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Geometrical directions and ends of a manifold, points of accumulation of a direction of a group in the hyperbolic space H^2

In: Jarolím Bureš and Vladimír Souček (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1996. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 39. pp. [37]--56.

Persistent URL: http://dml.cz/dmlcz/701564

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II - Suppl. 39 (1996), pp. 37-56

GEOMETRICAL DIRECTIONS AND ENDS OF A MANIFOLD, POINTS OF ACCUMULATION OF A DIRECTION OF A GROUP IN THE HYPERBOLIC SPACE H^2

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ABSTRACT. The basic notion in this paper is the definition of an end of a group. We express this notion in the language of sequences to obtain a definition equivalent to the standard one. In the set of these sesquences, we introduce an equivalence relation in such a way that all sequences representing the same end of the group divide into more subtle disjoint sets - directions of the group. The notion of a direction of a group gives us handful tools for studying the behaviour of groups acting on topological spaces.

For example, we are able to examine the interdependance between points of accumulation of the directions of a group of isometries acting on a manifold and the ends of the manifold, provided that the Dirichlet polygon of this group is bounded. It is the essence of the first theorem in this paper. The second main theorem shows that there exist infinitely many points of accumulation of the direction of a group of isometries acting on the hyperbolic space H^2, provided there are at least two. A suitable example describes the case where a direction of the group of isometries accumulates at at least two points.

0. Introduction

The basic notion in this paper is that of an end of a group, introduced in the 40's by H. Freudenthal. The combinatorical-topological approach to the theory of ends, presented by H. Freudenthal, shows that if a group acts in a proper way, then the structure of ends of a space depends only on the group.

In paper [Stall] J. Stallings studies 3-dimensional manifolds, properties of ends of topological spaces and relations between the ends and the homology of a space. The ends of the universal covering $\tilde{M}$ of a manifold M depend only on the first fundamental group $\pi_1(M)$ of M. All these studies lead to the definition of an end in an abstract finitely generated group. Also, J. Stallings introduces the notion of an end of a group by cohomologies with coeficients from the group $\mathbb{Z}_2$. D. E. Cohen in his papers [Coh1] and [Coh2] shows a new approach to the theory of end of a group which is purely algebraic in contradistinction to the combinatorical-topological approach of H. Freudenthal and J. Stallings.

The notion of a direction of a group, introduced in our paper, is a suitable tool allowing us to examine the action of a group on topological space in a more subtle way than that using only the theory of ends of a group.

Using a properly defined equivalence relation, we obtain that the space of ends of a group splits into abstract classes-directions of the group. The directions of the group allow us to study the behaviour of the group along a proper sequence of compositions of its generators. We are able to investigate the relations between the algebraic properties of a group and the geometry of the space where the group acts.

In paper [EbO'Ne] P. Eberlein and B. O'Neill show in a very elegant way that a group of isometries that is an algebraic structure, determines the geometry of the space where it acts. They consider a simply connected complete Riemannian manifold H with a sectional curvature $K \leq 0$ and a group D of isometries acting discontinuously on H. They define points at infinity for the manifold H. In set $\bar{H} = H \cup H(\infty)$, where $H(\infty)$ is a set of points at infinity, the cone topology is introduced in such a way that each isometry ϕ of H can be extended naturally to a homeomorphism of $\bar{H}$. Since D acts discontinuously on the manifold H, the invariant limit sets $L(D)$ appear in a natural way. The structure and the properties of the limit sets are considered also by W. Ballman [Ball] and A. Beardon [Bea]. Su Shing and P. Eberlein [ShiEb] study the interdependence between the algebraic structure of orbits of G in $H(\infty)$ and the geometry of the manifold H.

Moreover, we can recommend P. Nicholls' book [Nich] as a source of general information about the theory of limit points and a presentation of the results of this fruitful measure theory, describing limit sets of a discrete group.

The notion of a geometrical direction supplies us with a tool for more precise investigations of the acting of a group than those using only the orbit of the group. We can notice the way the orbit approaches points at infinity. We are able to study the structure of limit points of a direction in $H(\infty)$ as well as the relation between the structure of a geometric direction and ends of a space.

1. Points of accumulation of a direction on the hyperbolic space H^2.

Let G be a finitely generated group of transformations of a topological space X. Denote by G_1 a set of generators of the group G such that $G_1^{-1} \subset G_1$ and $\mathrm{id}_X \in G_1$. Put $G_n := \{g \in G : g = g_1 \cdot \ldots \cdot g_n, g_i \in G_1\}$.

