, G)$. When we combine $\widehat{r}$, the inclusion of O into g, and the adjoint from g to its dual g^* of the orthogonal structure on g, we obtain a smooth G-equivariant map μ from $\mathcal{H}(\mathcal{P}, G)$ to g^* which looks like a momentum mapping and has $\mathrm{Hom}(\pi, G)$ as its zero locus. It turns out that a G-invariant 2-form can be constructed on $\mathcal{H}(\mathcal{P}, G)$ which, on a neighborhood of $\mathrm{Hom}(\pi, G)$ in $\mathcal{H}(\mathcal{P}, G)$, amounts to a symplectic structure in such a way that (i) μ is a momentum mapping, and (ii) the holonomy identifies the moduli space of flat connections on the corresponding flat bundle with its stratified symplectic structure mentioned in Theorem 1 above with the reduced space $\mathcal{R} = \mathrm{Rep}(\pi, G)$ for μ with reference to zero which we now explain. This gives a purely finite dimensional treatment of the symplectic geometry of moduli spaces of flat connections. More generally, certain twisted analogues $\mathcal{R}_X$ associated to elements X in the Lie algebra of the centre of

G amount to moduli spaces $N(\xi)$ of central Yang-Mills connections. Goldman [13] constructed a symplectic structure on a smooth, open, connected, and dense subspace of $\mathcal{R}$; Karshon [34] gave the first proof using only finite dimensional techniques that the symplectic form is closed, and Weinstein [55] (whose construction we follow) reinterpreted Karshon's construction in terms of the de Rham-bar bicomplex. The construction to be presented extends the methods of [55] and gives a smooth finite dimensional space $\mathcal{M}$ together with a G-invariant symplectic form and momentum mapping in such a way that the moduli spaces $\mathcal{R}_X$ result from symplectic reduction with reference to X (or what corresponds to it in the dual of g). Such a construction is given by [27] (Section 5 and 6) and [33] (5.1).

To the Lie group G, we associate the *de Rham-bar bicomplex* $(C^{\cdot,\cdot}(G); d, \delta)$ defined by $C^{p,q}(G) = \Omega^q(G^p)$, the two operators d and δ being the de Rham differential $d\colon \Omega^*(G^*) \to \Omega^{*+1}(G^*)$ and the bar complex operator $\delta\colon \Omega^*(G^*) \to \Omega^*(G^{*+1})$, respectively. Its total cohomology gives the cohomology of the classifying space of G [8], [50]. For a discrete group Π — we shall take $\Pi = F$ or $\Pi = \pi$ — we shall denote the chain and cochain complexes of its inhomogeneous *reduced* normalized bar resolution $B\Pi$ by $(C_*(\Pi), \partial)$ and $(C^*(\Pi), \delta)$, respectively; see e. g. MAC LANE [38] for details. We may form another bicomplex $(\widetilde{C}^{\cdot,\cdot}; d, \delta)$ defined by $\widetilde{C}^{p,q} = C^p(\Pi) \otimes \Omega^q(\mathrm{Hom}(\Pi, G)$, the two operators d and δ being the de Rham differential on $\mathrm{Hom}(\Pi, G)$ and the bar complex operator $\delta\colon C^*(\Pi) \to C^{*+1}(\Pi)$, respectively. Notice when $\Pi = F$, the free group on 2ℓ generators, $\mathrm{Hom}(\Pi, G)$ amounts to $G^{2\ell}$. For general π, the evaluation maps $E\colon \Pi^p \times \mathrm{Hom}(\Pi, G) \to G^p$ give rise to a sequence of maps from $\widetilde{C}^{p,q}$ to $C^{p,q}$ which combine to give a morphism of bicomplexes.

Denote by τ the g-valued left-invariant 1-form on G which maps each tangent vector to the left invariant vector field having that value. The corresponding right-invariant form is denoted by $\overline{\tau}$. These are the Maurer-Cartan forms. Next, if α is any differential form on G, we denote by α_j the pullback of α to $G \times G$ by the projection p_j to the j'th component. Henceforth we denote the chosen invariant 2-form on g by "$\cdot$". Let

$$\Omega = \frac{1}{2}\tau_1 \cdot \overline{\tau}_2.$$

This is an alternating 2-form on $G \times G$. Then $E^*(\Omega) \in C^2(\Pi) \otimes \Omega^2(\mathrm{Hom}(\Pi, G))$. To any 2-chain c in $C_2(\Pi)$, Weinstein associated the 2-form $\omega_c = (c, E^*\Omega)$ on $\Omega^2(\mathrm{Hom}(\Pi, G))$, and he observed that this assignment induces a homomorphism from $\mathrm{H}_2(\Pi)$ to the closed 2-forms on $\mathrm{Hom}(\Pi, G)$ [55]. Now an element κ of $\mathrm{H}_2(\pi)$ may be expressed in terms of a representative $c \in C_2(F)$ having boundary ar, that is, an integral multiple of r determined by κ; this relies on the Schur-Hopf formula calculating the second homology group of a discrete group. We thus obtain a 2-cycle of π, which will still be denoted by c. The above construction with $\Pi = F$ yields the 2-form ω_c on $\mathrm{Hom}(F, G)$. Let

