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In: Jan Slovák (ed.): Proceedings of the 15th Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1996. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 43. pp. [189]--197.

Persistent URL: http://dml.cz/dmlcz/701586

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 43 (1996), pp. 189-197

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June 3, 1996

Abstract

It is possible, given any solution of the constant Yang-Baxter equation, to construct an algebra by replacing the standard relators of complex simple Lie algebras by their braided analogs. In particular the one-parameter deformations $U_q(g)$ of Drinfeld and Jimbo arise as images of the Yang-Baxter algebras.

The present text is based on a talk given at the winter school at Srni, January, 14^{th} to 21^{st}, 1995. It is an elaboration on some of the ideas presented in the author's thesis [3] and in [4]. The use of graphs and of a certain braid bimodule in these publications has been eliminated in favor of a simpler algebraic approach.

1 Yang-Baxter operators

In this first section, we gather some preliminaries on the Yang-Baxter equation. An overview on the Yang-Baxter equation can be found in [2]. For a definition of the braid groups B_n and the respective notation used in the following, the reader should consult the last section of the present text, where the definitions have been collected for convenience.

A Yang-Baxter space is a vector space V together with an invertible linear map $\Upsilon \in \mathrm{Aut}(V \otimes V)$ that satisfies the (permuting, unparametrised and quantised) Yang-Baxter-equation on $V \otimes V \otimes V$, $\Upsilon_1\Upsilon_2\Upsilon_1 = \Upsilon_2\Upsilon_1\Upsilon_2$. The index i indicates, Υ_i acts onto the tensorfactors $i, 1+i$.

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1 *Let $\Upsilon(e_a \otimes e_b) = q_{a,b} \cdot (e_b \otimes e_a)$ with $q_{a,b} \in \mathbf{C}\backslash\{0\}$. Then Υ is a Yang-Baxter operator for the space with base $\{e_a; a \in \{1, \ldots, D\}\}$. In particular, the permutation operator obtained by $q_{a,b} = 1$ is a Yang-Baxter operator.*

2 *If V is a Yang-Baxter space, for any $n \geq 1$ $V^{\otimes n}$ becomes a left module over the braid group B_n by the representation $\Upsilon^L \in Hom(B_n, Aut(V^{\otimes n}))$, mapping $\Upsilon^L : \tau_l \mapsto \Upsilon_l$. Similarly, a right action can be defined (that in general does not commute with the left action) with the anti-representation $\Upsilon^R = \Upsilon^L \circ Rev$, where $Rev : B_n \to B_n$ reverses braids, $Rev(\tau_l) = \tau_l$, $Rev(\alpha\beta) = Rev(\beta)Rev(\alpha)$.*

The right-action of a braid α onto a vector $u = \sum u^{i_1 \ldots i_n} e_{i_1 \ldots i_n}$ is given by a matrix (in matrix notation we sometimes omit the index 'R')

$$u\alpha = \Upsilon^R(\alpha)(u) = \sum u^{i_1 \ldots i_n} \Upsilon(\alpha)^{j_1 \ldots j_n}_{i_1 \ldots i_n} e_{j_1 \ldots j_n}, \tag{1}$$

with $\{e_{i_1,\ldots,i_n} = e_{i_1} \otimes \ldots \otimes e_{i_n}; i_k \in \{1, \ldots n\}\}$ being a base of $V^{\otimes n}$. The matrix representation (a homomorphism, not 'anti') of braids induced by the right representation (an anti-homomorphism) on the tensor product therefore is determined by

$$\begin{aligned}
\Upsilon(\tau_l)^{j_1,\ldots,j_n}_{i_1,\ldots,i_n} &= \delta^{j_1,\ldots,j_{l-1}}_{i_1,\ldots,i_{l-1}} \Upsilon^{j_l,j_{1+l}}_{i_l,i_{1+l}} \delta^{j_{2+l},\ldots,j_n}_{i_{2+l},\ldots,i_n}, &(2)\\
\Upsilon(\alpha\beta)^{j_1 \ldots j_n}_{i_1 \ldots i_n} &= \sum \Upsilon(\alpha)^{k_1 \ldots k_n}_{i_1 \ldots i_n} \Upsilon(\beta)^{j_1 \ldots j_n}_{k_1 \ldots k_n}. &(3)
\end{aligned}$$