Then we have

$$G = \bigcup_{n=1}^{\infty} G_n$$

and

$$G_1 \subset G_2 \subset G_3 \subset \ldots$$

Definition 1.1. We say that elements $f, g \in G \backslash G_n$ can be connected in $G \backslash G_n$ if

$$\exists_{k \in N} \ \exists_{g_1, g_2, \ldots, g_k \in G \backslash G_n} \ \forall_{1 \leq i \leq k} \ f = g_1 \ \wedge \ g_k = g \wedge g_{i+1} \cdot g_i^{-1} \in G_1.$$

We then write

$$f \mathrm{con}(n) g.$$

In the set S of all sequences (f_n) satisfying the conditions

$$\forall_{n \in N} f_n \in G_{n+1} \backslash G_n, \tag{1}$$

$$\forall_{n \in N} f_{n+1} \cdot f_n^{-1} \in G_1 \tag{2}$$

we introduce a relation "ϱ" in the following way:

$$(f_n)\varrho(g_n) \text{ if and only if } \forall_{n \in N} f_n \operatorname{con}(n) g_n.$$

The relation introduced above is an equivalence relation. The abstract classes of th relation "ϱ" are called ends of the group.

The equivalence between such a definition and the classical one of an end of a group can be found in [Bi]

We shall obtain subtler classes if we introduce in the set S a relation "$\sim$" i the following way:

$$(f_n) \sim (g_n) \text{ if and only if, for any } n \in N \; f_n \operatorname{con}(n) g_n \text{ and}$$

find elements $g_{n,1}, \ldots, g_{n,s(n)} \in G_1$ such that $f_n = g_{n,1} \cdot \ldots \cdot g_{n,s(n)} \cdot g_n$ in such a way that sequence $s(n)$ is bounded.

Definition 1.2. The abstract classes of the relation "$\sim$" are called algebraic directions in the group G.

Consider a complete, noncompact Riemannian manifold M with a metric d and negative curvature. Let G be a finitely generated group of isometries of the manifold M and let G_1 be a set of generators.

As above, let S be a set of sequences (f_n) satisfying conditions (1) and (2). In the set S we introduce a relation "$\approx$" in the following way:

$$(f_n) \approx (g_n) \text{ if and only if follwing conditions are satisfied}$$

1) $\exists_{p \in M} \exists_{A>0} \forall_{n \in N} \; d(f_n(p), g_n(p)) < A$,
2) $\forall_{n \in N} \; f_n \operatorname{con}(n) g_n$,
3) $\forall_{n \in N} \; \exists_{g_{n,1} \ldots g_{n,s(n)} \in G_1} \; f_n = g_{n,1} \cdot \ldots \cdot g_{n,s(n)} \cdot g_n$,
4) the sequence $s(n)$ is bounded.

Remark 1.3. We can notice that if condition (1) is satisfied by a point $p \in M$, then it is also satisfied by any other point of the manifold M.

In fact,

$$\begin{aligned}\underset{q\in M}{\forall}\ d(f_n(q),g_n(q)) &\le d(f_n(q),f_n(p))\\ &+ d(f_n(p),g_n(p) + d(g_n(p),g_n(q))\\ &= 2d(p,q) + d(f_n(p),g_n(q))\\ &< 2d(p,q) + A < \infty.\end{aligned}$$

Remark 1.4. The relation "$\approx$" is an equivalence relation.

Definition 1.5. The abstract class of a sequence $(f_n) \in S$ in the relation "$\approx$" is called a geometric direction.

Definition 1.6. Consider the maximal geodesics γ_v, γ_w of a manifold M, determined by vectors $v, w \in TM$. We say that the geodesics γ_v, γ_w are asymptotic if there exists a positive number C such that for a certain $t_0 \in \mathbb{R}$, the following implication

$$t \ge t_0 \Rightarrow d(\gamma_v(t), \gamma_w(t)) \le C$$

is true.
The above relation is an equivalence relation in the set of geodesics of the manifold M.

Definition 1.7. The abstract class of this relation determined by a geodesic γ is called a points at infinity and denoted by $\gamma(\infty)$. The set of all points at infinity is denoted by $M(\infty)$.

Let $\hat{M} := M \cup M(\infty)$. There are few ways of introducing a topology in the set $\hat{M}$ but, further, we consider only the cone topology (introduced in $\hat{M}$ by the family of topological cones) (see [Eb.O'Ne]).

Consider a group G of isometries of the manifold M which is finitely generated and which acts in a discontinuous way. For $q \in M$ consider an orbit $G(q) := \{g(q) : g \in G\}$ of the point q. The group acts in a discontinuous way, so the orbit $G(q)$ of the point q has no accumulation point in M, but such points can exist in a space $\hat{M}$ with the cone topology.

Let f by a geometric direction determined by a sequence of isometries (f_n) of an isometry group G acting on a manifold M. Consider the sets $\{f_1(a), \dots f_n(a), \dots\}$ and $\{f_1(b), \dots f_n(b), \dots\}$ with $a, b \in M$; then we have $d(f_k(a), f_k(b)) = d(a, b) < \infty$, so according to the cone topology, we obtain that the above-mentioned sets have the same accumulation points in the space $\hat{M}$.

