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Twistor theory, self-duality and integrability

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Introduction

Since these lectures were given, a book containing much of the material presented in these lectures has appeared, Mason and Woodhouse (1996). The following contribution will therefore just be a brief summary of the lectures and the reader is referred to the book for a full presentation and the details.

A basic aim of twistor theory is to study correspondences between solutions of physical field equations on space-time (such as the Yang-Mills or Einstein equations) and deformations of holomorphic structures on twistor space. The idea is to reformulate basic physics in terms of holomorphic structures on twistor space. The hope is that this formulation will lead to the theory that unifies quantum theory and gravity in much the same way that classical mechanics (at least with hindsight) can be seen to contain the imprint of quantum mechanics.

So far such correspondences only exist in satisfactory form for the self-dual Einstein and self-dual Yang-Mills equations, Penrose (1976) and Ward (1977) respectively. Indeed these constructions are to a large extent what has motivated the current form of the twistor programme. The twistor programme is still a long way from being realized in this form. However, these constructions have had substantial spin-offs in differential geometry and in the theory of nonlinear integrable or soluble equations. It is this latter spin-off that this article focuses on.

A first definition of an integrable or soluble nonlinear differential equation might be that such equations are those for which explicit procedures are available to find a dense class of solutions, and for which powerful methods exist for the analysis of the general solution. In this sense, the twistor constructions for the self-duality equations provide just such a general solution and to a large extent they provide the underlying reason for the phenomena of integrability. A reason for this arises from the fact, as a series of authors, Ward (1985), Hitchin (1987), Mason and Sparling (1989), Mason and Woodhouse (1993) have observed, that 'most' key integrable systems are symmetry reductions of the self-duality equations. Thus they each inherit a twistor correspondence, the appropriate symmetry reduction of that for the self-duality equations and this provides the theory underlying these equations.

These ideas have led to the following programme:

- Classify integrable systems as symmetry reductions of the self-duality equations (where possible).
- Obtain the theory of such equations from the symmetry reduction of the twistor correspondence for the self-duality equation.

This programme is the subject of the book Mason and Woodhouse (1996), and full details of the ideas sketched below can be found there.

What is an integrable system?

Unfortunately there is not a very systematic definition of integrability. There is no shortage of examples, or indeed definitions, but the equivalence and general validity of the definitions is far from established. The first four items following form part of the standard concept of integrability with the fifth being a definition suggested by the above programme.

1. As above, an integrable system is a system of (partial) differential equations that, in spite of their nonlinearity, are remarkably easy to solve; often there is a dense class of exact solutions and general constructions, such as the inverse scattering transform, for the analysis of the general solution. One can often reduce the construction of the general solution to the solution of linear auxiliary equations. Solutions tend to behave in a very regular way and the effects of the self-interaction of the field is mild (i.e. two lump or soliton solutions might pass over each other with only a phase shift as a memory of the interaction). In particular, if the system exhibits chaotic or ergodic behaviour it is usually taken not to be integrable. This definition is often what one means in practice despite its vagueness.

2. An integrable system is a Hamiltonian system in $2n$-dimensions on a symplectic manifold M^{2n} with Hamiltonian H and n explicitly given constants $H = H_1, \dots H_n$, of the motion in involution, $\{H_i, H_j\} = 0$ where $\{,\}$ is the Poisson bracket. For a general differential equation in $2n$ dimensions, one needs $2n-1$ constants of the motion to solve it by quadratures. However in symplectic geometry, constants of the motion in involution have a secondary role as symmetries of the original system, so one only needs half as many. This is formalized in Arnold's version of Liouville's theorem:

   Theorem 1 *Suppose that we are given a Hamiltonian system with n constants of the motion in involution as above, and suppose that the map* $H_i : M^{2n} \mapsto \mathbb{R}^n$

is regular and proper at $c_i \in \mathbb{R}^n$*, then the submanifold* $H_i = c_i$ *is a Lagrangian torus with a canonical linear structure and there is a coordinate transformation to 'action-angle' variables* (θ_i, I^i) *by quadratures such that (1) the coordinates are canonical,* $\{I^i, \theta_j\} = \delta^i_j$ *with all other Poisson brackets vanishing and* θ_i *are linear angle coordinates on the tori of constant* H_i *and (2) the flows are given by* $\dot{I}^i = 0$ *and* $\dot{\theta}_i$ *are constant.*