3 *If (V, Υ) is a Yang-Baxter space, the pair (V^*, Υ^*) with the dual space V^* and the pullback $\Upsilon^* \in Aut(V^* \otimes V^*)$, $\Upsilon^*(\phi \otimes \psi) = (\phi \otimes \psi) \circ \Upsilon$, for $\phi, \psi \in V^*$, also is a Yang-Baxter space. For the induced representations we have ($\alpha \in B_n$) $\Upsilon^{*L}(\alpha) = \Upsilon^R(\alpha)^*$ and $\Upsilon^{*R}(\alpha) = \Upsilon^L(\alpha)^*$.*

The left action of a braid α onto a covector $\phi = \sum e^{i_1 \ldots i_n} \phi_{i_1 \ldots i_n}$ is given by

$$\alpha\phi = \Upsilon^R(\alpha)^*(\phi) = \sum e^{j_1 \ldots j_n} \Upsilon(\alpha)^{i_1 \ldots i_n}_{j_1 \ldots j_n} \phi_{i_1 \ldots i_n} \tag{4}$$

with $\{e^{i_1,\ldots,i_n}; i_k \in \{1, \ldots n\}\}$ being the dual of the base of $V^{\otimes n}$.

4 *If (V, Υ) is a Yang-Baxter space, there is an operator $T(\Upsilon) \in Aut(T(V) \otimes T(V))$ that turns the tensor algebra $T(V)$ into a Yang-Baxter space. $T(\Upsilon)$ is uniquely determined by the components $T_{k,l}(\Upsilon) : V^{\otimes k} \otimes V^{\otimes l} \to V^{\otimes l} \otimes V^{\otimes k}$,*

$$\begin{aligned}
T_{k,l}(\Upsilon)(v \otimes w) &= (v \otimes w)\left(\prod_{m=1}^{k} \tau_{1+k-m,1+k+l-m}\right), &(5)\\
T_{0,l}(\Upsilon)(1 \otimes w) &= (w \otimes 1), &(6)\\
T_{k,0}(\Upsilon)(v \otimes 1) &= (1 \otimes v), &(7)
\end{aligned}$$

for $v \in V^{\otimes k}$, $w \in V^{\otimes l}$ (where the braid acts from the right onto the vector in $V^{\otimes(k+l)}$ via the action induced by Υ).

The braid successively transports the first k tensor factors to the right of the last l factors, starting with the k^{th} factor and proceeding to the left. This means, the first k and the last l strings are 'packed together' and it makes obvious the validity of the

Yang-Baxter equation. The operator is invertible, since it is a product of invertible ones. ♣

The braided commutator on the tensor algebra $T(V)$ is the map $[,]_{T(\Upsilon)} : T(V) \times T(V) \to T(V)$ given by $[v,w]_{T(\Upsilon)} = vw - T(\Upsilon)(v \otimes w)$. If Υ is chosen as the permutation of the tensor factors, the braided commutator coincides with the usual commutator in the tensor algebra. The braided commutator on the dual algebra $T^*(V)$ is the pullback of the braided commutator on $T(V)$, $[\phi,\psi]_{T(\Upsilon)^*} = \phi\psi - T(\Upsilon)^*(\phi \otimes \psi)$.

5 *The braided commutator is bilinear (therefore descends from $T(V) \times T(V)$ to $T(V) \otimes T(V)$), is braided skew-symmetric and is a braided derivation,*

$$\begin{aligned}
[T(\Upsilon)^{-1}(v \otimes w)]_{T(\Upsilon)} &= -[v,w]_{T(\Upsilon)^{-1}}, &(8)\\
[u, vw]_{T(\Upsilon)} &= [u,v]_{T(\Upsilon)} w + [T(\Upsilon)(u \otimes v) \otimes w]_{T(\Upsilon),2}, &(9)\\
[uv, w]_{T(\Upsilon)} &= u[v,w]_{T(\Upsilon)} + [u \otimes T(\Upsilon)(v \otimes w)]_{T(\Upsilon),1}, &(10)
\end{aligned}$$

(The index i on the bracket means the commutator acts onto the i^{th} and the $(1+i)^{th}$ factor in the product space $T(V) \otimes T(V) \otimes T(V)$.)