If the sequences (f_n) and (g_n) determine the same geometric direction, then, according to condition (3), for each point $a \in M$, there exists $A > 0$ such that $d(f_n(a), g_n(a)) < A$. So, the accumulation points of the sets $\{f_1(a), \dots, f_n(a), \dots\}$ and $\{g_1(a), \dots, g_n(a), \dots\}$ are the same.

That is why we can formulate

Definition 1.8. An accumulation point in the topological space $\hat{M}$ of the orbit $(f_n(a))$ of a point $a \in M$ is called an accumulation point of the geometric direction f determined by the sequence (f_n).

Remark 1.9. In the sequel, we shall consider only Fuchsian groups which act in a discontinuous way on the hyperbolic space H^2.

The main aim of this section is the following

Theorem 1.10. *Let H^2 be the two-dimensional hyperbolic space, G- a group of isometries acting on H^2 in a discontinuous way with a finite set of generators G_1. We assume that there exists a point $w_0 \in H^2$ such that a Dirichlet polygon $D(w_0)$ for the group G centered at the point $w_0 \in H^2$ is bounded. We assume that the geometric direction f has two distinct points of accumulation $q_1, q_2 \in \hat{M}$. Then the geometric direction f has infinitely many points of accumulation. Moreover, any point q from the segment q_1q_2 is an accumulation point of the geometric direction f.*

In the proof of theorem 1.10 we shall use the following definitions and lemmas.

Definition 1.11. Let G be a Fuschian group acting on the hyperbolic space H^2. A fundamental set F for the group G is a set $F \subset H^2$ which includes only one point of each orbit of the group G.

Definition 1.12. A subset D of the space H^2 is called a fundamental domain of the group G acting on H^2 if:

1) the set D is a domain,
2) there exists a fundamental set F such that $D \subset F \subset \bar{D}$,
3) the measure of the boundary of D is equal to zero.

Definition 1.13. We say that a fundamental domain of the group G is locally finite if each compact subset of H^2 intersects only a finite number of images of the set $\bar{D}$ in a mapping g where $g \in G$.

Definition 1.14. A set

$$D(w) := \bigcap_{g \in G \setminus \{\mathrm{Id}\}} \{z \in H^2 : d(w, z) < d(z, g(w))\}$$

is called a Dirichlet polygon for the group G centered at w.

Lemma 1.15. *The Dirichlet polygon is a locally finite fundamental domain for the group G.*

Proof - see [Bea], p.227.

Lemma 1.16. *Let D be a locally finite fundamental domain for a Fuchsian group G. Then the set*

$$G_0 := \{g \in G : g(\bar{D}) \cap \bar{D} \neq \emptyset\}$$

generates the group G.

Proof - can be found in [Bea], p.214.

Proof of theorem 1.10. Let a sequence (f_n) of isometries acting on the space H^2 determine a geometric direction with at least two different points of accumulation $p, q \in \hat{H}^2$. Let $r \in (p, q)$ and fix a point $w \in \hat{H}^2$. Let $\gamma_1, \gamma_2, \gamma_3$ be normalized geodesics comming, respectively, trough the points w and p, w and q, w and r. (fig. 1).

Let $D(w)$ be a Dirichlet polygon for the group G centered at w. Notice that the images of $D(w)$ in a mapping g, where $g \in G$, tesselate the whole space H^2. According to the assumption, the Dirichlet polygon $D(w)$ is bounded, so there exists a constant $c > 0$ such that $D(w)$ is contained in a ball $B(w, c)$ with center w and radius c. We also know that the images of the polygon $D(w)$ in a mapping g, where $g \in D$, are isometric to $D(w)$; that is why their diameters are bounded by $2c$.

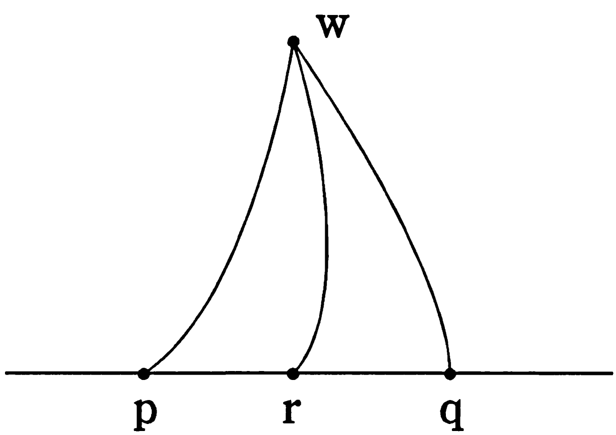

fig. 1

Consider three cones C_1, C_2, C_3 is the space $\hat{H}^2$ with the common vertex, determined by geodesics $\gamma_1, \gamma_2, \gamma_3$ and angles $\epsilon_1, \epsilon_2, \epsilon_3$.

