Miroslav Kureš

Natural lifts of classical linear connections to the cotangent bundle

In: Jan Slovák (ed.): Proceedings of the 15th Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1996. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 43. pp. [181]–187.

Persistent URL: http://dml.cz/dmlcz/701585

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 43 (1996), pp. 181-187

NATURAL LIFTS OF CLASSICAL LINEAR CONNECTIONS TO THE COTANGENT BUNDLE

Miroslav Kureš

Abstract. We determine all natural operators transforming classical torsion-free linear connections on a manifold M into classical linear connections on T^*M.

Keywords: Natural operator, connection, cotangent bundle
AMS Classification: 53C05, 58A20

0. Introduction

We study the problem of lifting of a classical torsion-free linear connection on a manifold M into a classical linear connection on the cotangent bundle T^*M. Some lifts of such a type were first considered by Yano and Patterson in [4], [5]. We recall them in Section 1 and we add an interesting generalization by Gancarzewicz, [1]. In Section 2 we give a complete description of all natural lifts of our type by finding of all natural difference tensors. Hence our main result can be formulated as the following assertion about natural operators in the sense of [3].

Proposition 1. *All natural operators transforming a classical torsion-free linear connection on a manifold M into a classical linear connection on the cotangent bundle T^*M are the sum of a classical lift from Section 1 with a natural difference tensor from the 21–parameter family determined in Proposition 2.*

In Section 3 we interpret generators of the natural difference tensors geometrically.—All manifolds and maps are assumed to be infinitely differentiable.

1. Classical lifts

Let ∇ be a classical torsion-free linear connection on M with the coordinate expression $dX^i = \nabla^i_{jk}(x)X^j dx^k$, where x^i, $X^i = dx^i$ are some coordinates on TM.

First we define the complete lift of ∇ to T^*M (x^i, p_i are the corresponding coor-

This paper is in final form and no version of it will be submitted for publication elsewhere.
Supported by GA CR, grant No. 201/93/2125.
The author is indebted to Prof. I. Kolář for suggesting the problem and for his help.

dinates on T^*M). We consider a (0,2)–tensor field g on T^*M with components

$$\begin{aligned} g_{ij} &= 2p_k\nabla^k_{ij} \\ g_i^j &= \delta_i^j \\ g_j^i &= \delta_j^i \\ g^{ij} &= 0 \end{aligned}$$

Clearly, g is symmetric and regular, i.e. g is a pseudo-Riemannian metric, $(ds)^2 = 2dx^i(dp_i + p_k\nabla^k_{ij}dx^j)$. We call g the *Riemann extension* of ∇ and denote it by ∇^R. Let ∇^C be the Levi-Civita connection determined by the Riemann extension ∇^R. We call ∇^C the *complete lift* of ∇ to T^*M. Some properties of the complete lift can be found in the above-mentioned paper [4] by Yano and Patterson, but it is necessary to remark that they are not too simple. For example $\nabla^C_{X^C}Y^C \neq (\nabla_X Y)^C$, where X^C, Y^C are the complete lifts of vector fields X, Y to T^*M. The coordinate expression of ∇^C is

$$\begin{aligned} dX^i &= \nabla^i_{jk}X^j dx^k \\ dP_i &= p_m(\nabla^m_{jk,i} - \nabla^m_{ij,k} - \nabla^m_{ik,j} - 2\nabla^m_{il}\nabla^l_{jk})X^j dx^k - \nabla^k_{ij}X^j dp_k - \nabla^j_{ik}P_j dx^k \end{aligned}$$

provided $X^i = dx^i$, $P_i = dp_i$, are the induced coordinates on TT^*M.

