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In: Jan Slovák (ed.): Proceedings of the 15th Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1996. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 43. pp. [85]–94.

Persistent URL: http://dml.cz/dmlcz/701578

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 43 (1996), pp. 85-94

# A high-order helicity invariant and the Rokhlin Theorem*

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**Abstract**

A high-order invariant for links in 3-dimensional space, called the Rokhlin invariant, is introduced. The invariant gives an integer modulo 2 for an arbitrary multicomponents link. The invariant is an extension modulo 2 of the Sato-Levine invariant of semi-boundary links.

## 1 Introduction

The classification problem of links in 3-dimensional space arises in different branches of natural sciences [9] [20]. Physical properties of magnetic fields in conductive medium can be investigated by topological methods. The spatial distribution of magnetic components of fields is given by vector $\vec{B}(\vec{r}), div(\vec{B}(\vec{r})) = 0$. The main topological characteristic of fields is called the helicity invariant. The simplest integral form of this invariant is the Gauss integral. The helicity invariant is given by the following expression

$$H = \int_{R^3} \vec{A}\vec{B} \quad dR^3, \tag{1}$$

where $rot\vec{A} = \vec{B}; \vec{A}|_\infty = \vec{B}|_\infty = 0$.

The freezing-in theorem holds true in the course of the motion of plasma with infinite conductivity [13]. In this case the following conception of magnetic tubes is natural. The union of closed oriented curves is surrounded by thin toroadal volume $U(i)$. Each volume $U(i)$ contains magnetic (divergence-free) fields $B(i)$ . Under this assumption the expression (1) can be presented in the form:

$$H = \sum_{i<j} \Phi_i \Phi_j Lk(i,j) + \sum_i (\Phi_i)^2 Lk(i,i), \tag{2}$$

*This paper is in final form and no version of it will be submitted for publication elswhere.

where $\Phi_i$ is the flow of the vector $\vec{B}(i)$ in the tube $U(i)$; $Lk(i,j)$ is the pair-wise linking number, $Lk(i,i)$ is the self-linking number [7], [14].

High-order topological invariants in Dynamo theory had been studied in [4], [6], [8], [16], [17]. Moffatt conjectures that every non-trivial high-order invariant of magnetic fields gives a lower boundary of the magnetic energy [15]. For the 2-order helicity invariant (1) this conjecture had been proved by M.Freedman [10]. The investigation of 4-order invariant, called the Robertello invariant, is related to a fluctuation of helicity under the reconnection of magnetic lines [3].

The generalization of (1) for high-order invariants of magnetic fields is not found. Note, that a number of magnetic tubes is sufficiently large, therefore we are interested in invariants of multi-components links.

We describe a new topological invariant of links, called the Rokhlin invariant. An application of classical invariants of links, such as Massey triple product and Sato-Lewine invariants is considered in [4], [8], [16]. Unfortunately, all the invariants are defined under strong conditions on linking numbers between tubes. More exactly, to define a high-order invariant an assumption $Lk(i,j) = 0$ for every pair $(i,j)$ of tubes is required. Such link is called semi-boundary link. Of course, configurations of magnetic tubs, presented by semi-boundary links, should be investigated, but the condition is not natural.

The Rokhlin invariant is a generalization of Sato-Lewine invariant. To construct the Rokhlin invariant we use the approach, proposed by V.I.Arnold in [5], and the conseption of Prem-maps, proposed by A.Szucs in [18], [19]. The generalisation is based on the Rokhlin theorem about signature of 4-dimension manifolds. The Sato-Lewine invariant, denoted below by $w$, is defined only if a link is semi-boundary. The Rokhlin invariant, denoted below by $W$, is defined without any assumption on linking numbers of components. On the other hand, the Sato-Levine invariant gives for an arbitrary semi-boundary link $L$ an integer number $w(L)$. The Rokhlin invariant gives for an arbitrary link $M$ an integer $W(M)(mod 2)$. If $M$ is a semi-boundary link, we have $W(M) = w(M)(mod 2)$. An analytical expression of $W$ is given by an integral with values $\{+1,-1\}$ on the circle $S^1$ .

Finally, note that the Casson invariant is direct generalization of the Rokhlin $\mu$ -invariant for homological 3-spheres. Casson invariant takes integer values, and we conjecture that $W$ can be defined as an integer.