At first, we shall study the case where

$$G_1 = G_0 = \{g \in G : g(\bar{D}) \cap \bar{D} \neq \emptyset\}.$$

Let s be a number such that the distance between the points lying on the opposite sides of the truncated cone $C_3 \backslash B(w, s)$ is greater than $2c$.

We shall study the centers of the Dirichlet polygon $D(w)$ in a mapping g where $g \in G$. By the assumption that p and q are distinct points of accumulation of the geometric direction f, we obtain that the f-orbit of the point w, that is the set $\{f_n(w) : n \in N\}$, has accumutation points at p and q. Let $w_k = f_k(w)$ and $w_l = f_l(w)$ be points such that $w_k \in C_1 \backslash B(w, s)$, $w_l \in C_2 \backslash B(w, s)$ and the elements $w_{k+1} = f_{k+1}(w), \ldots, w_{l-1} = f_{l-1}(w)$ do not belong to the ball $B(w, s)$.

There exists a sequence of Dirichlet polygons $D(w_k), \dots, D(w_l)$ such that the intersection of closures of any two neighbouring polygons is not empty. So, there exists $n \in \{k+1, \dots, l-1\}$ such that

$$w_n \in C_3 \backslash B(w, s).$$

According to the definition of cone topology, we obtain that the point r is an accumulation point of the f-orbit of the point w, so, in this case, the proof is completed.

We shall prove the case $G_1 \neq G_0$ reducing it to the previous one.

Let $G_1 = \{g_1, \dots, g_m\}$ and $G_0 = \{h_1, \dots, h_n\}$. An element $g_l \in G_1$ can be written as a composition of finitely many generators from the set G_0.

For any $l \in \{1, \dots, m\}$, we can write

$$g_l = h_{l_1} \dots h_{l_i}$$

where $h_{l_1}, \dots, h_{l_i} \in G_0$.

Put $E = \max\{i_l : l \in \{1, \dots, m\}\}$. Then we immediately obtain that for any $k_1, k_2 \in \{1, \dots, n\}$,

$$\begin{aligned} d(g_{k_1}(w), g_{k_2} \cdot g_{k_1}(w)) &\le d(g_{k_1}, w) + d(g_{k_2} \cdot g_{k_1}(w), w) \le \\ &\le E\,2c + 2E \cdot 2c < \infty. \end{aligned}$$

That is why the distance between the successive points of the sequence $(f_n(w))$ is less than or equal to $6Ec$.

The next part of this proof is very similar to the previous case with only one difference that instead of the number s we define a number s' in the following way: let s' be a number such that the distance beetween two points lying on the opposite sides of the truncated cone $C_3 \backslash B(w, s')$ is greater then $6Ec$.

Studying the sequence $(f_n(w))$ and using the assumption that the geometric direction $f = [(f_n)]_\approx$ accumulates at at least two points p and q, we obtain that there exist successive elements of the sequence $(f_n(w))$ such that $w_k \in C_1 \backslash B(w, s')$, $w_l \in C_2 \backslash B(w, s')$ and the elements $w_{k+1}, \dots w_{l-1}$ do not belong to the ball $B(w, s')$.

The distance between two successive elements $w_k, w_{k+1}, \dots, w_l$ is less than or equal to $6Ec$ and the distance between points on the opposite sides of the cone $C_3 \backslash B(w, s')$ is greater than $6Ec$, so there exists a point

$$w_n \in C_3 \backslash B(w, s').$$

In this case we also obtain that the point r is an accumulation point of the geometric direction f.

Proposition 1.17. *Let H^2 be a two-dimensional hyperbolic space and G an isometry group acting on H^2, finitely generated by a set G_1. We assume that the Dirichlet polygon $D(w_0)$ of this group, centered at a point $w_0 \in H^2$, is bounded.*

Then each algebraic direction f of the group G with a set of generators G_1 is a geometric direction.

Proof. . Let the sequences (f_n) and (g_n) determine the same algebraic direction in the group G. It is enough to show that there exists a constant $A > 0$ such that

$$\underset{n \in \mathbb{N}}{\forall} d(f_n(w_0), g_n(w_0)) < A.$$

Since $[(f_n)]_\sim = [(g_n)]_\sim$, therefore for each $n \in N$, there exist generators $g_{n,1}, \dots, g_{n,s(n)} \in G_1$ such that

$$f_n = g_{n,1} \cdots g_{n,s(n)} g_n$$

where the sequence $s(n)$ is bounded.

Each of the generators $g_{n,i}$ can be writen in the form of the composition of finitely many generators from the set G_0, defined in Lemma 1.16. So, we can find generators

$$h_{n,1}, \dots, h_{n,t(n)} \text{ such that } f_n = h_{n,1}, \dots, h_{n,t(n)} g_n,$$

and the sequence $t(n)$ is bounded.