Second we define the horizontal lift of ∇. The *horizontal lift* ∇^H of ∇ to T^*M is a unique classical linear connection on T^*M satisfying

$$\begin{aligned} \nabla^H_{\omega^V}\theta^V &= 0 \\ \nabla^H_{\omega^V}Y^H &= 0 \\ \nabla^H_{X^H}\theta^V &= (\nabla_X\theta)^V \\ \nabla^H_{X^H}Y^H &= (\nabla_X Y)^H \end{aligned}$$

where ω^V, θ^V are vertical lifts of 1-forms ω, θ and X^H, Y^H are horizontal lifts of vector fields X, Y with respect to ∇. A direct evaluation yields the following coordinate expression of ∇^H

$$\begin{aligned} dX^i &= \nabla^i_{jk}X^j dx^k \\ dP_i &= p_m(-\nabla^m_{ij,k} - \nabla^m_{lj}\nabla^l_{ik} - \nabla^m_{il}\nabla^l_{jk})X^j dx^k - \nabla^k_{ij}X^j dp_k - \nabla^j_{ik}P_j dx^k \end{aligned}$$

Gancarzewicz in [1] showed a way of generalizing ∇^H. For every linear connection D on T^*M (with the coordinate expression $dp_i = -D^j_{ik}(x)p_j dx^k$) and every classical linear connection ∇ on M there exists a unique classical linear connection (we call it *Gancarzewicz's lift*) $\nabla^G = \nabla^G(D, \nabla)$ on T^*M satisfying

$$\begin{aligned} \nabla^G_{\omega^V}\theta^V &= 0 \\ \nabla^G_{\omega^V}Y^H &= 0 \\ \nabla^G_{X^H}\theta^V &= (D_X\theta)^V \\ \nabla^G_{X^H}Y^H &= (\nabla_X Y)^H \end{aligned}$$

where the horizontal lifts of vector fields are taken with respect to D. If ∇^* is the dual connection to ∇, then $\nabla^G(\nabla^*, \nabla) = \nabla^H$. The coordinate expression of ∇^G is

$$\begin{aligned} dX^i &= \nabla^i_{jk} X^j dx^k \\ dP_i &= p_m(-D^m_{ij,k} - D^m_{lj} D^l_{ik} - D^m_{il} \nabla^l_{jk}) X^j dx^k - D^k_{ij} X^j dp_k - D^j_{ik} P_j dx^k \end{aligned}$$

2. All natural lifts

All natural lifts of classical torsion-free linear connections to T^*M are completely described by one lift together with all natural difference tensors. That is to say the difference of two classical linear connections on T^*M is a (1,2)–tensor field on T^*M and for finding all natural tensors of such a type we use standard methods stated in [2].

If we take the standard fiber $F_0 = TT_0^*\mathbb{R}^m$, then r-th order natural difference tensors correspond to the equivariant bilinear maps

$$L : F_0 \times F_0 \to F_0,\ L((x,p,X,P),(x,p,Y,Q)) = (x,p,\tilde{X},\tilde{P}),$$

in coordinates

$$\begin{aligned} \tilde{X}^i &= K^i_{jk} X^j Y^k + L^{ik}_j X^j Q_k + M^{ij}_k P_j Y^k + N^{ijk} P_j Q_k \\ \tilde{P}_i &= P_{ijk} X^j Y^k + Q^k_{ij} X^j Q_k + R^j_{ik} P_j Y^k + S^{jk}_i P_j Q_k \end{aligned} \tag{1}$$

Indeed, every such a natural operator has finite order, see [3], Prop.23.5., that is why all natural lifts of ∇ to T^*M have finite order r and for this reason functions $K^i_{jk}, L^{ik}_j, M^{ij}_k, N^{ijk}, P_{ijk}, Q^k_{ij}, R^j_{ik}, S^{jk}_i$ depend on p_i, ∇^i_{jk} and on partial derivatives $\nabla^i_{jk,\alpha}$ (α multiindex, $1 \le |\alpha| \le r$).

