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NATURAL OPERATORS LIFTING VECTOR FIELDS ON MANIFOLDS TO THE BUNDLES OF COVELOCITIES

W. M. Mikulski

Abstract. *We prove that for n-manifolds ($n \geq 3$) the sets of all natural operators $T \to (T_k^{r*}, T_l^{q*})$ and $T \to TT_k^{r*}$, respectively, are free finitely generated $C^\infty((\mathbf{R}^k)^r)$-modules. We construct explicitly the bases of the $C^\infty((\mathbf{R}^k)^r)$-modules.*

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0. Introduction. We show how a vector field X on M can induce a fibre bundle map $A_M(X) : T_k^{r*}M \to T_l^{q*}M$ over id_M (or a vector field $A_M(X)$ on $T_k^{r*}M$), where $T_k^{r*}M = J^r(M, \mathbf{R}^k)_0$ is the bundle of (k, r)-covelocities over M, cf.[6]. In Section 1 we present some constructions of such types. In Section 2 we remark that the idea of such constructions is reflected in the concept of natural operators $T \to (T_k^{r*}, T_l^{q*})$ (or $T \to TT_k^{r*}$), cf. [6]. The rest of the paper is dedicated to the proof of the following two theorems.

Theorem A *For n-manifolds ($n \geq 3$), the space of all natural operators $T \to TT_k^{r*}$ is a free finitely generated module over $C^\infty((\mathbf{R}^k)^r)$.*

Theorem B *For n-manifolds ($n \geq 3$), the space of all natural operators $T \to (T_k^{r*}, T_l^{q*})$ is a free finitely generated module over $C^\infty((\mathbf{R}^k)^r)$.*

[0] This paper is in final form and no version of it will be submitted for publication elsewhere.

In the proof of these theorems we construct explicitly the bases of these $C^\infty((\mathbf{R}^k)^r)$ -modules.

Since $T_l^{q*}M = T_1^{q*} \times_M ... \times_M T_1^{q*}M$ (l-times), without loss of generality we assume that $l = 1$. If $k = 1$ we reobtain the result of [10].

Similar problems have been studied by many authors, cf. [1], [4], [5]-[10] e.t.c.

All manifolds and maps in this paper are assumed to be smooth, i.e. infinitely differentiable.

1. Main examples. Let $n, r, k, q \geq 1$ be integers, M be an n-manifold and X be a vector field on M. The $C^\infty(M)$-module of all global vector fields on M is denoted by $\mathcal{X}(M)$.

In this section we present some examples of fibre bundle maps $T_k^{r*}M \to T_1^{q*}M$ and vector fields on $T_k^{r*}M$ induced by X.

Example 1.1. Let " $\leq$ " be the usual lexikographic ordering on $\mathbf{Z}^2$. Let $Q_{k,q}^r = \bigcup_{s=1}^q Q_{k,q}^{r,s}$, where for any $s \in \{1, ..., q\}$

$$(1.1) \qquad \begin{aligned} Q_{k,q}^{r,s} =&\{((\alpha_1, \overline{\alpha}_1), ..., (\alpha_s, \overline{\alpha}_s)) \in (\mathbf{Z}^2)^s : \\ &(0,1) \leq (\alpha_1, \overline{\alpha}_1) \leq ... \leq (\alpha_s, \overline{\alpha}_s) \leq (s-1+r-q, k)\}. \end{aligned}$$

We see that if $1 \leq s < max(1, q-r+1)$, then $Q_{k,q}^{r,s} = \emptyset$. Furthermore, if $q \geq s \geq max(1, q-r+1)$, then $card(Q_{k,q}^{r,s}) = \binom{(s+r-q)k+s-1}{s}$. For any

$$\alpha = ((\alpha_1, \overline{\alpha}_1), ..., (\alpha_s, \overline{\alpha}_s)) \in Q_{k,q}^r$$

we have a fibre bundle map $A_M^{\alpha;r,k,q}(X) : T_k^{r*}M \to T_1^{q*}M$ over id_M defined as follows.

Let $\gamma = (\gamma_1, ..., \gamma_k) : M \to \mathbf{R}^k$ and $x \in M$ be such that $\gamma(x) = 0$. Considering X as the differential operator $C^\infty(M) \to C^\infty(M)$ we define $b_{\gamma,x,\alpha} = b_\gamma : M \to \mathbf{R}$ by

$$b_\gamma = \prod_{i=1}^s (X^{\alpha_i}\gamma_{\overline{\alpha}_i} - X^{\alpha_i}\gamma_{\overline{\alpha}_i}(x)) = (X^{\alpha_1}\gamma_{\overline{\alpha}_1} - X^{\alpha_1}\gamma_{\overline{\alpha}_1}(x))...(X^{\alpha_s}\gamma_{\overline{\alpha}_s} - X^{\alpha_s}\gamma_{\overline{\alpha}_s}(x)),$$

where $X^{\alpha_i} = X \circ ... \circ X$ (α_i-times).

Since $\alpha \in Q_{k,q}^r$, the q-jet at x of b_γ depends only on the r-jet at x of γ. To see this we consider an arbitrary $\eta = (\eta_1, ..., \eta_k) \in (m_x^{r+1})^k$, where m_x is the ideal of maps $M \to \mathbf{R}$ vanishing at x. Let $\gamma' = \gamma + \eta$. Then $b_{\gamma'} - b_\gamma$ is a sum of terms of the form $\prod_{j=1}^s (X^{\alpha_j}\rho_j - X^{\alpha_j}\rho_j(x))$, where $\rho_j \in \{\eta_1, ..., \eta_k, \gamma_{\overline{\alpha}_1}, ..., \gamma_{\overline{\alpha}_s}\}$ and $\rho_{j_o} = \eta_{i_0}$ for some $j_o \in \{1, ..., s\}$ and $i_0 \in \{1, ..., k\}$. We see that $X^{\alpha_j}\rho_j - X^{\alpha_j}\rho_j(x) \in m_x$ and

$$X^{\alpha_{j_o}}\rho_{j_o} - X^{\alpha_{j_o}}\rho_{j_o}(x) = X^{\alpha_{j_o}}\eta_{i_0} \in m_x^{r+1-\alpha_{j_o}} \subset m_x^{q-s+2},$$

as $\alpha_{j_o} \le \alpha_s \le s-1+r-q \le r-1$. Therefore $b_\gamma - b_{\gamma'} \in m_x^{q+1}$, as well.

Any element from $T_k^{r*}M$ is of the form $j_x^r\gamma$, where $\gamma : M \to \mathbf{R}^k$ and $x \in M$ are such that $\gamma(x) = 0$. We put

$$(1.2) \qquad (A_M^{\alpha;r,k,q}(X))(j_x^r\gamma) := j_x^q(b_\gamma) = j_x^q\Big(\prod_{i=1}^{s}(X^{\alpha_i}\gamma_{\overline{\alpha}_i} - X^{\alpha_i}\gamma_{\overline{\alpha}_i}(x))\Big)$$

for any $j_x^r\gamma \in T_k^{r*}M$. Since $b_\gamma(x) = 0$, $j_x^q(b_\gamma) \in (T_1^{q*}M)_x$. Using the local coordinate argument it is easy to verify that $A_M^{\alpha;r,k,q}(X)$ is smooth. Of course, it is a fibre bundle map over id_M.