If $C > 0$ is the diameter of the Dirichlet polygon $D(w_0)$, then each of the generators $h_{n,1}, \dots, h_{n,t(n)} \in G_0$ transforms a point $p \in H^2$ into the point p' in such a way that

$$d(p, p') \leq C.$$

So,

$$d(f_n(w_0), g_n(w_0)) = d(h_{n,1}\cdot\ldots\cdot h_{n,t(n)}g_n(w_0), g_n(w_0)) \le C\cdot\max\{t(n) : n \in \mathbb{N}\} < \infty,$$

which completes the proof.

To finish with this section, we present an example of a geometric direction which accumulates at at least two points in H^2.

Example 1.18. Consider a disc model D of the hyperbolic space H^2 with a metric d. Let R_n be a rotation of the disc D around its centre O and let $\frac{\pi}{2^n}$ be the angle of this rotation, $n \in \mathbb{N}$. Let T be a hyperbolic isometry of H^2 such that its axis γ is a diameter of the disk D. Then $T|_\gamma : \gamma \to \gamma$ is a translation of the geodesic γ (see Lemma 6.5 in[BGSch]).

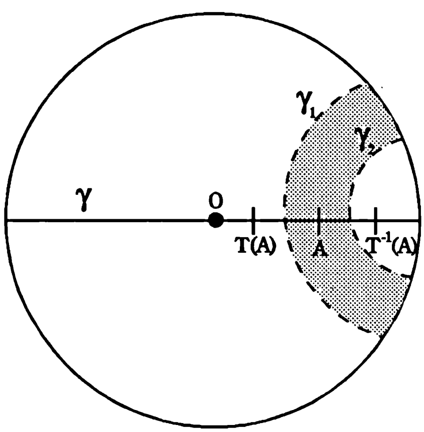

fig. 2

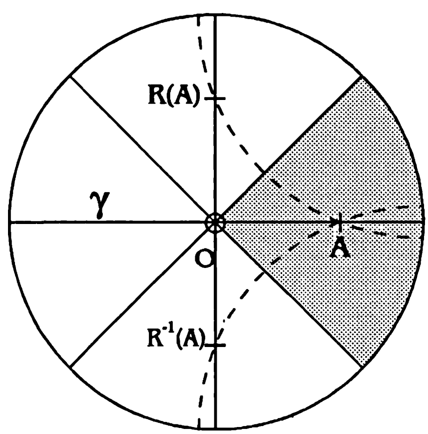

fig. 3

Let G be an isometry group of the disc D, generated by T and R_n. Choose a point $A \in \gamma$ such that $A \neq O$ and $T(A)$ is between O and $T^{-1}(A)$. Consider a Dirichlet polygon $D(A)$ with centre A for the group G.

Then we obtain

$$D(A) = \bigcap_{g\in G\setminus\{\mathrm{Id}\}} \{z \in D : d(z,A) < d(z,g(A))\} \subset B \cap C$$

where

$$B = \{z \in D : d(z,A) < d(z,T(A)) \wedge d(z,A) < d(z,T^{-1}(A))\},$$

$$C = \{z \in D : d(z,A) < d(z,R(A)) \wedge d(z,A) < d(z,R^{-1}(A))\}.$$

The set B is a "strip" on the disc D, bounded by two geodesics γ_1, γ_2 perpendicular to γ and intersecting γ at two points which are middle points of the geodesic intervals $\overline{T(A)A}$ and $\overline{AT^{-1}(A)}$ (fig. 2).The set C is a disc sector determined by the angle $\frac{\pi}{2^n}$ (fig. 3, for case n=1). Denote by α an angle with a vertex O determined by geodesic half- lines containing point $\gamma_2(\infty)$, $\gamma_2(-\infty) \in \bar{D}$. For a sufficiently large $m \in \mathbb{N}$,

$$\frac{\pi}{2^m} < \frac{\alpha}{2}.$$

Construct a geometric direction f in the following way: Let $f = [(f_n)]_\approx$ where $R_m = R$ and

$$f_1 = T, \quad f_2 = RT, \quad f_3 = RRT, \quad \ldots \quad , f_{2^m+1} = \underbrace{R\ldots R}_{2^m\times}T$$

$$f_{2^m+2} = T^{-1}R\ldots RT, \quad f_{2^m+3} = RT^{-1}R\ldots RT, \quad \ldots \quad ,$$

$$f_{2^{m+1}} + 2 = \underbrace{R\ldots R}_{2^m\times}T^{-1}\underbrace{R\ldots R}_{2^m\times}T$$

$$f_n = \begin{cases} T\,f_{n-1} & \text{if } (n-1) \mod (2^{m+1}+2) = 0 \\ R\,f_{n-1} & \text{if } 1 \le (n-1) \mod (2^{m+1}+2) \le 2^m \\ T^{-1}\,f_{n-1} & \text{if } (n-1) \mod (2^{m+1}+2) = 2^m+1 \\ R\,f_{n-1} & \text{if } 2^m+2 \le (n-1) \mod (2^{m+1}+2) \le 2^{m+1}+1. \end{cases}$$

Then $f_n(O) \in \gamma$ for infinitely many $n \in \mathbb{N}$.