The equivariant maps in question are to be found under the action of G^2_m on F_0 given by equations

$$\begin{aligned} \bar{p}_i &= \tilde{a}^j_i p_j \\ \bar{X}^i &= a^i_j X^j \\ \bar{P}_i &= \tilde{a}^j_i P_j - a^l_{jk} \tilde{a}^m_l \tilde{a}^j_i p_m X^k \end{aligned} \tag{2}$$

where $\tilde{a}^i_j, \tilde{a}^i_{jk}$ are coordinates of the inverse element a^{-1}; $a \in G^2_m$, and under the well-known action

$$\bar{\nabla}^i_{jk} = a^i_{lm} \tilde{a}^l_j \tilde{a}^m_k + a^i_l \nabla^l_{mn} \tilde{a}^m_j \tilde{a}^n_k$$

Now, (1) and (2) enable us to compute functions $K^i_{jk}, L^{ik}_j, M^{ij}_k, N^{ijk}, P_{ijk}, Q^k_{ij}, R^j_{ik}, S^{jk}_i$ (barred functions denote the transformed quantities).

By comparing of coordinates between the equation (1) for $\tilde{X}^i$ and the same one for $\tilde{\bar{X}}^i$ we obtain

$$\bar{N}^{ijk} = a^i_l N^{lmn} a^j_m a^k_n$$

The action of G^1_m on N^{ijk} is tensorial and the equivariancy with respect to homotheties in G^1_m (given by $\tilde{a}^i_j = k\delta^i_j, \tilde{a}^i_{jk} = 0$) yields

$$\frac{1}{k^3} N^{ijk}(p_i, \nabla^i_{jk}, \nabla^i_{jk,\alpha}) = N^{ijk}(kp_i, k\nabla^i_{jk}, k^{1+|\alpha|}\nabla^i_{jk,\alpha})$$

Letting $k \to 0$ we find $N^{ijk} = 0$. Next, on condition $N^{ijk} = 0$ we obtain

$$\begin{aligned} \bar{M}^{ij}_k &= a^i_l M^{lm}_n a^j_m \tilde{a}^n_k \\ \bar{L}^{ik}_j &= a^i_l L^{ln}_m \tilde{a}^m_j a^k_n \end{aligned}$$

and in the same way we find $M^{ij}_k = L^{ik}_j = 0$. Further, on condition $N^{ijk} = M^{ij}_k = L^{ik}_j = 0$ we obtain

$$\bar{K}^i_{jk} = a^i_l K^l_{mn} \tilde{a}^m_j \tilde{a}^n_k$$

The action of G^1_m on K^i_{jk} is tensorial and the equivariancy with respect to homotheties in G^1_m yields

$$kK^i_{jk}(p_i, \nabla^i_{jk}, \nabla^i_{jk,\alpha}) = K^i_{jk}(kp_i, k\nabla^i_{jk}, k^{1+|\alpha|}\nabla^i_{jk,\alpha})$$

By the homogeneous function theorem, [3], Th.24.1., and the invariant tensor theorem, [3], Th.24.4., we have

$$K^i_{jk} = a\delta^i_j p_k + b\delta^i_k p_j + A\nabla^i_{jk} + B\delta^i_j \nabla^l_{lk} + C\delta^i_k \nabla^l_{lj}$$

If we put $a^i_j = \delta^i_j$, the equivariancy reads

$$0 = Aa^i_{jk} + Ba^l_{lk} + Ca^l_{lj}$$

Hence $A = B = C = 0$ and $K^i_{jk} = a\delta^i_j p_k + b\delta^i_k p_j$.

By comparing of coordinates between the equation (1) for $\tilde{P}_i$ and the same one for $\bar{\tilde{P}}_i$ we obtain $S^{jk}_i = 0, Q^k_{ij} = c\delta^k_i p_j + d\delta^k_j p_i, R^j_{ik} = e\delta^j_i p_k + f\delta^j_k p_i$. The procedure is quite analogous and that is why we don't perform all these steps in detail here. However, the computation of P_{ijk} is more complicated. We have

$$\bar{P}_{ijk} = \tilde{a}^l_i P_{lmn} \tilde{a}^m_j \tilde{a}^n_k - \tilde{a}^l_i a^r_{lm} \tilde{a}^m_n \bar{p}_r \bar{K}^n_{jk} + \tilde{a}^l_i \bar{Q}^n_{lm} \tilde{a}^m_j a^r_{ns} \tilde{a}^s_k \bar{p}_r + \tilde{a}^l_i \bar{R}^m_{ln} a^r_{ms} \tilde{a}^s_j \bar{p}_r \tilde{a}^n_k$$