Thus the system can be completely solved by quadratures. This is one of the most attractive definitions, but is awkward to apply to partial differential equations as it requires a full phase space formulation. This brings the boundary conditions into the concept of integrability which seems unnatural, and furthermore makes it rather awkward to apply the idea to elliptic equations. Remarkably, however, this definition can be made precise in the infinite dimensional context of integrable nonlinear parabolic and hyperbolic partial differential equations.

3. An integrable system is a system of equations that admits a 'Lax pair'. That is, the equation can be expressed as the integrability condition for an auxiliary system of over-determined linear differential equations. This is most easily illustrated by means of the famous example of the Korteweg de Vries equation below.

   The existence of a Lax pair is not a sufficient condition for integrability. For example, the full Einstein equation admits a linear system representation in the form of the Rarita-Schwinger equations. A criterion that fails in this example but that is fulfilled for all integrable systems that I am familiar with is that the linear system should propagate data for its solutions across an initial data surface for the full nonlinear integrable equation. Thus, it should be so over-determined that its data surfaces have codimension-2 rather than 1.

4. An integrable system is a system of equations that satisfies the Painlevé property. This property in effect states that all non-characteristic singularities in the complex that are movable (in the sense that their location depends on the initial conditions) should be forced by the equations to be rational. This leads to what has become known as the Painlevé test for integrability. One form is to substitute in a power series in the time variable t into the equations to obtain recurrence relations on the coefficients. When the equation is integrable, the recurrence relations force the powers of t in the series to be integral and coefficients of terms that might lead to log terms in the expansion should also be forced to vanish. This test is very powerful in practice, but the theoretical underpinnings are still obscure.

5. My preferred definition of an integrable system is that it is a differential equation that admits a twistor correspondence. This raises the question of the definition of a twistor correspondence. If the definition is sufficiently loose one can certainly incorporate all integrable systems since, by the solubility criteria, an integrable equation must have some associated theory that one can exploit to solve the equations. What is remarkable is the wide applicability of symmetry reductions of the standard twistor correspondences. Indeed, these apply to the vast majority

of key examples of integrable systems. There are two families that require an extension of the standard correspondence, one based on the Landau-Lifschitz equation and the other based on the KP equations. The relevant extensions for the Landau-Lifschitz equations are discussed in Carey, Eastwood, Mason and Singer (1994) and the extensions for the KP and Davey Stewartson equations are discussed in §12.6 of Mason and Woodhouse (1996) and Mason (1995).

Part I: Classification

Because of the difficulties in the basic definition of integrability many presentations proceed by listing some of the key examples. In the following I list some of the more famous integrable systems and how they can be obtained as reductions of the self-dual Yang-Mills equations.

The first part of the above mentioned programme is to classify integrable systems as reductions of the self-duality equations. The classification proceeds by listing the ingredients required in the reduction. In the case of the self-dual Yang-Mills equations these are (a) a choice of symmetry subgroup of the conformal group (the symmetry group of the self-dual Yang-Mills equations) and action of the symmetry on the bundle (b) a choice of gauge group, (c) a choice of gauge and (d) a choice of certain constants of integration that arise when some of the reduced equations can be integrated directly.

Dimension 1: Ordinary differential equations.

These arise when one imposes a symmetry on the self-dual Yang-Mills equations with three-dimensional orbits. The reductions one obtains split roughly into two types according to whether one uses translational symmetries or more general conformal symmetries. In the first case, one obtains equations that are autonomous (i.e. invariant under time translation) that are generalizations of the spinning top equations and in the second one obtains time dependent equations that generalize the Painlevé equations. See chapter 7 of Mason & Woodhouse (1996).