Example 1.2. In general if $E \to M$ is a vector bundle and $A : E \to E$ is a fibre bundle map over id_M, then there exists the unique vertical vector field $A^+ \in \mathcal{X}(E)$ such that $A^+(y)$ is the velocity vector of $\mathbf{R} \ni t \to y + tA(y) \in E$ at 0, where $y \in E$, c.f. [2]. Hence for any fibre bundle map $A : T_k^{r*}M \to T_k^{r*}M$ over id_M there exists $A^+ \in \mathcal{X}(T_k^{r*}M)$. In particular, for any $(\alpha^1, ..., \alpha^k) \in (Q_{k,r}^r)^k = Q_{k,r}^r \times ... \times Q_{k,r}^r$ (k-times) there exists $A_M^{\alpha^1,...,\alpha^k;r,k}(X) \in \mathcal{X}(T_k^{r*}M)$ given by $A_M^{\alpha^1,...,\alpha^k;r,k}(X) = (A_M^{\alpha^1;r,k,r}(X), ..., A_M^{\alpha^k;r,k,r}(x))^+$, where $Q_{k,r}^r$ and $(A_M^{\alpha^1;r,k,r}(X), ..., A_M^{\alpha^k;r,k,r}(X)) : T_k^{r*}M \to T_k^{r*}M = T_1^{r*}M \times_M ... \times_M T_1^{r*}M$ (k-times) are defined in Example 1.1.

Example 1.3. In general, if G is a natural bundle, then we can define $G(X) \in \mathcal{X}(GM)$ (the complete lift of X to GM) via the prolongations of flows, c.f. [3],[6]. Hence we have $T_k^{r*}(X) \in \mathcal{X}(T_k^{r*}M)$ such that: if φ_t is the flow of X then $T_k^{r*}(\varphi_t)$ is the flow of $T_k^{r*}(X)$.

2. Natural operators $T \to (T_k^{r*}, T_1^{q*})$ and $T \to TT_k^{r*}$. It is well-known that the concept of geometrical constructions can be formulated in the form of natural operators, cf. [6].

Example 2.1. Let n, r, k, q be fixed natural numbers. Let $\alpha \in Q_{k,q}^r$ (or $(\alpha^1, ..., \alpha^k) \in (Q_{k,r}^r)^k$), where $Q_{k,q}^r$ is defined in (1.1). The family $A^{\alpha;r,k,q}$ (or $A^{\alpha^1,...,\alpha^k;r,k}$) of functions

$$\mathcal{X}(M) \ni X \to A_M^{\alpha;r,k,q}(X) \in C_M^\infty(T_k^{r*}M, T_1^{q*}M)$$
$$(\text{or } \mathcal{X}(M) \ni X \to A_M^{\alpha^1,...,\alpha^k;r,k}(X) \in \mathcal{X}(T_k^{r*}M)\ ),$$

for any n-manifold M, where $A_M^{\alpha;r,k,q}(X)$ (or $A_M^{\alpha^1,...,\alpha^k;r,k}(X)$) is described in Example 1.1 (or in Example 1.2) and $C_M^\infty(T_k^{r*}M, T_1^{q*}M)$ is the set of all fibre bundle maps over id_M, is a natural operator $T \to (T_k^{r*}, T_1^{q*})$ (or $T \to TT_k^{r*}$). Similarly, the family T_k^{r*} of functions

$$\mathcal{X}(M) \ni X \to T_k^{r*}(X) \in \mathcal{X}(T_k^{r*}M)$$

for any n-manifold M, where $T_k^{r*}(X)$ is described in Example 1.3, is a natural operator $T \to TT_k^{r*}$.

3. The main result. Let n be a natural number. Let $G : \mathcal{M}f \to \mathcal{VB}$ and $F : \mathcal{M}f_n \to \mathcal{VB}$ be bundle functors, cf.[6], where $\mathcal{VB}$ is the category of vector bundles and vector bundle maps. Let $E = (G|\mathcal{M}f_n)^* : \mathcal{M}f_n \to \mathcal{VB}$ be the bundle functor dual to $(G|\mathcal{M}f_n)$ i.e. $EM = (GM)^*$ and $E\varphi = (G(\varphi^{-1}))^*$ for any $M \in obj(\mathcal{M}f_n)$ and any $\varphi \in morph(\mathcal{M}f_n)$. . Then the set of all natural operators $T \to (E, F)$ (or $T \to TE$), cf. [6, p.174], is a $C^\infty((G_0\mathbf{R})^*)$-module. Actually, for any $A, B : T \to (E, F)$ (or $A, B : T \to TE$ and $f, g \in C^\infty((G_0\mathbf{R})^*)$ the natural operator $fA + gB : T \to (E, F)$ (or $fA + gB : T \to TE$) is defined by

$$((fA + gB)_M(X))(\omega) = f(\omega_X)((A_M(X))(\omega)) + g(\omega_X)((B_M(X))(\omega)) ,$$

where M is an n-manifold, $X \in \mathcal{X}(M)$, $\omega \in E_xM$, $x \in M$, and where $\omega_X \in (G_0\mathbf{R})^*$ is defined as follows. Let $\{\varphi_t^X\}$ be the flow of X. Let $\Phi_x^X : \mathbf{R} \to M$ be a map defined on some neighbourhood of $0 \in \mathbf{R}$ by $\Phi_x^X(t) = \varphi_t^X(x)$. Then we put $\omega_X := \omega \circ (G_0(\Phi_x^X))$, cf [10].

In particular, $T_k^{r*} : \mathcal{M}f_n \to \mathcal{VB}$ is naturally isomorphic to $(T_k^{(r)}|\mathcal{M}f_n)^*$, where $T^{(r)} : \mathcal{M}f \to \mathcal{VB}$ is the linear r-th order tangent bundle functor, see [6, p.123], and $T_k^{(r)}M = T^{(r)}M \times_M ... \times_M T^{(r)}M$ (k-times). Now using the isomorphism $T_0^{(r)}\mathbf{R} = \mathbf{R}^r$, $\omega \to (\omega(j_0^r(x^s)))_{s=1}^r$, we have the $C^\infty((\mathbf{R}^k)^r)$-modules of all natural operators $T \to TT_k^{r*}$ and $T \to (T_k^{r*}, T_1^{q*})$ respectively. It is easy to verify that for every natural operators $A, B : T \to (T_k^{r*}, T_1^{q*})$ (or $A, B : T \to TT_k^{r*}$ and $f, g \in C^\infty((\mathbf{R}^k)^r)$ the natural operator $fA + gB : T \to (T_k^{r*}, T_1^{q*})$ (or $fA + gB : T \to TT_k^{r*}$) is given by

$$\begin{aligned}((fA + gB)_M(X))(j_x^r\gamma) =& f(X\gamma(x), ..., X^r\gamma(x))((A_M(X))(j_x^r\gamma)) \\ &+ g(X\gamma(x), ..., X^r\gamma(x))((B_M(X))(j_x^r\gamma)),\end{aligned}$$

where M is an n-manifold, $X \in \mathcal{X}(M)$, $X^s = X \circ ... \circ X$ (s-times) and $j_x^r\gamma \in T_k^{r*}M$.

The main result of this paper is formulated in the following two theorems, corresponding to Theorem A and Theorem B, respectively.

Theorem 3.1. *Let r, k, n be natural numbers. If $n \geq 3$, then (for n-manifolds) the natural operators T_k^{r*} and $A^{\alpha^1,...,\alpha^k;r,k}$ for $(\alpha^1, ..., \alpha^k) \in (Q_{k,r}^r)^k$ (described in Example 2.1) form a basis of the $C^\infty((\mathbf{R}^k)^r)$-module (described above) of all natural operators $T \to TT_k^{r*}$.*
In particular, if $n \geq 3$, then the $C^\infty((\mathbf{R}^k)^r)$-module of all natural operators $T \to TT_k^{r*}$ is isomorphic to $(C^\infty((\mathbf{R}^k)^r))^{1+k\sum_{s=1}^r \binom{sk+s-1}{s}}$.