The action of G^1_m on P_{ijk} is tensorial and the equivariancy with respect to homotheties in G^1_m yields

$$k^3 P_{ijk}(p_i, \nabla^i_{jk}, \nabla^i_{jk,\alpha}) = P_{ijk}(kp_i, k\nabla^i_{jk}, k^{1+|\alpha|}\nabla^i_{jk,\alpha})$$

By the homogeneous function theorem we see that admissible degrees of $(p_i, \nabla^i_{jk}, \nabla^i_{jk,l}, \nabla^i_{jk,lm})$ are $(3,0,0,0)$, $(2,1,0,0)$, $(1,2,0,0)$, $(1,0,1,0)$, $(0,3,0,0)$, $(0,1,1,0)$, $(0,0,0,1)$. Putting $a^i_j = \delta^i_j$ yields

$$\begin{aligned} \bar{P}_{ijk} = P_{ijk} - a^l_{im}\bar{p}_l(a\delta^m_j \bar{p}_k + b\delta^m_k \bar{p}_j) + \\ a^l_{mk}\bar{p}_l(c\delta^m_i \bar{p}_j + d\delta^m_j \bar{p}_i) + a^l_{mj}\bar{p}_l(e\delta^m_i \bar{p}_k + f\delta^m_k \bar{p}_i) \end{aligned} \tag{3}$$

and by the invariant tensor theorem we obtain
I. (3,0,0,0): $\bar{P}_{ijk} = g\bar{p}_i\bar{p}_j\bar{p}_k$, i.e. $P_{ijk} = gp_ip_jp_k$.
II. (2,1,0,0): $\bar{P}_{ijk} = D\bar{p}_i\bar{p}_l\bar{\nabla}^l_{jk} + E\bar{p}_j\bar{p}_l\bar{\nabla}^l_{ik} + F\bar{p}_k\bar{p}_l\bar{\nabla}^l_{ij} + G\bar{p}_i\bar{p}_j\bar{\nabla}^l_{lk} + H\bar{p}_i\bar{p}_k\bar{\nabla}^l_{lj} + I\bar{p}_j\bar{p}_k\bar{\nabla}^l_{li}$. We compare a^i_{jk}–elements with (3) and we obtain $G = H = I = 0$, $D = d + f, E = c - b, F = e - a$. This means $P_{ijk} = (d+f)p_ip_l\nabla^l_{jk} + (c-b)p_jp_l\nabla^l_{ik} + (e-a)p_kp_l\nabla^l_{ij}$.
III. To the remaining cases we apply the following lemma.

Lemma. *Let E, F be G^1_m–spaces. For every G^{r+2}_m–map $f : E \times T^r_m Q \to F$ there exists a unique G^1_m–map $g : E \times K^{r-1} \to F$ satisfying $f = g \circ C^{r-1}$, where $C^{r-1} = (C, C_1, ..., C_{r-1})$ is the formal curvature and its absolute differentials up to order $r-1$ and $K^{r-1} = C^{r-1}(T^r_m Q)$ is $(r-1)$-th order curvature space.*

Proof is only a slight modification of the proof of Theorem 28.6., [3]. □

In our case, we have factorization through $g : \mathbb{R}^{m*} \times K^1 \to \bigotimes^3 \mathbb{R}^{m*}$, where the canonical coordinates in K^1 are R^i_{jkl}, R^i_{jklm}. (R^i_{jkl} are skew-symmetric in the last two subscripts.) It holds

$$k^3 P_{ijk}(p_i, R^i_{jkl}, R^i_{jklm}) = P_{ijk}(kp_i, k^2 R^i_{jkl}, k^3 R^i_{jklm})$$