1. Spinning tops. These equations describe the motion of a rigid body spinning about some fixed point. The most celebrated example is that of the Euler top which is fixed at its centre of mass. In line with the Hamiltonian definition of integrability, the phase space is the cotangent bundle of SO(3). For integrability we require three constants of motion. One always has the total energy and the total angular momentum about the point of suspension, so we only require one 'unexpected' constant. In the case of the Euler top the angular momentum along the z-axis in space is the third constant.

   For the Euler top, the equations for the components of the angular velocity relative to the body, $(\omega_1, \omega_2, \omega_3)$ decouple from those for the configuration variables and we obtain
   $$\dot{\omega}_1 = (\lambda_2 - \lambda_3)\omega_2\omega_3$$
   and its cyclic permutations where λ_i are the reciprocal moments of inertia. The total energy and the total angular momentum are quadratics in the ω_i. Thus, if

we complexify and compactify $\mathbb{R}^3$ to $\mathbb{CP}^3$, we see that the flow is restricted to the intersection* of two quadrics, an elliptic curve. Furthermore, the field equations separate to give

$$dt = \frac{d\omega_1}{(\lambda_2 - \lambda_3)\omega_2\omega_3}$$

and cyclic permutations and this defines a global non-vanishing 1-form on the elliptic curve. Thus the solutions are given by elliptic functions.

The other integrable tops are the Lagrange and Kovalevskaya tops, the latter found by what has subsequently become known as Painlevé analysis. Again the general solution can be obtained in terms of (hyper)-elliptic functions.

These arise from the self-dual Yang-Mills equations by imposing three translational symmetries tangent to non-null hyper-planes. The Euler top arises from the simplest gauge group, $SO(3)$, the Lagrange top from $SO(3,1)$ after an additional $\mathbb{Z}_2$ symmetry has been imposed and the Kovalevskaya top from $SO(3,2)$, again after a $\mathbb{Z}_2$ symmetry has been imposed. The Euler equations can also be obtained from 3 translational symmetries tangent to a null hyper-plane also and this reduction generalizes to the integrable case of rigid rotations of an n-dimensional top about its centre of mass when a larger gauge group is chosen.

2. The Painlevé equations. Painlevé classified all the second order ordinary differential equations rational in the dependent variable and its first derivative. Of the 50 or so equations in the classification, six new equations required new transcendental functions for their solution (the others being soluble with known functions).

   To obtain these from the self-dual Yang-Mills equations, one imposes symmetry under an abelian three dimensional symmetry group that is non-degenerate in an appropriate sense. The different choices of symmetry group classify the different Painlevé equations except for the first and second which are distinguished by a choice of constant of integration. See Mason & Woodhouse (1996) chapter 7 for full details.

Dimension 2.

The most celebrated integrable systems are those in 2-dimensions. Those that are autonomous (i.e. admit a full translation symmetry group) are obtained from the self-dual Yang-Mills equations by symmetry reduction by two-dimensional translational symmetry groups. Such symmetries can be classified by the rank (and signature) of the metric restricted to the two-plane spanned by the translations.

If the two-plane is non-degenerate, one obtains various forms of the harmonic map, or wave map equation with values in the gauge group or complexified gauge group divided by the real gauge group. With further discrete symmetries, one can obtain harmonic maps into Riemannian symmetric spaces (when the discrete symmetry is $\mathbb{Z}_2$) or the periodic Toda lattice (when the gauge group is $SL(n)$ and the discrete symmetry is $\mathbb{Z}_n$ acting appropriately on the bundle).

If the two-plane is null but with non-vanishing metric, one obtains equations that are Galilean invariant. For gauge group contained in $SL(2,\mathbb{C})$ we obtain the key examples of the nonlinear Schrodinger equation and the Korteweg de Vries equation.

This latter equation describes the evolution with respect to time t of water waves with height $u(x,t)$ in a shallow channel parametrized by x. The equation arises from the compatibility conditions between the operators

$$L = \partial_x^2 + u + \lambda\,, \qquad M = \partial_t - \partial_x^3 - \frac{3}{2}u\partial_x - \frac{3}{4}u_x.$$

We have

$$[L, M] = 0 \quad \Longleftrightarrow \quad 4u_t = u_{xxx} + 6uu_x,$$

the Korteweg de Vries equation.