Theorem 3.2. *Let r, k, q, n be natural numbers. If $n \geq 3$, then (for n-manifolds) the natural operators $A^{\alpha;r,k,q}$ for $\alpha \in Q^r_{k,q}$ (described in Example 2.1) form a basis of the $C^\infty((\mathbf{R}^k)^r)$-module of all natural operators $T \to (T^{r*}_k, T^{q*}_1)$.*

Since $T^{q*}_l M = T^{q*}_1 \times_M \dots \times_M T^{q*}_1 M$ (l-times), we have the following consequence of Theorem 3.2.

Corollary 3.1. *Let r, k, q, l, n be natural numbers. If $n \geq 3$, then (for n-manifolds) the natural operators $(A^{\alpha^1;r,k,q}, \dots, A^{\alpha^l;r,k,q})$ for $(\alpha^1, \dots, \alpha^l) \in (Q^r_{k,q})^l$ form a basis of the $C^\infty((\mathbf{R}^k)^r)$-module of all natural operators $T \to (T^{r*}_k, T^{q*}_l)$, where $(A^{\alpha^1;r,k,q}, \dots, A^{\alpha^l;r,k,q})_M(X) := (A^{\alpha^1;r,k,q}_M(X), \dots, A^{\alpha^l;r,k,q}_M(X)) : T^{r*}_k M \to T^{q*}_l M$.*

In particular, if $n \geq 3$, then the $C^\infty((\mathbf{R}^k)^r)$-module of all natural operators $T \to (T^{r*}_k, T^{q*}_l)$ is isomorphic to $(C^\infty((\mathbf{R}^k)^r))^{l\sum_{s=max(1,q-r+1)}^{q}\binom{(s+r-q)k+s-1}{s}}$.

4. A preparatory proposition.

The following (decomposition) lemma for $E = T^{r*}_k$ shows that Theorem 3.1 is a simple consequence of Theorem 3.2.

Lemma 4.1. *Let $E = (G|\mathcal{M}f_n)^*$, where $G : \mathcal{M}f \to \mathcal{VB}$ is a bundle functor. Let $A : T \to TE$ be a natural operator. Then there exist $h \in C^\infty((G_0\mathbf{R})^*)$ and a natural operator $B : T \to (E, E)$ such that $A = hE + B^+$, i.e. $(A_M(X))(\omega) = h(\omega_X)((E(X))(\omega)) + (B_M(X))^+(\omega)$ for any n-manifold M, $X \in \mathcal{X}(M)$, $\omega \in E_x M$ and $x \in M$, where $\omega_X \in (G_0\mathbf{R})^*$ is defined in Section 3, $E(X)$ is the complete lift of X to E (see Example 1.3) and the operation $(\ )^+$ is described in Example 1.2.*

A proof of Lemma 4.1 one can find in [10].

The proof of Theorem 3.2 will be given in Section 5. In the proof of Theorem 3.2 we shall use some technical facts proved in this section.

From now on the usual coordinates on $\mathbf{R}^n$ are denoted by $x^1, \dots, x^n$. The canonical vector fields $\frac{\partial}{\partial x^i}$ on $\mathbf{R}^n$ are denoted by ∂_i. The $C^\infty((\mathbf{R}^k)^r)$-module of all natural operators $T \to (T^{r*}_k, T^{q*}_1)$ for n-manifolds is denoted by $\mathcal{T}(r, k, q, n)$.

For any integers $r, k, q, n \geq 1$ we have a homomorphism of $C^\infty((\mathbf{R}^k)^r)$-modules

$$\mathcal{T}(r, k, q+1, n) \ni A \to \pi^{q+1}_q \underline{\circ} A \in \mathcal{T}(r, k, q, n), \tag{4.1}$$

where $(\pi^{q+1}_q \underline{\circ} A)_M(X) := \pi^{q+1}_q \circ (A_M(X))$ for any n-manifold M and any $X \in \mathcal{X}(M)$ and where for arbitrary $\overline{r}, \overline{k}$

$$\pi^{\overline{r}+1}_{\overline{r}} : T^{\overline{r}+1*}_{\overline{k}} M \to T^{\overline{r}*}_{\overline{k}} M,\ \pi^{\overline{r}+1}_{\overline{r}}(j^{\overline{r}+1}_x \gamma) = j^{\overline{r}}_x \gamma, \tag{4.2}$$

is the jets projection. We also have a homomorphism of $C^\infty((\mathbf{R}^k)^r)$-modules

$$\mathcal{T}(r,k,q,n) \ni A \to A \underline{\circ} \pi_r^{r+1} \in \mathcal{T}(r+1,k,q,n)\ , \tag{4.3}$$

where $(A\underline{\circ}\pi_r^{r+1})_M(X) := (A_M(X)) \circ \pi_r^{r+1}$. A $C^\infty((\mathbf{R}^k)^r)$-module structure in $\mathcal{T}(r+1,k,q,n)$ is induced from the $C^\infty((\mathbf{R}^k)^{r+1})$-structure by the homomorphism

$$C^\infty((\mathbf{R}^k)^r) \ni f \to f \circ p_r \in C^\infty((\mathbf{R}^k)^{r+1})$$

of rings, where $p_r : (\mathbf{R}^k)^{r+1} = (\mathbf{R}^k)^r \times \mathbf{R}^k \to (\mathbf{R}^k)^r$ is the projection.

In this section we will prove the following proposition.

Proposition 4.1. *Let r,k,q,n be natural numbers. If $n \geq 3$, then the natural operators $A^{\alpha;r,k,q}$ for $\alpha \in Q^{r,q}_{k,q}$ (see (1.1) and Example 2.1) form a basis of the $C^\infty((\mathbf{R}^k)^r)$-submodule*

$$\mathcal{T}^0(r,k,q,n) = \{A \in \mathcal{T}(r,k,q,n) : \pi^q_{q\dot{-}1}\underline{\circ}A = 0\} \tag{4.4}$$

where $\pi^1_0 := 0$.

Proposition 4.1 will be a consequence of the following lemma.

Lemma 4.2. *Let r,k,q,n be natural numbers. For any $C \in \mathcal{T}^0(r,k,q,n)$ define $h^C : (\mathbf{R}^k)^r \times (\mathbf{R}^k)^r \to \mathbf{R}$ by*

$$h^C(\nu,\mu) = (\Phi \circ (C_{\mathbf{R}^n}(\partial_1)))(j^r_0(\sum_{m=1}^{r} \frac{1}{m!}\nu_m (x^1)^m + \sum_{m=0}^{r-1} \frac{1}{m!}\mu_m (x^1)^m x^2)), \tag{4.5}$$

where $\nu = (\nu_1,\ldots,\nu_r)$, $\mu = (\mu_0,\ldots,\mu_{r-1}) \in (\mathbf{R}^k)^r$ and

$$\Phi : (T^{q*}_1\mathbf{R}^n)_0 \to \mathbf{R},\ \Phi(j^q_0\eta) = \frac{1}{q!}(\partial_2)^q\eta(0). \tag{4.6}$$

If $n \geq 3$, then the function

$$\mathcal{T}^0(r,k,q,n) \ni C \to h^C \in C^\infty((\mathbf{R}^k)^r \times (\mathbf{R}^k)^r) \tag{4.7}$$

is a $C^\infty((\mathbf{R}^k)^r)$-module monomorphism, provided a $C^\infty((\mathbf{R}^k)^r)$-module structure in $C^\infty((\mathbf{R}^k)^r \times (\mathbf{R}^k)^r)$ is given by $(f+g)(y,z) = f(y,z) + g(y,z)$ and $(\lambda f)(y,z) = \lambda(y) f(y,z)$,where $\lambda \in C^\infty((\mathbf{R}^k)^r)$, $f,g \in C^\infty((\mathbf{R}^k)^r \times (\mathbf{R}^k)^r)$ and $y,z \in (\mathbf{R}^k)^r$.