## 2 Construction of the Rokhlin invariant

Let $L = \bigcup_{i=1}^{n} S^1(i), 1 \leq i \leq n$ be a finite union of circles, $\varphi : L \to R^2$ be a map ( immersion ) in general position. We denote a set of double points of the immersed curve $\varphi(L)$, by $\Delta_2(\varphi) \subset R^2$.

$$\bigcup_{i,j} \Delta(i,j) = \Delta_2(\varphi), \quad \Delta(i,j) = \varphi(S^1(i)) \cap \varphi(S^1(j)).$$

We consider a function $o : \Delta_2(\varphi) \to Z/_2 = \{+1; -1\}$.

Let M be an oriented close surface, $\Phi : M^2 \to R^3$ be a map in general position with finite number of singular points. We denote by $\Delta_2(\Phi)$ a curve of double self-intersection points of $\Phi(M^2)$ , by $\Delta_3(\Phi)$ the set of triple self-intersection points of $\Phi(M^2)$, $\Delta_3(\Phi) \subset \Delta_2(\Phi)$. We denote by $O$ an orientation on the curve $\Delta_2(\Phi)$ .

For an arbitrary pair $(\Phi, O)$, we divide the points $\Delta_3(\Phi)$ into two types, denoted below by A and B. Let $L_1, L_2, L_3$ be three sheets of the surface $\phi(M^2)$ in a small neighborhood of a triple point $x \in \Delta_3(\Phi)$ . We choose an order of the branches of the curve $\Delta_2(\phi)$, such that $O_i \perp L_i$ . For each i, we define an integer $sign(i) = +1$, if $O_i$ is a positive normal vector to $L_i$ and $sign(i) = -1$ otherwise.

### 2.1 Definition

A point $x \in \Delta_3(\Phi)$ is a point of the type A, if the integers $sign(i)$, $i = \{1,2,3\}$ form the following two combinations $(+,+,-)$ or $(+,-,-)$. A point $x$ is a point of the type B, if the integers form a combination $(+,+,+)$ ( the variety $B,+$) or a combination $(-,-,-)$ ( the variety $B,-$).

Let us assume that $M^2 = \bigcup_j M^2(j)$ is decomposed on components. A point $x \in \Delta_3(\Phi)$ of the type B is called point of subtype B2, if two sheets $L_1, L_2$ of the surface $\Phi(M^2)$ are at the same component of $M$, $L_1, L_2 \subset M^2(j)$ and the last sheet $L_3$ is at the component $M^2(j), j \neq k$ . If three sheets $L_1, L_2, L_3$ are at the same component $M^2(j)$, the point x is called point of of subtype B1.

We define two cobordism relations on the set of pairs $(\varphi, o)$.

## 2.2 Definition

Let $(\varphi_0, \leq_0); (\varphi_1, \leq_1)$ be two pairs. The pairs are AB-cobordant, $(\varphi_0, o_0) \sim_{AB} (\varphi_1, o_1)$, if there exist a homotopy $\Phi : L \times I \to R^2 \times I$ in general position and an orientation $O$ on the curve $\Delta_2(\Phi)$ under the following condition

$$\Phi \mid_{L\times\{t\}} = \varphi_t; \quad O \mid_{L\times\{t\}} = -o_t, \quad t \in \{0; 1\}.$$

## 2.3 Definition

Pairs $(\varphi_0; o_0), (\varphi_1; o_1)$ are called A-cobordant, $(\varphi_0, o_0) \sim_A (\varphi_1, o_1)$, if there exists a pair $(\Phi; O)$ that gives AB-cobordism between the pairs, and the set $\Delta_3(\Phi)$ contains no points of the type B2.

## 2.4 Denotion

We denote by $ImmAB(n)$, $(ImmA(n))$ the set of AB- ( A-) cobordism classes of n-components pairs.

Note that the symmetric group $\Sigma(n)$ acts on the sets by renumeration of components of immersed curve.

We describe the relation between the constructed cobordism sets and links in three-dimensional space. Let $\bar{\varphi} : L \to R^3$ be n-component link, $\pi : R^3 \to R^2$ be the standard projection onto the plane, $\varphi = \pi \circ \bar{\varphi} : L \to R^2$ be the projection in general position of the link $\bar{\varphi}$. Consider the double points set of the projection. A function $o : \Delta_2(\phi) \to Z/_2$ is defined in the following way. For every $x \in \Delta_2(\varphi)$ let $(\xi_1, \xi_2)$ be a base on the plane, composed by the tangent vectors along the branches of $\varphi(L)$ at x. The order of the vectors corresponds to coordinates of two inverse images $x_1, x_2$ of the point x, $x_1, x_2 \in \bar{\varphi}(L)$. We define $o(x) = +1$, if the base $(\xi_1, \xi_2)$ is positive and $o(x) = -1$ otherwise. The pair $(\varphi, o)$ determs an element of the set ImmA(n) ( ImmB(n) ).