$$\lambda = \frac{1}{12}[\tau, \tau] \cdot \tau \in \Omega^3(G).$$

This is a closed invariant 3-form so that

$$d\omega_c = ar^*\lambda \in \Omega^3(\mathrm{Hom}(F, G)) \tag{5.1}$$

([27] (18); [33] Proposition 4.1). The choice of generators induces an embedding of the space $\mathrm{Hom}(\pi, G)$ into $\mathrm{Hom}(F, G)$ and (5.1) entails that the 2-form ω_c, restricted to $\mathrm{Hom}(\pi, G)$, is closed, suitably interpreted in its singularities.

We now suppose that κ is a (suitably chosen) generator. In view of (5.1), $d\eta^*\omega_c$ then equals $d\widehat{r}^*(\alpha)$ where α is a closed form on O. Using the homotopy operator h that enters the usual proof of the Poincaré Lemma, let $\beta = h\alpha$; then $\alpha = d\beta$ on g and hence on O. Thus we have:

Proposition 5.2. *The 2-form $\widetilde{\omega} = \eta^*\omega_c - \widehat{r}^*(\beta)$ on $\mathcal{H}(\mathcal{P}, G)$ is closed and G-invariant.*

As usual we write i for the contraction operator on forms. An equivariant version of the de Rham-bar bicomplex finally provides a proof of the following [27], [33].

Proposition 5.3. *The above map μ from $\mathcal{H}(\mathcal{P}, G)$ to g^* satisfies, for every $Y \in g$, the formula*

$$i_{Y_M}\widetilde{\omega} = d(\mu^Y)$$

where as before $\mu^Y : M = \mathcal{H}(\mathcal{P}, G) \to \mathbf{R}$ refers to the composite of μ with Y, viewed as a linear map on g^*.

Thus formally the momentum mapping property is satisfied, cf. [6], except that $\widetilde{\omega}$ is not necessarily symplectic since it need not be non-degenerate. By adding μ to $\widetilde{\omega}$ on $\mathcal{H}(\mathcal{P}, G)$ we obtain in a standard way an extension of $\widetilde{\omega}$ to an equivariantly closed 2-form on $\mathcal{H}(\mathcal{P}, G)$.

Now $\mathrm{H}_2(\pi)$ is infinite cyclic, generated by an element represented by a 2-chain c of F whose boundary equals r. The subspace O of g contains the centre z of g, and Poincaré duality in the cohomology of π implies that, on a neighborhood $\mathcal{M}$ of the pre-image $(\widehat{r}^*)^{-1}(z)$ in $\mathcal{H}(\mathcal{P}, G)$, the 2-form $\widetilde{\omega}$ has maximal rank, that is, is symplectic. Symplectic reduction with reference to zero then yields the usual space $\mathcal{R}$ of representation of π in G, while symplectic reduction with reference to suitable non-zero central values X corresponding to topologically non-trivial G-bundles yields certain twisted moduli spaces $\mathcal{R}_X$. Application of the SJAMAAR-LERMAN result [51] then yields the following, cf. [27].

Theorem 5.4. *With respect to the decomposition according to G-orbit types, the space $\mathcal{R}$ and, more generally, each twisted representation space $\mathcal{R}_X$ inherits a structure of stratified symplectic space.*

Each possible non-zero central value X is precisely of the kind X_ξ coming into play in Section 3 above, where ξ refers to a suitable principal bundle, and the Wilson loop mapping, that is, the operation of taking holonomies with reference to suitable paths, yields a diffeomorphism of stratified symplectic spaces from $N(\xi)$ to $\mathcal{R}_X$. In other words, we have recovered the moduli space $N(\xi)$ of central Yang-Mills connections as a stratified symplectic space by a finite dimensional construction. In particular, for $G = U(n)$, the unitary group, the moduli spaces of holomorphic semi stable rank n vector bundles of arbitrary degree arise in this way by finite dimensional symplectic reduction.

Theorem 5.5. *Each stratum of the space $\mathcal{R}$ and, more generally, each stratum of a twisted representation space $\mathcal{R}_X$ has finite symplectic volume.*

The finiteness of the symplectic volumes may also be derived from the local models, see [21].

Theorem 5.6. *The reduced Poisson algebra is symplectic, that is, its only Casimir elements are the constants.*

Again this may as well be derived from the local models.

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