By the homogeneous function theorem we see that admissible degrees of $(p_i, R^i_{jkl}, R^i_{jklm})$ are $(3,0,0)$, $(1,1,0)$, $(0,0,1)$. The first case was described in I. We look into remaining cases.
IIIA. (1,1,0): By the invariant tensor theorem and by the first Bianchi identity we find

$$P_{ijk} = hp_lR^l_{ijk} + ip_lR^l_{kij} + jp_iR^l_{jkl} + kp_iR^l_{klj} + lp_jR^l_{kli} + mp_jR^l_{lik} + np_kR^l_{ijl} + op_kR^l_{lij}$$

IIIB. (0,0,1): By the invariant tensor theorem, by the absolute derivative of the first Bianchi identity and by the second Bianchi identity we find

$$P_{ijk} = pR^l_{jikl} + qR^l_{jlik} + rR^l_{kijl} + sR^l_{klij} + tR^l_{lijk} + uR^l_{lkij}$$

Thus, we have proved our main result.

Proposition 2. *All natural difference tensors on T^*M form a 21–parameter family*

$$\begin{aligned}
\tilde{X}^i &= (a\delta^i_j p_k + b\delta^i_k p_j)X^jY^k \\
\tilde{P}^i &= (gp_ip_jp_k + (d+f)p_ip_l\nabla^l_{jk} + (c-b)p_jp_l\nabla^l_{ik} + (e-a)p_kp_l\nabla^l_{ij} + hp_lR^l_{ijk} + \\
&\quad ip_lR^l_{kij} + jp_iR^l_{jkl} + kp_iR^l_{klj} + lp_jR^l_{kli} + mp_jR^l_{lik} + np_kR^l_{ijl} + op_kR^l_{lij} + \\
&\quad pR^l_{jikl} + qR^l_{jlik} + rR^l_{kijl} + sR^l_{klij} + tR^l_{lijk} + uR^l_{lkij})X^jY^k + \\
&\quad (c\delta^k_i p_j + d\delta^k_j p_i)X^jQ_k + (e\delta^j_i p_k + f\delta^j_k p_i)P_jY^k
\end{aligned}$$

Let us remark that if $i = 1$ and all other coefficients are zero, we obtain just the difference between the complete lift and the horizontal lift.

3. Geometrical interpretation

We give a geometrical interpretation of the individual generators of the 21–parameter family of natural difference tensors.

We consider the canonical projections $\pi : T^*M \to M$, $\sigma : TT^*M \to T^*M$, $\tau = T\pi : TT^*M \to TM$. Further, the identification $T^*M \times_M T^*M = VT^*M$ determines an injection $\iota : T^*M \to TT^*M$, $\iota(x,p) = (x,p,0,p)$.

Let $H = (x,p,X,P), J = (x,p,Y,Q) \in TT_0^*\mathbb{R}^m$. Then $X = \tau(H), Y = \tau(J), p = \sigma(H) = \sigma(J)$.

$\underline{a}$. (such an underbarred letter means $a = 1$ and all other coefficients are zero)
The coordinate expression

$$\begin{aligned}\tilde{X}^i &= X^i(p_kY^k)\\ \tilde{P}_i &= -p_l\nabla^l_{ij}X^j(p_kY^k)\end{aligned}$$

represents a lift of X with respect to dual connection ∇^* multiplied by the evaluation $\langle X,p\rangle$, i.e. the operator $\nabla^*(X)\langle Y,p\rangle$.

$\underline{b}$. as $\underline{a}$.; $\nabla^*(Y)\langle X,p\rangle$.

$\underline{c}$. The coordinate expression

$$\begin{aligned}\tilde{X}^i &= 0\\ \tilde{P}_i &= (p_l\nabla^l_{ik}Y^k + Q_i)(p_jX^j)\end{aligned}$$

represents a difference between J and a lift of Y with respect to ∇^* multiplied by $\langle X,p\rangle$, i.e. the operator $(J - \nabla^*(Y))\langle X,p\rangle$.