2 The Yang-Baxter algebra

Now we will define what we call the 'Yang-Baxter algebra'. Consider the complex tensor algebra Y on the set

$$\{T_a^b, \bar{T}_a^b, U_c^d, \bar{U}_c^d, F_e, E^f; 1 \le a,b,c,d,e,f \le D\}. \tag{11}$$

On the subalgebra generated by the elements F_a (E^b, respectively) define the braided commutators

$$\begin{aligned}
[F_{a_1}, F_{a_2} \ldots F_{a_n}] &= F_{a_1} \ldots F_{a_n} - \sum \Upsilon^R(\tau_{1,n})_{a_1 \ldots a_n}^{b_1 \ldots b_n} \cdot F_{b_1} \ldots F_{b_n} &(12)\\
&= \Upsilon^R(1 - \tau_{1,n})(F_{a_1} \ldots F_{a_n}), &(13)\\
[E^{a_n} \ldots E^{a_2}, E^{a_1}] &= E^{a_n} \ldots E^{a_1} - \sum \Upsilon^R(\tau_{n,1})_{b_1 \ldots b_n}^{a_1 \ldots a_n} \cdot E^{b_n} \ldots E^{b_1} &(14)\\
&= \Upsilon^R(1 - \tau_{n,1})^*(E^{a_n} \ldots E^{a_1}). &(15)
\end{aligned}$$

The ordering of the indices in the dual representation has been reversed compared to the equations in the last section. In the representation to be introduced later on, the rightmost operator acts onto the leftmost vector in a tensor product. This is in contrast to the action of tensor products of dual vectors onto tensor products of vectors, which gave rise to the equation as given before.

6 *Let $z_{1+n} = \prod_{k=0}^{n-1}(1 - \tau_{n-k,n}\tau_n^2) braid!(n)$ (see the last section for notation). Let Y_Υ be the quotient of the complex tensor algebra Y by the ideal generated by the set $\sum \bar{T}_b^c T_a^b - \delta_a^c$, $\sum T_b^c \bar{T}_a^b - \delta_a^c$, $\sum \bar{U}_b^c U_a^b - \delta_a^c$, $\sum U_b^c \bar{U}_a^b - \delta_a^c$, and*

$$\begin{aligned}
\sum \Upsilon_{d,b}^{h,g} T_c^d T_a^b &- \sum T_d^h T_b^g \Upsilon_{c,a}^{d,b}, &(16)\\
\sum \Upsilon_{b,d}^{f,h} U_c^d U_a^b &- \sum U_d^h U_b^f \Upsilon_{a,c}^{b,d}, &(17)
\end{aligned}$$

$$\begin{aligned}
\sum T^f_g \Upsilon^{d,g}_{a,c} U^c_b &= \sum U^d_e \Upsilon^{e,f}_{c,b} T^c_a, & (18)\\
\sum \bar{T}^e_b E^c T^b_a &= \sum \Upsilon^{c,e}_{a,d} E^d, & (19)\\
\sum U^e_a E^d \bar{U}^a_b &= \sum \bar{\Upsilon}^{d,e}_{b,c} E^c, & (20)\\
\sum \bar{T}^d_c F_a T^c_b &= \sum F_c \bar{\Upsilon}^{d,c}_{a,b}, & (21)\\
\sum U^c_b F_a \bar{U}^b_d &= \sum F_e \Upsilon^{c,e}_{a,d}, & (22)\\
E^b F_a - \sum F_c \Upsilon^{b,c}_{a,d} E^d &= (\delta^b_a - \sum U^b_c T^c_a). & (23)
\end{aligned}$$