Each isotopy between links $\bar{\varphi}_0, \bar{\varphi}_1$ in $R^3$ produces a cobordism pair $(\Phi, O)$ without triple points of B-type. Therefore each invariant on the sets ImmA(n), ImmB(n) gives the corresponded invariant of isotopy classe of links.

## 2.5 Preposition

There exists a natural bijection

$$Lk = \oplus_{i<j} Lk(i,j) : ImmAB(n) \longrightarrow \oplus_{i<j} Z(i,j), 1 \leq i < j \leq n,$$

which commuts with the natural action of $\Sigma(n)$ on the set $ImmAB(n)$ and the group $\oplus_{i<j} Z(i,j)$. The map Lk(i,j) is given by the formula

$$Lk(i,j) = \frac{1}{2} \sum_{x_k \in \Delta(i,j)} O(x_k), 1 \leq i < j \leq n.$$

The main result is the following theorem.

## 2.6 Theorem

1. The set ImmA(n) is included into the following exact sequence

$$0 \longrightarrow KImmA(n) \longrightarrow ImmA(n) \longrightarrow ImmAB(n) \longrightarrow 0,$$

where $p : ImmA(n) \longrightarrow ImmAB(n)$ is a natural projection.

2. There exists a $\Sigma(n)$ -equivariant map

$$W : ImmA(n) \longrightarrow Z/_2,$$

called the Rokhlin invariant. The restriction $W\mid_{KImmA(n)}$ is given by the formula

$$W(\varphi, o) = \sum_{i<j)} w(i,j) \quad (mod 2), \quad (\varphi, o) \in KImmA(n), \tag{3}$$

where $w(i,j)$ is the Sato-Levine invariant for two-components semi-boundary links.

## 2.7 Remark

The set KImmB(n) is composed of projections of semi-boundary n-components links. The definition and properties of the Sato-Levine invariant $w : KImmA(n) \longrightarrow Z$ can be taken from [4], [17]. For an arbitrary two-components semi-boundary link $\bar{\varphi}(L_1 \cup L_2)$, there is the following formula

$$w(1,2) = R(1) + R(2) + R(1,2),$$

where $R$ is the Robertello invariant, see [1], [3].

## 2.8 Construction

We start by the construction of the Rokhlin invariant in the case $n = 2$. By Preposition 7 we have $ImmA(2) = Z$. In each cobordism class $(z) \in Z$ we choose the simplest link. The link can be presented by an embedding

$$\bar{\varphi} : S^1(1) \cup S^1(2) \in R^3,$$

where $\bar{\varphi}(S^1(1))$ is the standard inclusion, $\bar{\varphi}(S^1(2))$ is a (1,z)-toric winding along the component $\bar{\varphi}(S^1(1))$ .

Let $(\varphi, o) \in ImmB(2)$, $p(\varphi, o) = (z)$ . Let $(\Phi, O)$ be a cobordism, which joins the elements $p(\varphi, o)$ and $(z) \in ImmA(2)$. We define

$$W(\varphi, o) = ord(B2) \qquad (mod 2),$$

where ord(B2) is a number of triple points of the subtype B2 for the cobordism. In general case $n \geq 3$, we define W by the following formula

$$W(\varphi, o) = \sum_{i<j} W(i, j),$$

where W(i,j) is the invariant for two-components links, constructed above.

The extention of W from two-components links to multy-components links is proposed by A.Ruzmaikin.

# 3 The Rokhlin invariant and the Rokhlin theorem.

In this section we prove, that W is well-defined. We recall the Rokhlin theorem from [11], [12].

## 3.1 The Rokhlin Theorem.

Let $K^4$ be a closed 4-dimensional manifold with $H_1(K^4; Z) = 0$. Let $M^2 \in K^4$ be an oriented characteristic surface, $\langle M, M \rangle$ be the integer self-intersection number of $M$, $Arf(M)$ be the Arf-number $(mod 2)$ of $M$, $\sigma(K)$ be the signature of $K^4$ ( see [11] ). The following expression holds

$$\langle M, M \rangle - \sigma(K^4) = 8Arf(M) \quad (mod 16). \qquad (4)$$

We recall some constructions from [2]. Let $\varphi : M^2 \to R^3$ be a map in general position of an oriented surface $M$, $O$ be an orientation on the curve