So, the sequence accumulates at points $\gamma(\infty)$, $\gamma(-\infty) \in \bar{D}$. The direction f accumulates at at least two points. The group G acts on the hyperbolic space in a discontinuous way, its Dirichlet polygon $D(A)$ centered at the point A is included in the bounded set $B \cap C$.

2. Geometric directions and ends of manifolds.

Definition 2.1. Let X be a topological space and G a homeomorphism group acting on X. We say that G acts in a discontinuous way on the space X if and only if for any compact set $K \subset X$,

$$g(K) \cap K \neq \emptyset$$

only for finitely many elements $g \in G$.

Remark 2.2. If $x \in X$ and $g_1, g_2, \dots$ are distinct elements of G, then the sequence $g_1(x), g_2(x), \dots$ has no accumulation points in X.

In fact, if the subsequence $(g_{n_k}(x))$ of $(g_n(x))$ accumulates at a point x_0 and $K = \{x_0, x, g_{n_1}(x), g_{n_2}(x) \dots\}$, then:
$g_{n_k}(x) \in K \cap g_{n_k}(K)$ for infinitely many distinct elements $g_{n_k} \in G$, which contradicts the assumption about the discontinuity of the group G.

In this paper we consider only a finitely generated isometry group G acting in a discontinuous way on a complete simply connected Riemannian manifold M with a metric d. We call the set $G(x) := \{g(x) : g \in G\}$ an orbit of a point $x \in M$. Also, we say that points $x, y \in M$ are G-equivalent if both of them belong to the same orbit.

Definition 2.3. A set $F \subset M$ which consists only one point from each orbit is called a fundamental set for the group G.

Definition 2.4. A subset $D \subset M$ is called a fundamental domain for the group G if and only if:

1) D is a domain,
2) there exists a fundamental set F such that $D \subset F \subset \bar{D}$,
3) $\mathrm{vol}(\delta D) = 0$.

Directly from definitions 2.3 and 2.4 we obtain that for any $g \in G \backslash \{\mathrm{Id}\}$, $g(D) \cap D = \emptyset$ and $\bigcup_{g \in G} g(\bar{D}) = M$. Let G be a group acting on M (with the assumptions as above). Choose a point $w \in M$ which is not a fixed point of any isometry of $G \backslash \{\mathrm{Id}\}$.

We show that there exists a point $p \in M$ which is not a fixed point of any isometry from the group G acting in a discontinuous way. Then we assume to the contrary that

$$\forall_{x \in M} \exists_{\mathrm{Id} \neq g_x \in G} \; g_x(x) = x.$$

Let K be a closed ball of radius $r > 0$ centred at x_0. By the assumption that the group acts in a discontinous way, we obtain that there exist only finitely many elements $g \in G$ such that

$$g(K) \cap K \neq \emptyset.$$

Denote them by $g_1, g_2, \ldots, g_n$. Let

$$M_{g_i} = \{x \in M : g_i(x) = x\}, \quad i = 1, 2, \ldots, n.$$

Then we can write

$$K \subset \bigcup_{i=1}^{n} M_{g_i}.$$

Using Baire's Theorem, we obtain that there exists $i_0 \in \{1, 2, \ldots, n\}$ such that

$$\mathrm{Int} M_{g_{i_0}} \neq \emptyset.$$

So, the isometry g_{i_0} is the identity mapping which contradicts our assumption.

Put

$$L_g(w) = \{z \in M : d(z, w) = d(z, gw)\},$$

$$H_g(w) = \{z \in M : d(z, w) < d(z, gw)\}.$$

Definition 2.5. A set

$$D(w) := \bigcap_{g \in G \setminus \{\mathrm{Id}\}} H_g(w)$$

is called Dirichlet set with centre w for the group G.

Definition 2.6. We say that a fundamental domain D for a group G is locally finite if and only if a compact set $K \subset M$ intersects only finitely many sets $g(\bar{D})$ where $g \in G$.

Proposition 2.7. *The set $D(w)$ is a locally finite fundamental domain for the group G.*

Proof. At first, we should notice that only a finite number of sets $L_g(w)$ can intersect a fixed compact set $K \subset M$. In fact, if $G = \{g_1, g_2, \dots\}$, then using a discontinuous action of the group G we obtain

$$d(w, L_{g_n}(w)) = \frac{1}{2} d(w, g_n(w)) \underset{n \to \infty}{\longrightarrow} \infty.$$

Let $z \in \overline{D(w)}$. For any element $g \in G$, there exists a ball centered at z such that $K \subset H_g(w)$ or $z \in L_g(w)$. Moreover, the last condition can be satisfied only by finitely many elements $g \in G$. The boundary of the set $D(w_0)$ is contained in a union of the sets $L_g(w)$, $g \in G \backslash \{\mathrm{Id}\}$. Therefore

$$\mathrm{vol}(D(w)) = 0$$

because, for any $g \in G \backslash \{\mathrm{Id}\}$, we have $\mathrm{vol}(L_g(w)) = 0$.