First we prove the following lemma.

Lemma 4.3. *Let $A,B \in \mathcal{T}^0(r,k,q,n)$ be such that $g^A = g^B$, where $g^A : (T^{r*}_k\mathbf{R}^n)_0 \to \mathbf{R}$ is defined by $g^A(j^r_0\gamma) = (\Phi\circ(A_{\mathbf{R}^n}(\partial_1)))(j^r_0\gamma)$. If $n \geq 2$, then $A = B$.*

Proof of Lemma 4.3. Let $\gamma_\zeta : \mathbf{R}^n \to \mathbf{R}$ and $\eta_\zeta : \mathbf{R}^n \to \mathbf{R}$ be such that $j_0^q(\eta_\zeta) = (A_{\mathbf{R}^n}(\partial_1))(\zeta)$ and $j_0^q(\gamma_\zeta) = (B_{\mathbf{R}^n}(\partial_1))(\zeta)$ for any $\zeta \in (T_k^{r*}\mathbf{R}^n)_0$. By the assumption we have $D^\alpha\gamma_\zeta(0) = D^\alpha\eta_\zeta(0) = 0$ for any $\alpha \in (\mathbf{N} \cup \{0\})^n$ with $0 \le |\alpha| \le q-1$ and any $\zeta \in (T_k^{r*}\mathbf{R}^n)_0$ ($D^\alpha := (\partial_1)^{\alpha_1} \circ ...(\partial_n)^{\alpha_n}$). It is sufficient to show that $D^\alpha\gamma_\zeta(0) = D^\alpha\eta_\zeta(0)$ for any $\alpha \in (\mathbf{N} \cup \{0\})^n$ with $|\alpha| = q$. (Then $(A_{\mathbf{R}^n}(\partial_1))(\zeta) = (B_{\mathbf{R}^n}(\partial_1))(\zeta)$, i.e. $A_{\mathbf{R}^n}(\partial_1) = B_{\mathbf{R}^n}(\partial_1)$ over 0, and consequently $A = B$ because of the Frobenius theorem.)

By the polarization formula, it is sufficient to show that $X^q\eta_\zeta(0) = X^q\gamma_\zeta(0)$ for any constant vector field $X \in \mathcal{X}(\mathbf{R}^n)$ and for any $\zeta \in (T_k^{r*}\mathbf{R}^n)_0$. We can assume that $X \notin span(\partial_1)$. Since $n \ge 2$, there exists a linear isomorphism $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ preserving ∂_1 such that $T\varphi \circ \partial_2 = X \circ \varphi$. Using the invariancy (of A and B) with respect to φ we obtain

$$X^q\eta_\zeta(0) = (\partial_2)^q(\eta_\zeta \circ \varphi^{-1})(0) = q!g^A(T_k^{r*}\varphi(\zeta)) = q!g^B(T_k^{r*}\varphi(\zeta)) = X^q\gamma_\zeta(0).$$

Lemma 4.3 is proved. □

Proof of Lemma 4.2. It is easy to see that the function given in (4.7) is a homomorphism of $C^\infty((\mathbf{R}^k)^r)$-modules. Let $A, B \in \mathcal{T}^0(r,k,q,n)$ be such that $h^A = h^B$. We shall prove that $A = B$.

By the invariancy of A with respect to $b_t = (x^1, x^2, tx^3, ..., tx^n)$, $t \ne 0$, (preserving ∂_1 and ∂_2) we obtain that $g^A(j_0^r\gamma) = g^A(j_0^r(\gamma \circ b_t))$ for any $j_0^r\gamma \in (T_k^{r*}\mathbf{R}^n)_0$ and $t \in \mathbf{R} - \{0\}$, where g^A is defined in Lemma 4.3. If $t \to 0$ then

$$g^A(j_0^r\gamma) = g^A(j_0^r(\gamma \circ (x^1, x^2, 0, ..., 0))) \tag{4.8}$$

for any $j_0^r\gamma \in (T_k^{r*}\mathbf{R}^n)_0$. Let $P = \{(m,s) \in (\mathbf{N} \cup \{0\})^2 : \ 1 \le m+s \le r\}$. Define $f^A : (\mathbf{R}^k)^P \to \mathbf{R}$ by

$$f^A(\mu_{m,s};\ (m,s) \in P) = g^A(j_0^r(\sum_{(m,s)\in P} \frac{1}{m!s!}\mu_{m,s}(x^1)^m(x^2)^s)).$$

We show that f^A is independent of $\mu_{m,s}$ for all $(m,s) \in P$ with $s \ge 2$.

Assume the contrary. Let $s_o \ge 2$ be the maximal number for which there exists m such that $(m, s_o) \in P$ and f^A depends on μ_{m,s_o}. Since $n \ge 3$, the diffeomorphism $\varphi = (x^1, x^2, x^3 + (x^2)^{s_o}, x^4, ..., x^n) : \mathbf{R}^n \to \mathbf{R}^n$ preserves ∂_1. Since $j_0^1\varphi = j_0^1(id)$, then $T_1^{q*}\varphi^{-1}(j_0^q\eta) = j_0^q\eta$ for any $j_0^q\eta \in (T_1^{q*}\mathbf{R}^n)_0$ with $j_0^{q-1}\eta = 0$. Hence using the invariancy of A with respect to φ and the assumption $\pi_{q-1}^q \circ (A_{\mathbf{R}^n}(\partial_1)) = 0$ we deduce that

$$\begin{aligned} g^A \circ (T_k^{r*}\varphi^{-1})_0 &= \Phi \circ (A_{\mathbf{R}^n}(\partial_1)) \circ (T_k^{r*}\varphi^{-1})_0 = \Phi \circ (T_1^{q*}\varphi^{-1})_0 \circ (A_{\mathbf{R}^n}(\partial_1)) \\ &= \Phi \circ (A_{\mathbf{R}^n}(\partial_1)) = g^A. \end{aligned}$$

Then, using (4.8), we have

$$f^A(\mu_{m,s};(m,s)\in P)=g^A(j_0^r(\sum_{m=1}^{r}\frac{1}{m!}\mu_{m,0}(x^1)^m+\sum_{s=1}^{s_o}\sum_{m=0}^{r-s}\frac{1}{m!s!}\mu_{m,s}(x^1)^m(x^2)^s))$$

$$=g^A(j_0^r(\sum_{m=1}^{r}\frac{1}{m!}\mu_{m,0}(x^1)^m+\sum_{s=1}^{s_o}\sum_{m=0}^{r-s}\frac{1}{m!s!}\mu_{m,s}(x^1)^m(x^2)^s$$
$$+\sum_{m=0}^{r-s_o}\frac{1}{m!s_o!}\mu_{m,s_o}(x^1)^m x^3))$$

$$=g^A(j_0^r((\sum_{m=1}^{r}\frac{1}{m!}\mu_{m,0}(x^1)^m+\sum_{s=1}^{s_o-1}\sum_{m=0}^{r-s}\frac{1}{m!s!}\mu_{m,s}(x^1)^m(x^2)^s+$$
$$\sum_{m=0}^{r-s_o}\frac{1}{m!s_o!}\mu_{m,s_o}(x^1)^m x^3)\circ\varphi))$$

$$=g^A(T_k^{r*}\varphi^{-1}(j_0^r(\sum_{m=1}^{r}\frac{\mu_{m,0}}{m!}(x^1)^m+\sum_{s=1}^{s_o-1}\sum_{m=0}^{r-s}\frac{1}{m!s!}\mu_{m,s}(x^1)^m(x^2)^s$$
$$+\sum_{m=0}^{r-s_o}\frac{1}{m!s_o!}\mu_{m,s_o}(x^1)^m x^3)))$$