Furthermore, for all (dual) vectors $v = \sum v^{i_1 \ldots i_{1+n}} e_{i_1 \ldots i_{1+n}} \in V^{\otimes(1+n)}$, $\phi = \sum e^{i_1 \ldots i_{1+n}} \phi_{i_1 \ldots i_{1+n}} \in V^{*\otimes(1+n)}$, *obeying*

$$\begin{aligned}
\Upsilon^R(z_{1+n})(v) &= 0, & (24)\\
\Upsilon^R(z_{1+n})^*(\phi) &= 0 & (25)
\end{aligned}$$

let the generalized Serre-relators

$$\sum_i v^{i_1 \ldots i_{1+n}} \cdot [F_{i_1}, [F_{i_2}, \ldots, [F_{i_n}, F_{i_{1+n}}] \ldots]], \qquad (26)$$

$$\sum [[\ldots [E^{i_{1+n}}, E^{i_n}], \ldots, E^{i_2}], E^{i_1}] \cdot \phi_{i_1 \ldots i_{1+n}} \qquad (27)$$

be in the ideal, respectively. Then Y_Υ *is not the trivial algebra* $\{0\}$ *nor the unital algebra* $\mathbf{C}$.

In the next section, we will construct a representation $Y_\Upsilon \to \mathrm{End}(V_\Upsilon)$ of this algebra, where the vector space V_Υ has dimension bigger than D. The subalgebra of $\mathrm{End}(V_\Upsilon)$ obtained in this way is strictly larger than the one generated by $0, 1 \in \mathrm{End}(V_\Upsilon)$. ♣

The ideal defining the Yang-Baxter algebra as a quotient can be constructed for any map $\Upsilon \in \mathrm{End}(V \otimes V)$. It does not necessarily need to obey the Yang-Baxter equation nor does it need to be invertible, either. But these requirements will make it possible to construct a representation of the algebra and to prove its non-triviality.

3 The representation of the Yang-Baxter algebra

Here we will prove non-triviality of the algebra defined in the previous section by constructing a linear representation for it. In the course of the proof, we will, without derivation, refer to several combinatorial identities holding in the ring of the braid group. These identities have been collected in the last section.

7 *Let* $V^{\otimes 0} = \mathbf{C}1$, $z_0 = 1$, $V_\Upsilon = \bigoplus_{n=0}^\infty \Upsilon^R(z_n)(V^{\otimes n})$. *Then there are linear operators in* $\mathrm{End}(V_\Upsilon)$, *uniquely determined by* ($v = uz_n \in V_\Upsilon^{(n)}$, $u \in V^{\otimes n}$)

$$\begin{aligned}
T^b_a(v) &= \Upsilon^R(z_n)\phi^b_{1+n}\Upsilon^R(\tau_{1,1+n})(e_a \otimes u) & (28)\\
\bar{T}^b_a(v) &= \Upsilon^R(z_n)\phi^b_1\Upsilon^R(\tau^{-1}_{1,1+n})(u \otimes e_a) & (29)\\
U^b_a(v) &= \Upsilon^R(z_n)\phi^b_1\Upsilon^R(\tau_{1+n,1})(u \otimes e_a) & (30)\\
\bar{U}^b_a(v) &= \Upsilon^R(z_n)\phi^b_{1+n}\Upsilon^R(\tau^{-1}_{1+n,1})(e_a \otimes u) & (31)\\
F_b(v) &= \Upsilon^R(z_{1+n})(e_b \otimes u) & (32)\\
E^c(v) &= \phi^c_1\Upsilon^R(z_n)(u) & (33)
\end{aligned}$$

ϕ^b_j is the dual vector $\phi^b = (e_b)^*$ acting onto the j^{th} factor of the tensor product and by convention, $\phi^b(1) = 0$.