Choose exactly one point $z' \in G(z)$ which satisfies the condition

$$d(w, z) \leq d(w, gz) \text{ for all } g \in G.$$

Such a choice can be done since the set $G(z)$ does not accumulate at the point w.

Denote by F the set of points described above. Using the description of the set $D(w)$ we immediately obtain that the condition $z \in D(w)$ yields $z = z'$. So, $D(w) \subset F$.

Choose $z \in F$ and consider the geodesic segment $[w, z)$.

If $L_g(w) \cap (w, z) \neq \emptyset$, then we have the inequality

$$d(z, w) > d(z, gw) = d(g^{-1} z, w),$$

which contradicts the assumption that $z \in F$.

So, for any $g \in G \backslash \{\mathrm{Id}\}$, we have $L_g(w) \cap (w, z) = \emptyset$; that is why $(w, z) \subset D(w)$ and, in consequence, $F \subset \overline{D(w)}$.

We should show that $D(w)$ is a locally finite set. Let K be a ball centred at w and with radius r. Assume that $g(\overline{D(w)} \cap K \neq \emptyset$. There exists $z \in \overline{D(w)}$, such that $d(gz, w) \leq r$. If $z \in \overline{D(w)}$, then

$$\begin{aligned} d(w, gw) &\leq d(w, gz) + d(gz, gw) \\ &\leq r + d(z, w) \\ &\leq r + d(gz, w) \\ &\leq 2r. \end{aligned}$$

The above condition can be true only for finitely many elements $g \in G$ because the group G acts in a discontinuous way.

Under the assumption as above we obtain:

Proposition 2.8. *If $D(w)$ is a Dirichlet set for the group G, the set*

$$G_0 = \{g \in G : g(\overline{D(w)}) \cap \overline{D(w)} = \emptyset\}$$

generates the group G.

Proof. Let G^* be a group generated by G_0. For any $z \in M$, there exists $g \in G$ such that $g(z) \in \overline{D(w)}$. We may assume that $h(z) \in \overline{D(w)}$. Then $h(z) \in \overline{D(w)} \cap hg^{-1}\overline{D(w)}$, so $hg^{-1} \in G_0$ and $G^*h = G^*g$. That is why the following mapping is properly defined:

$$\begin{aligned} &\Phi : M \to G\backslash_{G^*}, \\ &\Phi(z) = G^* g \quad \text{where} \quad g(z) \in \overline{D(w)}. \end{aligned}$$

Consider $z \in M$. If $D(w)$ is a locally finite domain, there exist finitely many elements $g_1, \ldots, g_m \in G$ such that the point z belongs to each set $g_1(\overline{D(w)}), \ldots, g_m(\overline{D(w)})$ and there exists an open neighbourhood N of z which satisfies the following condition:

$$N \subset \bigcup_{j=1}^{m} g_j(\overline{D(w)}).$$

If $w \in \mathbb{N}$, there exists $j \in \{1, \ldots, m\}$ such that $w \in g_j(\overline{D(w)})$.

We have proved that each point of M has a neighbourhood N such that $\Phi|_N$ is a constant mapping.

Considering $\Phi(M)$ with the discrete topology, we obtain that Φ is a continuous mapping, so the connected set $\Phi(M)$ consists of one point. Therefore $\Phi(z) = \Phi(w)$ for any $z \in M$.

For any element $g \in G$, choose a point $z \in D(w)$ and a point $v \in g^{-1}(D(w))$. Then

$$\Phi(z) = \Phi(v);$$

according to the difinition of the mapping Φ,

$$\Phi(z) = G^* \quad \text{and} \quad \Phi(v) = G^*g,$$

which yields that $g \in G^*$ and $G = G^*$.

Definition 2.9. Let (K_n) be an ascending sequence of compact subsets of the Riemannian manifold M, such that

$$\bigcup_{n=1}^{\infty} \operatorname{Int} K_n = M.$$

A descending sequence of connected components of the sets $M \backslash K_n$ is called an end of the manifold M.

Definition 2.10. We say that an orbit of a point $q \in M$ along the sequence (f_n) determining the geometric direction in a finitely generated group G acting in a discontinuous way on the Riemannian manifold M accumulates at an end $E = (E_i)$ of M if and only if

$$\underset{i \in \mathbb{N}}{\forall} \ \underset{n \in \mathbb{N}}{\forall} \ \underset{m \geq n}{\exists} \ f_m(q) \in E_i.$$

Definition 2.11. We say that a geometric direction $F = [(f_n)]_{\approx}$ accumulates at an end $E = (E_i)$ of a manifold M if, for any represectation (f_n) of F and any point $q \in M$, an orbit of q along the sequence (f_n) accumulates at the end $E = (E_i)$ of M.