$$=g^A(j_0^r(\sum_{m=1}^{r}\frac{1}{m!}\mu_{m,0}(x^1)^m+\sum_{s=1}^{s_o-1}\sum_{m=0}^{r-s}\frac{1}{m!s!}\mu_{m,s}(x^1)^m(x^2)^s$$
$$+\sum_{m=0}^{r-s_o}\frac{1}{m!s_o!}\mu_{m,s_o}(x^1)^m x^3))$$

$$=g^A(j_0^r(\sum_{m=1}^{r}\frac{1}{m!}\mu_{m,0}(x^1)^m+\sum_{s=1}^{s_o-1}\sum_{m=0}^{r-s}\frac{1}{m!s!}\mu_{m,s}(x^1)^m(x^2)^s)),$$

i.e. f^A is independent of μ_{m,s_o} for all $m=0,...,r-s_o$. This is a contradiction. Thus f^A is independent on $\mu_{m,s}$ for all $(m,s)\in P$ with $s\geq 2$. Hence

$$\begin{aligned}
g^A(j_0^r\gamma)&=g^A(j_0^r(\gamma\circ(x^1,x^2,0,...,0)))\\
&=g^A(j_0^r(\sum_{(m,s)\in P}\frac{1}{m!s!}(\partial_1)^m(\partial_2)^s\gamma(0)(x^1)^m(x^2)^s))\\
&=f^A((\partial_1)^m(\partial_2)^s\gamma(0);\ (m,s)\in P)\\
&=g^A(j_0^r(\sum_{m=1}^{r}\frac{1}{m!}(\partial_1)^m\gamma(0)(x^1)^m+\sum_{m=0}^{r-1}\frac{1}{m!}(\partial_1)^m\partial_2\gamma(0)(x^1)^m x^2))\\
&=h^A((\partial_1)^1\gamma(0),...,(\partial_1)^r\gamma(0),\ (\partial_1)^0\partial_2\gamma(0),...,(\partial_1)^{r-1}\partial_2\gamma(0))\\
&=h^B((\partial_1)^1\gamma(0),...,(\partial_1)^r\gamma(0),\ (\partial_1)^0\partial_2\gamma(0),...,(\partial_1)^{r-1}\partial_2\gamma(0))=g^B(j_0^r\gamma)
\end{aligned}$$

for any $j_0^r\gamma \in (T_k^{r*}\mathbf{R}^n)_0$. Therefore $A = B$ because of Lemma 4.3. □

Proof of Proposition 4.1. Let $A \in \mathcal{T}^0(r,k,q,n)$. Using the invariancy of A with respect to the homotheties $c_t = (x^1, tx^2, x^3, ..., x^n) : \mathbf{R}^n \to \mathbf{R}^n$, $t \neq 0$, (preserving ∂_1) we obtain

$$h^A(\nu, t\mu) = t^q h^A(\nu, \mu)$$

for any $\nu = (\nu_1, ..., \nu_r)$, $\mu = (\mu_0, ..., \mu_{r-1}) = (\mu_{0,1}, ..., \mu_{0,k}, ..., \mu_{r-1,1}, ..., \mu_{r-1,k}) \in (\mathbf{R}^k)^r$ and $t \in \mathbf{R} - \{0\}$, where h^A is given by (4.5). Therefore, by the homogeneous function theorem, cf.[6], h^A is a linear combination of monomials in $\mu_{0,1}, ..., \mu_{0,k}$, $..., \mu_{r-1,1}, ..., \mu_{r-1,k}$ of degree q with coefficients being C^∞ maps depending on ν.

On the other hand, if $B = A^{\alpha;r,k,q}$, where $\alpha = ((\alpha_1, \overline{\alpha}_1), ..., (\alpha_q, \overline{\alpha}_q)) \in Q_{k,q}^{r,q}$, then $h^B(\nu, \mu) = \mu_{\alpha_1,\overline{\alpha}_1} ... \mu_{\alpha_q,\overline{\alpha}_q}$ for all $\nu = (\nu_1, ..., \nu_r)$, $\mu = (\mu_0, ..., \mu_{r-1})$ $= (\mu_{0,1}, ..., \mu_{0,k}, ..., \mu_{r-1,1}, ..., \mu_{r-1,k}) \in (\mathbf{R}^k)^r$. It follows from the fact that $h^B(\nu, \mu)$ is the coefficient corresponding to $(x^2)^q$ of the polynomial (in x^1, x^2)

$$\prod_{j=1}^{q} [(\partial_1)^{\alpha_j} (\sum_{m=1}^{r} \frac{1}{m!} \nu_{m,\overline{\alpha}_j} (x^1)^m + \sum_{m=0}^{r-1} \frac{1}{m!} \mu_{m,\overline{\alpha}_j} (x^1)^m x^2)$$
$$- (\partial_1)^{\alpha_j} (\sum_{m=1}^{r} \frac{1}{m!} \nu_{m,\overline{\alpha}_j} (x^1)^m + \sum_{m=0}^{r-1} \frac{1}{m!} \mu_{m,\overline{\alpha}_j} (x^1)^m x^2)(0)]$$

(which is of the form $\prod_{j=1}^q (\mu_{\alpha_j,\overline{\alpha}_j} x^2 + x^1 w_j(x^1, x^2))$, where w_j are polynomials), see (1.2).

Now applying Lemma 4.2 we end the proof. □

5. Proof of the main result. We will prove Theorem 3.2 by induction with respect to q. The first step is Proposition 6.1 for $q = 1$. Now, we assume that Theorem 3.2 is true for q and for all r, k. We prove the theorem for $q + 1$ and all r, k.

It follows from (1.1) that

(5.1) $$Q_{k,q+1}^{r+1} = Q_{k,q}^r \cup Q_{k,q+1}^{r+1,q+1},\ Q_{k,q+1}^r - Q_{k,q+1}^{r,q+1} \subset Q_{k,q}^r \text{ and } Q_{k,q}^r \subset Q_{k,q}^{r+1}.$$

By the inductive assumption $A^{\alpha;r,k,q}$, for $\alpha \in Q_{k,q}^r$, are $C^\infty((\mathbf{R}^k)^r)$-linearly independent. By Proposition 4.1, $A^{\alpha;r,k,q+1}$, for $\alpha \in Q_{k,q+1}^{r,q+1}$, are also $C^\infty((\mathbf{R}^k)^r)$-linearly independent. We see that $\pi_q^{q+1} \circ A^{\alpha;r,k,q+1} = 0$ if $\alpha \in Q_{k,q+1}^{r,q+1}$, and $= A^{\alpha;r,k,q}$ if $\alpha \in Q_{k,q+1}^r - Q_{k,q+1}^{r,q+1}$. Moreover, the function (4.1) is a $C^\infty((\mathbf{R}^k)^r)$ -module homomorphism. Therefore, $A^{\alpha;r,k,q+1}$, for $\alpha \in Q_{k,q+1}^r$, are $C^\infty((\mathbf{R}^k)^r)$-linearly independent.