It has to be shown that the action of T^b_a, $\bar{T}^b_a$, U^d_c, $\bar{U}^d_c$ and F_e does not depend on the choice of the representative u, which is defined only up to elements in the kernel of $\Upsilon^R(z_n)$. Also E^c must leave invariant the subspace $V_\Upsilon \leq T(V)$. Due to the properties of z_n (see the last section) we have $T^b_a(uz_n) = \phi^b_{1+n}\Upsilon^R(\tau_{1,1+n})(e_a \otimes (uz_n))$, and similar for $\bar{T}, U, \bar{U}$. The equation $F_b(uz_n) = (e_b \otimes (uz_n))\sum_{i=1}^n \tau_{1,i}(1-\mu_{i,1+n})$, shows the independence from the choice of the representative u. Again due to the properties of z_n we have $E^c(uz_n) = (\phi^c_1(u\sum_{i=1}^{n-1}(1-\mu_{i,n})\tau_{i,1}))z_{n-1}$, such that the image under E^c is in V_Υ. ♣

8 *There is a homomorphism $Y_\Upsilon \to End(V_\Upsilon)$ uniquely determined by mapping the generators of the Yang-Baxter algebra to the linear operators introduced above.*

As has already been indicated before, we will make use of several identities in the ring of the braid group. In order to shorten the present proof, these identities have been collected in the last section. It has to be shown that the map defined on the generators of the algebra sends the generating elements of the defining ideal of Y_Υ to zero in $\mathrm{End}(V_\Upsilon)$. We therefore check the relations on an arbitrary vector $v = uz_m \in V_\Upsilon^{(m)}$. The relations showing that $T, \bar{T}$ and $U, \bar{U}$ are mutual inverses are clear from the definition.

$$\begin{aligned}
&\Upsilon^{d,g}_{a,c}T^f_g U^c_b(v) = && (34)\\
&= (uz_n)^{a_1\ldots a_n}\Upsilon^R(\tau_{1+n,1})^{cb_1\ldots b_n}_{a_1\ldots a_n b}\Upsilon^{d,g}_{a,c}\Upsilon^R(\tau_{1,1+n})^{c_1\ldots c_n f}_{gb_1\ldots b_n}e_{c_1\ldots c_n} && (35)\\
&= (uz_n)^{a_1\ldots a_n}\Upsilon^R(\tau_{n+2,2}\tau_1\tau_{2,n+2})^{dc_1\ldots c_n f}_{aa_1\ldots a_n b}e_{c_1\ldots c_n} && (36)\\
&= (uz_n)^{a_1\ldots a_n}\Upsilon^R(\tau_{1,1+n}\tau_{1+n}\tau_{1+n,1})^{dc_1\ldots c_n f}_{aa_1\ldots a_n b}e_{c_1\ldots c_n} && (37)\\
&= (uz_n)^{a_1\ldots a_n}\Upsilon^R(\tau_{1,1+n})^{c_1\ldots c_n c}_{aa_1\ldots a_n}\Upsilon^{e,f}_{c,b}\Upsilon(\tau_{1+n,1})^{dd_1\ldots d_n}_{c_1\ldots c_n e}e_{d_1\ldots d_n} && (38)\\
&= \Upsilon^{e,f}_{c,b}U^d_e T^c_a(v). && (39)
\end{aligned}$$

The proof of the remaining relations involving only T, $\bar{T}$ and U, $\bar{U}$ proceeds in a similar fashion and is omitted.

$$\begin{aligned}
&E^c T^b_a(v) = && (40)\\
&= (uz_n)^{a_1\ldots a_n}\Upsilon^R(\tau_{1,1+n})^{cb_2\ldots b_n b}_{aa_1\ldots a_n}e_{b_2\ldots b_n} && (41)\\
&= (uz_n)^{a_1\ldots a_n}\Upsilon^{c,e}_{a,a_1}\Upsilon^R(\tau_{1,n})^{b_2\ldots b_n b}_{ea_2\ldots a_n}e_{b_2\ldots b_n} && (42)\\
&= \Upsilon^{c,e}_{a,a_1}T^b_e(uz_n)^{a_1\ldots a_n}e_{a_2\ldots a_n} && (43)\\
&= \Upsilon^{c,e}_{a,d}T^b_e E^d(uz_n)^{a_1\ldots a_n}e_{a_1\ldots a_n}. && (44)
\end{aligned}$$