For larger gauge group one obtains various generalizations including parts of the Drinfeld-Sokolov and Zakharov-Shabat hierarchies.

When the translation symmetry group is tangent to totally null anti-self-dual two-planes, one obtains systems such as the Wess-Zumino-Witten equations, the n-wave equations and various other parts of the Drinfeld-Sokolov and Zakharov-Shabat hierarchies.

Full details of these reductions, and others, are discussed in chapter 6 of Mason and Woodhouse 1996. Chapter 5 contains reductions by a single symmetry, which yields various forms of the Bogomolny equations for magnetic monopoles and degenerate analogues of the Davey Stewartson and KP equations. Chapter 13 studies reductions of the various self-duality equations on a conformal structure. These also yield many familiar examples of integrable systems, although the procedure there is restricted to giving systems with gauge group contained in $\mathrm{SL}(2,\mathbb{C})$.

Part II: theory of integrable equations

The direct relationship with the self-duality equations allows one to unify the theory of integrable equations. One can establish a body of theory for the self-duality equations and then check that the results and structures survive symmetry reduction to yield the analogous theory for the reduced equation. There is a large body of theory and many techniques to be understood in this way. For example, one characterization of integrability in the Hamiltonian context is the existence of a recursion operator satisfying appropriate conditions. This can be established for the self-dual Yang-Mills equations and it can be checked that the standard recursion operators for its reductions descend from it. See chapter 8 of Mason and Woodhouse (1996).

The core of the theory of integrable systems comes down to the various different methods for representing general solutions to the equations. The twistor construction for the self-duality equations is the most general such method for those equations, giving a representation for the general local solution. Each symmetry reduction of the self-duality equations inherits a reduced twistor correspondence which will also yield the general local solution. Global methods, such as the ADHM construction or, in the context of reductions, the inverse scattering transform, are obtained by globalizing the twistor construction. This often leads to useful simplifications of the twistor construction. For example, in the case of the inverse scattering transform, it is possible to obtain a natural gauge fixing of the Ward transform using the boundary conditions. This removes the gauge freedom which can be awkward when trying to get a good presentation of the solution space. These topics are discussed in chapters 10 and 11 of Mason and Woodhouse.

Conclusions

The tone of this piece has perhaps been somewhat territorial, suggesting that integrability is perhaps a subfield of the theory of the self-duality equations and their associated twistor theory. However, this is far from the case. The theory underlying integrable systems is much too multifaceted for it to be limited in this way. In the long term, the most important pay-offs of these ideas will probably result from the feedback into twistor theory of ideas from the theory of integrable systems. There are a number of areas where twistor theory can be extended and improved using techniques from the theory of integrable systems. One example arises from the inverse scattering transform alluded to above. An appropriate twistor formulation leads to a new family of twistor correspondences that has application to an interesting family of questions arising from twistor theory in split signature. These applications were presented at the 1996 Srni Winter School.

Another area where the theory of integrable systems goes beyond conventional twistor theory is in the theory of the KP equations. This has lead to a new extension of the standard twistor construction in which one, in effect, replaces the $\bar{\partial}$-operator with a Dirac operator, see Mason (1995) and chapter 12.6 of Mason and Woodhouse (1996). This should have application to many other interesting problems.

However, the most important problem in twistor theory itself at the moment is that of finding a twistor construction for the full Einstein vacuum equations. These do have a linear system in the form of the Rarita-Schwinger equations and indeed Penrose has proposed that these might provide an appropriate vehicle for the definition of twistors in vacuum space-time, Penrose (1990), Mason and Penrose (1994). The Einstein vacuum equations are certainly not integrable, but nevertheless, the Rarita-Schwinger equations do have a scattering theory. Perhaps it will be possible to set up an inverse scattering theory along the lines of that for integrable sytems (although perhaps with a nonlinear inverse scattering transform so that it no longer implies integrability of the equations). The connection with twistors that Penrose points out might well then amount to the needed twistor construction for vacuum space-times.

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