Now we prove that $A^{\alpha;r,k,q+1}$, for $\alpha \in Q^r_{k,q+1}$, generate the $C^\infty((\mathbf{R}^k)^r)$-module $\mathcal{T}(r,k,q+1,n)$. Let us consider an arbitrary natural operator $A \in \mathcal{T}(r,k,q+1,n)$. By the inductive assumption for any $\alpha \in Q^r_{k,q}$ there is $f_\alpha \in C^\infty((\mathbf{R}^k)^r)$ such that

$$\pi^{q+1}_q \underline{\circ} A = \sum_{\alpha \in Q^r_{k,q}} f_\alpha A^{\alpha;r,k,q}.$$

We see that

$$\pi^{q+1}_q \underline{\circ}(A \underline{\circ} \pi^{r+1}_r - \sum_{\alpha \in Q^r_{k,q}} (f_\alpha \circ p_r) A^{\alpha;r+1,k,q+1}) = 0, \tag{5.2}$$

where $p_r : (\mathbf{R}^k)^{r+1} = (\mathbf{R}^k)^r \times \mathbf{R}^k \to (\mathbf{R}^k)^r$ is the obvious projection, because $\pi^{q+1}_q \underline{\circ} A^{\alpha;r+1,k,q+1} = A^{\alpha;r+1,k,q} = A^{\alpha;r,k,q} \underline{\circ} \pi^{r+1}_r$ for all $\alpha \in Q^r_{k,q}$. Then it follows from Proposition 4.1 (with $q+1$ and $r+1$ instead of q and r) and from (5.1) and (5.2) that

$$A \underline{\circ} \pi^{r+1}_r = \sum_{\alpha \in Q^{r+1}_{k,q+1}} g_\alpha A^{\alpha;r+1,k,q+1} \tag{5.3}$$

for some maps $g_\alpha \in C^\infty((\mathbf{R}^k)^{r+1})$.

For any $B \in \mathcal{T}(r+1,k,q+1,n)$ and $s \in \{1,\dots,q+1\}$ with $s-1+r-q \geq 0$ define $H^{B,s} : (\mathbf{R}^k)^{r+1} \times (\mathbf{R}^k)^{r+1} \times \mathbf{R}^k \to \mathbf{R}$ by

$$H^{B,s}(\nu,\mu,\lambda) = (\Psi_s \circ (B_{\mathbf{R}^n}(\partial_1)))(j^{r+1}_0(\gamma_{\nu,\mu,\lambda,s})),$$

where $\nu = (\nu_1,\dots,\nu_{r+1})$, $\mu = (\mu_0,\dots,\mu_r) \in (\mathbf{R}^k)^{r+1}$, $\lambda \in \mathbf{R}^k$,

$$\gamma_{\nu,\mu,\lambda,s} = \sum_{m=1}^{r+1} \frac{1}{m!} \nu_m (x^1)^m + \sum_{m=0}^{r} \frac{1}{m!} \mu_m (x^1)^m x^3 + \frac{1}{(s-1+r-q)!} \lambda (x^1)^{s-1+r-q} (x^2)^{q-s+2},$$

and $\Psi_s : (T^{q+1*}_1 \mathbf{R}^n)_0 \to \mathbf{R}$ is given by

$$\Psi_s(j^{q+1}_0 \eta) = \frac{1}{(q-s+2)!(s-1)!} (\partial_2)^{q-s+2} (\partial_3)^{s-1} \eta(0).$$

By the invariancy of B with respect to the homotheties $c_t = (x^1, tx^2, x^3, \dots, x^n)$, $t \neq 0$, we deduce that $H^{B,s}(\nu,\mu,0) = t^{q-s+2} H^{B,s}(\nu,\mu,0)$, i.e. $H^{B,s}(\nu,\mu,0) = 0$ for any $\nu,\mu \in (\mathbf{R}^k)^{r+1}$. In particular,

$$H^{A \underline{\circ} \pi^{r+1}_r, s}(\nu,\mu,0) = 0. \tag{5.4}$$

If $B = A^{\alpha;r+1,k,q+1}$, where $\alpha = ((\alpha_1, \overline{\alpha}_1), ..., (\alpha_a, \overline{\alpha}_a)) \in Q^{r+1}_{k,q+1}$, and $s \in \{1, ..., q+1\}$ with $s-1+r-q \geq 0$, then $H^{B,s}(\nu, \mu, \lambda)$ is the coefficient corresponding to $(x^2)^{q-s+2}(x^3)^{s-1}$ of the polynomial

$$\prod_{j=1}^{a}((\partial_1)^{\alpha_j}\gamma^{\overline{\alpha}_j}_{\nu,\mu,\lambda,s} - (\partial_1)^{\alpha_j}\gamma^{\overline{\alpha}_j}_{\nu,\mu,\lambda,s}(0))$$

in x^1, x^2, x^3; or (equivalently) it is the coefficient corresponding to $(x^2)^{q-s+2}(x^3)^{s-1}$ of the polynomial $\prod_{j=1}^{a}(\partial_1)^{\alpha_j}\gamma^{\overline{\alpha}_j}_{0,\mu,\lambda,s}$, where $\gamma_{\nu,\mu,\lambda,s} = (\gamma^1_{\nu,\mu,\lambda,s}, ..., \gamma^k_{\nu,\mu,\lambda,s})$. Hence, if $\alpha \in Q^{r+1,s}_{k,q+1} - Q^{r,s}_{k,q+1}$, then $a = s$ and $\alpha_s = s-1+r-q$, and then $H^{B,s}(\nu, \mu, \lambda)$ $= \lambda_{\overline{\alpha}_s}\mu_{\alpha_1,\overline{\alpha}_1}...\mu_{\alpha_{s-1},\overline{\alpha}_{s-1}}$ for any $\nu, \mu = (\mu_{0,1}, ..., \mu_{0,k}, ..., \mu_{r-1,1}, ..., \mu_{r-1,k}) \in (\mathbf{R}^k)^{r+1}$ and $\lambda = (\lambda_1, ..., \lambda_k) \in \mathbf{R}^k$. If $\alpha \in Q^{r,s}_{k,q+1}$, then $a = s$ and $\alpha_s < s-1+r-q$, and then $H^{B,s}(\nu, \mu, \lambda) = 0$ for any $\nu, \mu \in (\mathbf{R}^k)^{r+1}$ and $\lambda \in \mathbf{R}^k$. If $\alpha \in Q^{r+1}_{k,q+1} - Q^{r+1,s}_{k,q+1}$, then $H^{B,s}(\nu, \mu, \lambda)$ is a polynomial in $\lambda_1, ..., \lambda_k, \mu_{0,1}, ..., \mu_{0,k}, ..., \mu_{r,1}, ..., \mu_{r,k}$ each term of which is a monomial of degree $a \neq s$.

Then using (5.3), (5.4) and the equality $j^r_0(\gamma_{\nu,\mu,\lambda,s}) = j^r_0(\gamma_{\nu,\mu,0,s})$ we have

$$\begin{aligned} 0 = H^{A\circ\pi^{r+1}_r,s}(\nu, \mu, 0) &= H^{A\circ\pi^{r+1}_r,s}(\nu, \mu, \lambda) \\ &= \sum_{\alpha\in Q^{r+1,s}_{k,q+1} - Q^{r,s}_{k,q+1}} g_\alpha(\nu)\lambda_{\overline{\alpha}_s}\mu_{\alpha_1,\overline{\alpha}_1}...\mu_{\alpha_{s-1},\overline{\alpha}_{s-1}} + ... \end{aligned}$$

for any $s \in \{1, ..., q+1\}$ with $s-1+r-q \geq 0$ and any $\mu = (\mu_{0,1}, ..., \mu_{0,k}, ..., \mu_{r-1,1}, ..., \mu_{r-1,k}), \nu \in (\mathbf{R}^k)^{r+1}$, $\lambda = (\lambda_1, ..., \lambda_k) \in \mathbf{R}^k$, where the dots denote a polynomial in $\mu_{0,1}..., \mu_{0,k}, ..., \mu_{r,1}, ..., \mu_{r,k}, \lambda_1, ..., \lambda_k$ each term of which is a monomial of degree $\neq s$. Then $g_\alpha = 0$ for any $\alpha \in Q^{r+1,s}_{k,q+1} - Q^{r,s}_{k,q+1}$. Hence (by (5.3))

$$\begin{aligned} (A_{\mathbf{R}^n}(\partial_1))(j^r_0\gamma) &= ((A_{\mathbf{R}^n}(\partial_1)) \circ \pi^{r+1}_r)(j^{r+1}_0\gamma) \\ &= ((\sum_{\alpha\in Q^r_{k,q+1}} g_\alpha(.,0)A^{\alpha;r,k,q+1})_{\mathbf{R}^n}(\partial_1))(j^r_0\gamma) \end{aligned}$$

for any $\gamma : \mathbf{R}^n \to \mathbf{R}^k$ with $\gamma(0) = (\partial_1)^{r+1}\gamma(0) = 0$ (i.e. for an arbitrary $j^r_0\gamma \in (T^{r*}_k\mathbf{R}^n)_0$). Therefore $A = \sum_{\alpha\in Q^r_{k,q+1}} g_\alpha(.,0)A^{\alpha;r,k,q+1}$, as well. □