$$\begin{aligned}
&U^c_b F_a(v) = && (45)\\
&= U^c_b(u^{a_1\ldots a_n}\Upsilon^R(z_{1+n})^{b_1\ldots b_{1+n}}_{aa_1\ldots a_n}e_{b_1\ldots b_{1+n}}) && (46)\\
&= u^{a_1\ldots a_n}\Upsilon^R(\tau_{n+2,1})^{cb_1\ldots b_{1+n}}_{aa_1\ldots a_n b}\Upsilon^R(z_{1+n})^{d_1\ldots d_{1+n}}_{b_1\ldots b_{1+n}}e_{d_1\ldots d_{1+n}} && (47)\\
&= u^{a_1\ldots a_n}\Upsilon^R(\tau_{1+n,1})^{db_2\ldots b_{1+n}}_{a_1\ldots a_n b}\Upsilon^{c,b_1}_{a,d}\Upsilon^R(z_{1+n})^{d_1\ldots d_{1+n}}_{b_1\ldots b_{1+n}}e_{d_1\ldots d_{1+n}} && (48)\\
&= \Upsilon^{c,e}_{a,d}F_e(u^{a_1\ldots a_n}\Upsilon^R(\tau_{1+n,1})^{db_2\ldots b_{1+n}}_{a_1\ldots a_n b}\Upsilon^R(z_n)^{d_2\ldots d_{1+n}}_{b_2\ldots b_{1+n}}e_{d_2\ldots d_{1+n}}) && (49)\\
&= \Upsilon^{c,e}_{a,d}F_e U^d_b(v). && (50)
\end{aligned}$$

$$
\begin{aligned}
& E^b F_a(v) = && (51)\\
& = E^b(u^{a_1\ldots a_n}\Upsilon^R(T_2(z_n)\sum_{i=1}^{n}\tau_{1,i}(1-\mu_{i,1+n}))^{b_1\ldots b_{1+n}}_{aa_1\ldots a_n}e_{b_1\ldots b_{1+n}}) && (52)\\
& = E^b(u^{a_1\ldots a_n} && (53)\\
& \quad \Upsilon^R\{T_2(z_n)[1-\mu_{1,1+n}+\tau_1\sum_{i=2}^{n}\tau_{2,i}(1-\mu_{i,1+n})]\}^{b_1\ldots b_{1+n}}_{aa_1\ldots a_n} && (54)\\
& \quad e_{b_1\ldots b_{1+n}}) && (55)\\
& = (\delta^b_a - U^b_c T^c_a)(v) + && (56)\\
& \quad + u^{a_1\ldots a_n}\Upsilon^R(z_n)^{c_1\ldots c_n}_{a_1\ldots a_n}\Upsilon^R(\tau_1\sum_{i=2}^{n}\tau_{2,i}(1-\mu_{i,1+n}))^{bb_2\ldots b_{1+n}}_{ac_1\ldots c_n}e_{b_2\ldots b_{1+n}} && (57)\\
& = (\delta^b_a - U^b_c T^c_a)(v) + && (58)\\
& \quad + u^{a_1\ldots a_n}\Upsilon^R(z_n)^{c_1\ldots c_n}_{a_1\ldots a_n}\Upsilon^{b,c}_{a,c_1}\Upsilon^R(\sum_{i=1}^{n-1}\tau_{1,i}(1-\mu_{i,n}))^{b_1\ldots b_n}_{cc_2\ldots c_n}e_{b_1\ldots b_n} && (59)\\
& = (\delta^b_a - U^b_c T^c_a)(v) + \Upsilon^{b,c}_{,c_1}F_c(u^{a_1\ldots a_n}\Upsilon^R(z_n)^{c_1\ldots c_n}_{a_1\ldots a_n}e_{c_2\ldots c_n}) && (60)\\
& = (\delta^b_a - U^b_c T^c_a)(v) + \Upsilon^{b,c}_{a,d}F_c E^d(u^{a_1\ldots a_n}\Upsilon^R(z_n)^{c_1\ldots c_n}_{a_1\ldots a_n}e_{c_1\ldots c_n}). && (61)
\end{aligned}
$$