$\underline{d}$. The coordinate expression

$$\begin{aligned}\tilde{X}^i &= 0\\ \tilde{P}_i &= (p_l\nabla^l_{jk}X^jY^k + X^jQ_j)p_i\end{aligned}$$

represents a contraction of the vertical vector $J - \nabla^*(Y)$ by X multiplied by $\iota(p)$, i.e. the operator $\langle X, J - \nabla^*(Y)\rangle\iota(p)$.

$\underline{e}$. as $\underline{c}$.; $(H - \nabla^*(X))\langle Y,p\rangle$.

$\underline{f}$. as $\underline{d}$.; $\langle Y, H - \nabla^*(X)\rangle\iota(p)$.

$\underline{g}$. The coordinate expression

$$\begin{aligned}\tilde{X}^i &= 0\\ \tilde{P}_i &= (p_jX^j)(p_kY^k)p_i\end{aligned}$$

represents the operator $\langle X,p\rangle\langle Y,p\rangle\iota(p)$.

$\underline{h}$. The coordinate expression

$$\begin{aligned}\tilde{X}^i &= 0\\ \tilde{P}_i &= p_lR^l_{ijk}X^jY^k\end{aligned}$$

represents a contraction of the curvature tensor R by p, denoted by $\langle R_1, p\rangle$.
$\underline{\text{i}}$. as $\underline{\text{h}}$.; $\langle R_2, p\rangle$.
$\underline{\text{j}}$. The coordinate expression

$$\tilde{X}^i = 0$$
$$\tilde{P}_i = (R^l_{jkl} X^j Y^k) p_i$$

represents a contraction of the curvature tensor R into itself multiplied by $\iota(p)$, which is an operator denoted by $\widehat{R}_1 \iota(p)$.
$\underline{\text{k}}$.$\underline{\text{l}}$.$\underline{\text{m}}$.$\underline{\text{n}}$.$\underline{\text{o}}$. as $\underline{\text{j}}$.; $\widehat{R}_2\iota(p)$, $\widehat{R}_3\iota(p)$, $\widehat{R}_4\iota(p)$, $\widehat{R}_5\iota(p)$, $\widehat{R}_6\iota(p)$.
$\underline{\text{p}}$. The coordinate expression

$$\tilde{X}^i = 0$$
$$\tilde{P}_i = R^l_{jikl} X^j Y^k$$

represents a contraction of the absolute differential of the curvature tensor into itself, which will be denoted by $\widehat{\nabla R_1}$.
$\underline{\text{q}}$.$\underline{\text{r}}$.$\underline{\text{s}}$.$\underline{\text{t}}$.$\underline{\text{u}}$. as $\underline{\text{p}}$.; $\widehat{\nabla R_2}$, $\widehat{\nabla R_3}$, $\widehat{\nabla R_4}$, $\widehat{\nabla R_5}$, $\widehat{\nabla R_6}$.

References

1. Gancarzewicz, J., *Horizontal lifts of linear connections to the natural vector bundle*, Research Notes in Math. 121, Pitman (1985), 318–341.
2. Kolář, I., *Some natural operators in differential geometry*, Proc. "Conf. Diff. Geom. and Its Appl.", Brno 1986, Dordrecht (1987), 91–110.
3. Kolář, I., Michor, P.W., Slovák, J., *Natural Operations in Differential Geometry*, Springer Verlag, 1993.
4. Yano, K., Patterson, E.M., *Vertical and complete lifts from a manifold to its cotangent bundle*, J. Math. Soc. Japan **19** (1967), 91–113.
5. ———, *Horizontal lifts from a manifold to its cotangent bundle*, J. Math. Soc. Japan **19** (1967), 185–198.

INSTITUTE OF MATHEMATICS, FACULTY OF ENGINEERING, TECHNICAL UNIVERSITY, TECHNICKÁ 2, 616 69 BRNO, CZECH REPUBLIC