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References

1. Doupovec, M. *Natural operators transforming vector fields to the second order tangent bundle*, Cas. pest. mat. 115 (1990), 64-72.
2. Gancarzewicz, J. *Horizontal lifts of linear connections to the natural vector bundle*, In: Research Notes in Math. 121, pp. 318-334, Boston: Pitman. 1985.
3. Gancarzewicz, J. *Liftings of functions and vector fields to natural bundles*, Warsaw 1983, Dissertationes Mathematicae CCXII.
4. Kobak, P. *Natural liftings of vector fields to tangent bundles of bundles of 1-forms*, Math. Bohemica 116 (3) (1991), 319-326.
5. Kolář, I. *On the natural operators on vector fields*, Annals of Global Anal. and Geom. 6 (1988), 109-117.
6. Kolář, I. Michor, P.W. Slovák, J. *Natural operations in differential geometry*, Springer Verlag, Berlin, 1993.
7. Kolář, I. Slovák, J. *Prolongation of vector fields to jet bundles*, Suppl. Rendiconti Circolo Mat. Palermo, Serie II 22 (1990), 103-111.
8. Mikulski, W.M. *Natural transformations of the covelocities functors into some natural bundles*, Math. Bohemica 118(3)(1993), 277-280.
9. Mikulski, W.M. *Some natural operations on vector fields*, Rend. Math. (Serie VII) Vol. 12, Roma (1992), 783-803 .
10. Mikulski, W.M. *Some natural constructions on vector fields and higher order cotangent bundles*, Mh Math. (to appear).

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SOME INTEGRAL FORMULAS FOR A RIEMANNIAN 3-MANIFOLD EQUIPPED WITH A SYSTEM OF ORTHOGONAL FOLIATIONS.

JACEK ROGOWSKI

1. INTRODUCTION.

All objects in the paper are assumed to be of claas C^∞.

Let M be an orientable Riemannian manifold of dimension 3. Two codimension 1 foliations $\mathcal{F}$ and $\mathcal{G}$ of M are *orthogonal* if and only if for any $p \in M$ the 1-dimensional complements $(T_p\mathcal{F})^\perp$ and $(T_p\mathcal{G})^\perp$ of $T_p\mathcal{F}$ and $T_p\mathcal{G}$ in T_pM are orthogonal. Here $T_p\mathcal{F}$ denotes the tangent space of the leaf of $\mathcal{F}$ through p.

D. Hardorp in [H] has shown that every 3-dimensional compact orientable manifold M admits a system of three mutually transverse foliations $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ of codimension one. Taking any Riemannian metric which makes these foliations pairwise orthogonal and choosing unit vector fields X_1, X_2, X_3 such that X_i is tangent to the 1-dimensional foliation $\mathcal{F}_j \cap \mathcal{F}_k$ we get some Riemannian structure g with three mutually orthogonal foliations. If this construction is made for a Riemannian manifold (M, g'), then in general $g \neq g'$. It leads to the following question: Which closed orientable Riemannian manifolds of dimension 3 admit three mutually orthogonal foliations of codimension 1? The full answer to this problem is not known, anyway to the author. Local version of the question was considered by E. Cartan [C], who proved that for each point of any analytic manifold of dimension 3 there exists a neighbourhood with three mutually orthogonal foliations. This result was extended in 1984 by DeTurck and Yang [DY] to the case of C^∞-manifolds.

In the present paper we prove some integral formulas for a 3-dimensional closed oriented Riemannian manifold equipped with a system of mutually orthogonal foliations of codimension 1 (shortly: SMOF). The first formula (Theorem 1) is some upper estimation for total scalar curvature of the manifold with a SMOF, and the estimation is stronger then this one which follows immediately from formulas obtained by F. G. Brito and P. G. Walczak in [BW]. Two other formulas give some relations between principal curvatures of leaves of foliations in a SMOF. They seem to be useful for further investigations. For example, if M is of constant non-zero sectional curvature, then no foliation with at least one constant pricipal curvature can be raised up to a SMOF (Corollary 2).

This paper is in final form and no version of it will be submitted for publication elsewhere.

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2. Main results.

Let M be a closed oriented Riemannian manifold of dimension 3. In [W], P. Walczak has proved that if $\mathcal{F}$ is a codimension-one foliation of a closed oriented Riemannian manifold M, then

$$\int_M (Ric(N) - 2k_2(\mathcal{F})) = 0, \tag{1}$$

where N is the unit vector field on M orthogonal to $\mathcal{F}$, Ric(N) is the Ricci curvature of M in the direction of N, and $k_2(\mathcal{F})$ is the second mean curvature of (the leaves of) $\mathcal{F}$. The integral is computed with respect to the standard volume form defined by the metric tensor of M. We shall always omit a symbol of this form.

Now, let M be a closed oriented Riemannian manifold of dimension 3. Suppose that $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ is a SMOF of M, and X_1, X_2, X_3 are unit vector fields on M such that $X_i \perp \mathcal{F}_i$, $i = 1,2,3$. Hence $\mathcal{F}_i$ is the integral foliation of the distribution spanned by $\{X_j, X_k\}$, $i \neq j \neq k \neq i$. From (1) we get immediately the following inequality

$$\int_M Ric(X_i) \leq \int_M h_i^2, \quad i = 1,2,3,$$

where h_i is the mean curvature of $\mathcal{F}_i$. Indeed, if $\lambda_j^{(i)}$ denotes the j-th principal curvature of $\mathcal{F}_i$, then $k_2(\mathcal{F}_i) = \lambda_j^{(i)}\lambda_k^{(i)}$ and $h_i = \lambda_j^{(i)} + \lambda_k^{(i)}$ for $i \neq j \neq k \neq i$. Summing up left and right sides of the last inequalities we get

$$\int_M s \leq \int_M \sum_{i=1}^{3} h_i^2,$$

where s is the scalar curvature of M. We shall prove that, in fact, some stronger inequality holds.

Theorem 1. *With the above notation and assumptions*

$$\int_M \sum_{i=1}^{3} h_i^2 \geq 2 \int_M s. \tag{2}$$

Moreover, if the equality holds in (2), *then all foliations $\mathcal{F}_i$ are totally umbilical.*

Proof. Let $\Gamma_{AB}^C = \langle \nabla_{X_C} X_A, X_B \rangle$, where $\langle\ ,\ \rangle$ denotes the metric tensor on M and ∇ is the Levi-Civita connection of M.