Here again we used certain properties of the braid ring element z_n, which are explicitly stated in the last section. Let us turn to the Serre-relations.

$$
\begin{aligned}
& [F_{i_1},[\ldots,[F_{i_n},F_{i_{1+n}}]\ldots]](v) && (62)\\
& = (\Upsilon^R(1-\tau_{1,1+n})\ldots\Upsilon^R(1-\tau_{n,1+n})(F_{i_1}\ldots F_{i_{1+n}}))(v) && (63)\\
& = (\Upsilon^R(\prod_{l=1}^{n}(1-\tau_{1+n-l,1+n}))(F_{i_1}\ldots F_{i_{1+n}}))(v) && (64)\\
& = (e_{i_1\ldots i_{1+n}}\otimes u)(1-\tau_{n,1+n})\ldots(1-\tau_{1,1+n})z_{n+m+1} && (65)\\
& = (e_{i_1\ldots i_{1+n}}\otimes u)z_{1+n}\epsilon, && (66)
\end{aligned}
$$

for suitable ϵ by the properties of z_n. Thus we obtain,

$$[F_{i_1},[\ldots,[F_{i_n},F_{i_{1+n}}]\ldots]](v) = (\Upsilon^R(z_{1+n})(e_{i_1\ldots i_{1+n}})\otimes u)\epsilon, \qquad (67)$$

such that the condition (24) guarantees validity of the first Serre relation. In order to prove the second Serre relation let $m > 1+n$. Then

$$
\begin{aligned}
& [[\ldots[E^{i_{1+n}},E^{i_n}],\ldots,E^{i_2}],E^{i_1}](v) && (68)\\
& = [\Upsilon^R(1-\tau_{1+n,1})^*\ldots\Upsilon^R(1-\tau_{1+n,n})^*(E^{i_{1+n}}\ldots E^{i_1})](v) && (69)\\
& = [\Upsilon^R(\prod_{j=1}^{n}(1-\tau_{1+n,j}))^*(E^{i_{1+n}}\ldots E^{i_1})](v) && (70)\\
& = E^{i_{1+n}}\ldots E^{i_1}(uz_m\prod_{j=1}^{n}(1-\tau_{1+n,j})) && (71)\\
& = E^{i_{1+n}}\ldots E^{i_1}(u(\prod_{l=1}^{m-1}(1-\tau_{m-l,m-1}\tau^2_{m-1}) && (72)
\end{aligned}
$$

$$\cdot \mathrm{braid}(m-1, m-n-2)T_{n+2}(\mathrm{braid!}(m-n-2))\mathrm{braid!}(1+n) \tag{73}$$
$$\cdot \prod_{j=1}^{n}(1-\tau_{1+n,j}))) \tag{74}$$

In the last step we used a property of z_m and an identity in the braid ring. A further identity for z_n shows,

$$[[\ldots[E^{i_{1+n}}, E^{i_n}], \ldots, E^{i_2}], E^{i_1}](v) \tag{75}$$
$$= E^{i_{1+n}} \ldots E^{i_1}(u(\prod_{l=1}^{m-1}(1-\tau_{m-l,m-1}\tau^2_{m-1})\cdot \tag{76}$$
$$\cdot \mathrm{braid}(m-1, m-n-2)T_{n+2}(\mathrm{braid!}(m-n-2))\cdot \tag{77}$$
$$\cdot z_{1+n})) \tag{78}$$

such that the second Serre relation holds under the condition (25). ♣

At least the preimage of the Yang-Baxter algebra, in which the Serre relators have not yet been set to zero, carries the structure of a bialgebra.