$\Delta_2(\varphi)$. By the construction 6, there exists a map $\bar{\varphi} : M^2 \to R^4$, such that $\Pi \circ \bar{\varphi} = \varphi$, $\Pi : R^4 \to R^3$, where $\Pi$ is the standart projection. The map $\bar{\varphi}$ is an embedding everywhere, except a finite number of small disks $D_j^4 \in R^4$. Projections $\Pi(D_j^4) \in R^3$ contain triple points $\{x_j\} \in \Delta_3(\varphi)$ of the type $B$ on the surface $\varphi(M^2$. Each boundary $\partial D_j^4 = S_j^3$ intersects the surface $\bar{\varphi}(M^2)$ along 3-components link $L_j \in S_j^3$, called "Borromeo Rings".

We attach three 2-handles $(D^2 \times D^2)_i$, $i = 1, 2, 3$ to $S_j^3$ by maps $\xi_i : (D^2 \times S^1)_{i,j} \to S_j^3$. Here the map $\xi_i$ gives a parameterization of a little neighborhood of $L_j \in S_j^3$. A frame of the embedding $\xi_i$ is equal to +1 or -1, with respect to the positive or negative subtype of the point $x_j$. The result of the surgery is denoted by $(K^4, \partial K^4)$. The boundary $\partial K^4$ is an union of 3-dimensional homology Poincare spheres, corresponded to the points $x_j$.

We have an embedding $\bar{\varphi} : M^2 \to K^4$. The surface $\bar{\varphi}(M^2)$ is a characteristic surface for $K^4$. For this surface we have

$$\langle M, M \rangle = -\sigma(K^4). \tag{5}$$

Using (4), we get

$$ord\{x_j\} = Arf(\bar{\varphi}(M^2) \quad (mod 2). \tag{6}$$

We prove that $W$ is well-defined in the case $n = 2$. Let $(\Phi, O)$ be a cobordism, such that

$$(\Phi, O)|_{L\times\{0\}} = (\varphi, o), \quad (\Phi, O)|_{L\times\{1\}} = (-\varphi, -o).$$

We identify $\Phi$ along $L \times 0$ and $L \times 1$ and obtane an immersion

$$\Phi' = \Phi'_1 \cup \Phi'_2 : T_1^2 \cup T_2^2 \to R^3$$

of two copies $T_1^2, T_2^2$ of oriented tori with the corresponded orientation $O'$ on the curve of double points. Using (6) for $\Phi'_1; \Phi'_2; \Phi'$, we obtain

$$ord\{x_j\} = Arf(\Phi'_1) + Arf(\Phi'_2) + Arf(\Phi') \quad (mod 2),$$

where $\{x_j\}$ be a set of points of the B2- type for the immersion $\Phi'$. By the evident reason, we have

$$Arf(\Phi'_1) = Arf(\Phi'_2) = 0.$$

By direct computations we have $Arf(\Phi') = 0$, because the quadratic form for the characteristic surface $\Phi'$ is represented by the direct sum of two isomorphical forms. We have proved that in case $n = 2$, the number of triple points of the type B2 doesn't depended on the choise of a cobordism $(\Phi, O)$ ( see Construction 7). The prove for $n > 2$ is evident. The invariant $W$ is well-defined.

We prove the formula ( 3 ). Let $t_i \in I$ be a value of parameter, such that the restriction $(\varphi_{t_i}, o_{t_i}) = (\Phi, O)|_{t=t_i}$ contains a triple point $x$ of the type B2. For the Sato-Levine invariant we hawe

$$w(t|_i - \varepsilon) = w(t|_i + \varepsilon) + 1 \quad (mod 2).$$

If t is a critical value such that the paire $(\varphi_t, o_t)$ doesn't containe triple point of the type B2, we have

$$w(t - \varepsilon) = w(t + \varepsilon) \quad (mod Z).$$

This provs the formula (3). The Theorem is proved.

The proof of the Preposition 7 follows from the following remark.

## 3.2 Remark

Let $(\varphi_0, o_0)$, $(\varphi_1, o_1)$ be two pairs, such that $\varphi_0 = \varphi_1 = \varphi$, $o_0 = o_1$ for every $x \in \Delta_2(\varphi)$ , exept two points $x_1, x_2$, $o_0(x_1) = o_1(x_2) = +1$; $o_0(x_1) = o_1(x_2) = -1$. Than $(\varphi_0, o_0)$ and $(\varphi_1, o_1)$ are AB-cobordant.

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