Theorem 2.12. *Let G be a finitely generated isometry group acting in a discontinuous way on a complete connected Riemannian manifold M with a metric d. Denote by G_1 a set of generators of G. If there exists a point $w \in M$ such that the Dirichlet set $D(w)$ centred at w is bounded for the group G, then:*

1) an orbit of a point $p \in M$ along the sequence (f_n) determining the geometric direction F accumulates at one end of M,

2) if an orbit of a point $p \in M$ along the sequence (f_n) determining a geometric direction accumulates at an end E of M, then the orbit of another point $q \in M$ along the sequence (f_n) accumulates at the same end E,

3) for two sequences of isometries determinig the same geometric direction and for a point $p \in M$, we obtain that the sequences $(f_n(p))$ and $(g_n(p))$ accumulate at the same end of M.

Proof of 1). At first consider case 1) when a finite set G_1 of gererators of the group G is equal to the set $G^* := \{f \in G : f(\overline{D(w)}) \cap \overline{D(w)} \neq \emptyset\}$. Denote by D the diameter of the Dirichlet set $D(w)$. According to our assumption, we have that D is less than infinity.

Contrary to the assertion, assume that the orbit $(f_n(q))$ of the point q accumulates at at least two distinct ends E and E' of M. Let $E = (E_i)$ and $E' = (E_i')$ where E_i and E_i' are the connected components of the set $M \backslash K_i$. Since $E \neq E'$ there exists $i_0 \in \mathbb{N}$ such that $E_{i_0} \cap E_{i_0}' = \emptyset$. From definition 2.10 we obtain that each of the sets E_i and E_i' includes infinitely many elements of the sequence $(f_n(q))$.

Consider the set $K_{i_0}^D := \{x \in M : d(x, K_{i_0}) \leq D\}$. There exists $j \in \mathbb{N}$ such that the compact set $K_{i_0}^D$ is included in K_j.

The distance between the succesive points of the orbit $(f_n(q))$ is less than or equal to D. Therefore, infinitely many elements of $(f_n(q))$ are included in a compact set K_j, which contradicts the assumption that the group G acts in a discontinuous way.

In the case when $G_1 \neq G^*$ we have that for any $g_i \in G_1$ there exist $g_{i,1}, \ldots,$ $g_{i,k_i} \in G^*$ such that

$$g_i = g_{i,1} \cdot \ldots \cdot g_{i,k_i}.$$

Let $s = \max\{k_i\}$. Then the distance between the succesive points of $(f_n(q))$ is less than or equal to $s \cdot D$. Take $i_0 \in \mathbb{N}$ such that the intersection of the connected components E_{i_0} and E'_{i_0} of the set $M \backslash K_{i_0}$ is empty. The compact set $K_{i_0}^{sD}$ is included in K_j for some $j \in \mathbb{N}$. The distance between E_{i_0} and E'_{i_0} is greater than or equal to $2sD$, so the orbit $(f_n(q))$ intersects the compact set $K_{i_0}^{sD}$ infinitely many times, which contradicts the assumption that G acts in a discontinuous way.

Proof of 2). According to the above proof, we can say that the orbit $(f_n(p))$ of a point p accumulates at an end E. Assume that the orbit $(f_n(q))$ of a point q accumulates at an end E'. Since G is a group of isometries, therefore

$$\underset{n \in \mathbb{N}}{\forall} \; d(f_n(p), f_n(q)) = d(p, q).$$

On the other hand, if $E \neq E'$, then there exists $i_0 \in \mathbb{N}$ such that $E_{i_0} \cap E'_{i_0} = \emptyset$ where E_{i_0} and E'_{i_0} are the connected components of the set $M \backslash K_{i_0}$. Taking an $A > 0$, we obtain, that for a compact set $K_{i_0}^{A} \subset K_j$, so the distance between E_j and E'_j is not less than $2A$. Since A is an arbitrary number, we have that the distances $d(f_n(p), f_n(q))$ are unbounded, where $n \in \mathbb{N}$. So, $E = E'$.

Proof of 3). follows immediately from those of 1) and 2).

Using the above theorem, we can derive

Corollary 2.13. *If G is a finitely generated group of isometries acting in a discontinuous way on a complete Riemannian manifold M and if there exists a point $w \in M$ such that the Dirichlet set $D(w)$ is bounded for the group G, then each geometric direction in G accumulates at one end of M.*

Proposition 2.14. *Let M be a connected complete Riemannian manifold and G a finitely generated group of isometries acting on M, such that the Dirichlet set $D(w)$ is bounded for any $w \in M$.*

Then each algebraic direction in G generated by a finite set G_1 is geometric direction in G generated by G_1.

Proof. - analogous to that of Proposition 1.17.

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