In [BW], the authors proved the following formula

$$2\int_M (\Gamma_{13}^1\Gamma_{23}^2 + \Gamma_{12}^1\Gamma_{32}^3 + \Gamma_{21}^2\Gamma_{31}^3) = \int_M (s + \Gamma_{21}^3\Gamma_{31}^2 + \Gamma_{32}^1\Gamma_{12}^3 + \Gamma_{13}^2\Gamma_{23}^1)$$

in the case, when $\{X_i, X_j\}$, $, i \neq j$, span a distribution on M which need not be integrable.

Further, we shall always assume that $i \neq j \neq k \neq i$ is some permutation of $(1,2,3)$.

In our situation, by integrability of $\{X_i, X_j\}$, we get

$$\Gamma_{ij}^k = \Gamma_{ki}^j,$$

what leads to the conclusion that

$$\Gamma_{ij}^k = 0.$$

On the other hand

$$2\Gamma^i_{ik}\Gamma^j_{jk} = h_k^2 - (\Gamma^i_{ik})^2 - (\Gamma^j_{jk})^2,$$

hence

$$\int_M \sum_{i=1}^3 h_i^2 = \int_M \sum_{i=1}^3 k_i^2 + \int_M s,$$

where k_i is the geodesic curvature of X_i. By the main theorem of [BW]

$$\int_M \sum_{i=1}^3 k_i^2 \geq \int_M s,$$

what implies (2).

Finally, equality in (2) holds if and only if

$$\int_M \sum_{i=1}^3 k_i^2 = \int_M s.$$

In this case, by [BW], all $\mathcal{F}_i$ are totally umbilical. □

Corollary 1. *If the sectional curvature c of M is constant, then*

$$\int_M \sum_{i=1}^3 h_i^2 = 2\int_M s,$$

if and only if $c = 0$ and all foliations $\mathcal{F}_i$ have only planar leaves in M.

PROOF. By the assumptions we get

$$\int_M \sum_{i=1}^3 h_i^2 = 6\, c\, vol(M),$$

and all foliations are totally umbilical. It is known that there are no totally umbilical foliations in the closed 3-dimensional Riemannian manifold of constant sectional curvature $c > 0$. It implies that $c = 0$ and $h_i = 0$. In consequence $\Gamma^j_{ji} = -\Gamma^k_{ki}$ and, by umbilicity, $\Gamma^j_{ji} = 0$ for all $i \neq j$. □

Remark. If M is the flat torus $T^3 = S^1 \times S^1 \times S^1$ (endowed with the standard product metric) with three mutually orthogonal foliations $\{t \times S^1 \times S^1\}, \{S^1 \times t \times S^1\}, \{S^1 \times S^1 \times t\}$, then all leaves are planar and equality in Corollary 1 holds.

In a little different way, one can obtain some other integral formulas for systems $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ of mutually orthogonal foliations on a closed oriented 3-dimensional manifold M of constant sectional curvature. In the theorem below we present some of them, which seem to be usefull in further investigations.

Theorem 2. *If M has a constant sectional curvature c, then for any natural number n the following formulas hold.*

$$n\int_M (\lambda_i^{(j)})^{n-1}\lambda_i^{(k)}(\lambda_i^{(j)} - \lambda_k^{(j)}) = \int_M (\lambda_i^{(j)})^n h_k, \tag{3}$$

$$\int_M [(n-1)(\omega_{kk})^n h_k + n(\omega_{kk})^{n-1}\omega] = 0, \tag{4}$$

where ω is the determinant of the matrix

$$\begin{bmatrix} 0 & \lambda_1^{(2)} & \lambda_1^{(3)} \\ \lambda_2^{(1)} & 0 & \lambda_2^{(3)} \\ \lambda_3^{(1)} & \lambda_3^{(2)} & 0 \end{bmatrix}$$

and ω_{kk} are subdeterminants obtained by ignoring the k-th row and k-th column.

PROOF. Let X_1, X_2, X_3 be unit pairwise orthogonal vector fields on M such that for every permutation $i \neq j \neq k \neq i$ of $(1,2,3)$ the fields X_i, X_j span the foliation $\mathcal{F}_k$. Then $\lambda_i^{(j)}$ is the principal curvature of leaves of $\mathcal{F}_j$ along X_i. Hence, Codazzi equations yield:

$$X_j\lambda_k^{(i)} = \lambda_k^{(j)}(\lambda_k^{(i)} - \lambda_j^{(i)}).$$

On the other hand

$$div(fX_k) = X_kf - fh_k$$

for any smooth function f on M. Since M is closed, then

$$\int_M X_kf = \int_M fh_k.$$

This fact and Codazzi equations give the formula (3) for $f = (\lambda_i^{(j)})^n$ and the formula (4) for $f = (\lambda_i^{(j)}\lambda_j^{(i)})^n$. □

Corollary 2. *If M has a constant sectional curvature c and one of the foliations $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ has at least one constant principal curvature, then $c = 0$.*

PROOF. Let $G_k = \lambda_i^{(k)}\lambda_j^{(k)}$ be the Gaussian curvature of $\mathcal{F}_k$. Suppose that $\lambda_2^{(3)} = const.$ Then by Codazzi equations

$$\lambda_2^{(1)}(\lambda_2^{(3)} - \lambda_1^{(3)}) = 0.$$

For $n = 2$, from (3) we get

$$\int_M (\lambda_2^{(1)})^2(\lambda_2^{(3)} - \lambda_1^{(3)}) = \int_M G_2\lambda_2^{(3)}.$$

By our assumption

$$\lambda_2^{(3)} \int_M G_2 = 0.$$

From the Asimov theorem (see [A]) it follows

$$\int_M G_2 = c,$$

so $\lambda_2^{(3)}c = 0$. Hence $c = 0$ or $\lambda_2^{(3)} = 0$. In the second case $G_3 = 0$ and using again the Asimov theorem we get $c = 0$. □

3. Final remarks.

1) If the sectional curvature of M is not constant, then the formulas (3) and (4) take the following form:

$$n\int_M (\lambda_i^{(j)})^{n-1}\lambda_i^{(k)}(R_{kii}^j + \lambda_i^{(j)} - \lambda_k^{(j)}) = \int_M (\lambda_i^{(j)})^n h_k, \tag{5}$$

$$\int_M [(n-1)(\omega_{kk})^n h_k + n(\omega_{kk})^{n-1}\omega - \lambda_j^{(i)} R_{kii}^j - \lambda_i^{(j)} R_{kjj}^i] = 0,$$

where $R_{ABC}^D =< R(X_A, X_B)X_C, X_D >$ and R is the curvature tensor on M.

2) Formulas (3) and (5) remain true if $\dim M = m$ and there is a system of mutually orthogonal foliations of codimension one on M.

3) Except some integral formulas as above, there are not known (anyway to the author) any other global properties of manifolds equipped with a system of mutually orthogonal foliations. In particular, it seems to be interesting to find all closed 3-dimensional Riemannian manifolds of constant curvature, which admit such systems.

References

[A] Asimov D. *Average Gaussian curvature of leaves of foliation*, Bull. A. M. S. **84** (1978), 131-133.

[BW] Brito F. G., Walczak P. G. *Total curvature of orthogonal vector fields on three-manifolds*, Bull. Polish A. S. **35** (1987), 553-556.

[C] Cartan E. *Les systèmes différentiels extérieurs et leurs applications géométriques*, Paris 1945.

[DY] DeTurck D., Yang D. *Existence of elastic deformations with prescribed principal strains and triply orthogonal systems*, Duke Math. Journal **51** (1984), 243-260.

[H] Hardorp D. *All compact orientable three-dimensional manifolds admit total foliations*, Memoirs Amer. Math. Soc. **223** (1980).

[W] Walczak P. G. *An integral formula for a Riemannian manifold with two orthogonal complementary distributions*, Coll. Math. **58** (1990), 243-252.

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