9 (Conjecture) *The maps Δ and ϵ, with $\Delta(T^b_a) = \sum T^b_c \otimes T^c_a$, $\Delta(\bar{T}^b_a) = \sum \bar{T}^c_a \otimes \bar{T}^b_c$, $\Delta(U^b_a) = \sum U^c_a \otimes U^b_c$, $\Delta(\bar{U}^b_a) = \sum \bar{U}^b_c \otimes \bar{U}^c_a$, $\Delta(F_a) = 1 \otimes F_a + \sum F_b \otimes T^b_a$, $\Delta(E^a) = 1 \otimes E^a + \sum E^b \otimes U^a_b$ and $\epsilon(F_a) = \epsilon(E^b) = 0$, $\epsilon(T^b_a) = \epsilon(U^b_a) = \delta^b_a\cdot$extend to a comultiplication and counit, respectively, for the Yang-Baxter algebra.*

It must be checked that the maps extend to homomorphisms of algebras, $\Delta \in \mathrm{Hom}(Y_\Upsilon, Y_\Upsilon \otimes Y_\Upsilon)$ and $\epsilon \in \mathrm{Hom}(Y_\Upsilon, \mathrm{C})$. Furthermore it must hold, $(\Delta \otimes id)\circ\Delta = (id \otimes \Delta)\circ\Delta$ and $m\circ(id\otimes\epsilon)\circ\Delta = m\circ(\epsilon\otimes id)\circ\Delta = id$ (m is the product in Y_Υ, id the identity map). ϵ indeed extends to a homomorphism, since it maps the defining ideal I to $\{0\}$, as can be seen from the relations. Also Δ extends homomorphically, sending I to $I \otimes I$, at least without consideration of the Serre relations. The restrictions on Δ and ϵ are verified straightforwardly. ♣

4 q-deformed Lie algebras

Here we show that the well-known q-deformed complex simple Lie algebras arise as homomorphic images of the Yang-Baxter algebras, if the Yang-Baxter solution Υ is chosen in a simple way. For an overview on q-deformed Lie algebras, one might consult [2].

10 *Let a Yang-Baxter operator be given by $\Upsilon^{c,d}_{a,b} = q_{a,b}\cdot\delta^d_a\cdot\delta^c_b$ with $q_{a,b} = q_{b,a} \in \mathrm{C}\backslash\{0\}$, $q_{a,a} \neq 1$. If the map on the set of generators of Y_Υ $T^b_a, U^b_a \mapsto K_a\cdot\delta^b_a$, $E^a \mapsto (1-q^{-1}_{a,a})X^+_a K^{1/2}_a$, $F_a \mapsto X^-_a K^{1/2}_a$ extends to a homomorphism $Y_\Upsilon \to Z$ of algebras, then in Z we have $K^-_a K_c K_a = K_c$, $K^-_a X^{+(-)}_c K_a = q^{+(-)1}_{a,c}\cdot X^{+(-)}_c$, $X^-_b X^+_a - X^+_a X^-_b = \delta_{a,b}\frac{K_a - K^{-1}_a}{q^{1/2}_{a,a} - q^{-1/2}_{a,a}}$. If it extends to a homomorphism of bialgebras, we find $\Delta X^{+(-)}_a = X^{+(-)}_a \otimes K^{1/2}_a + K^{-1/2}_a \otimes X^{+(-)}_a$, $\Delta K_a = K_a \otimes K_a$, $\epsilon(X^{+(-)}_a) = 0$, $\epsilon(K_a) = 1$.*

If $q_{a,b} = q^{(\lambda_a,\lambda_b)/2}$, with the scalarproduct $(\cdot,\cdot)$ in the root space of a complex simple Lie algebra with simple roots $\{\lambda_a\}$ and the Cartan matrix $C_{a,b} = 2\frac{(\lambda_a,\lambda_b)}{(\lambda_a,\lambda_a)}$, then the

Serre relations are valid in Z,

$$[F_a,\ldots]^{1-C_{a,b}}(F_b) = 0, \text{ if } a \neq b, \quad (79)$$
$$[\ldots,E^a]^{1-C_{a,b}}(E^b) = 0, \text{ if } a \neq b. \quad (80)$$

Here the commutator is the deformed one as defined before.

The relations between the Cartan generators K_a, the eigenvalue relations between the Cartan elements and the ladder operators and the relations between the ladder operators $X_a^{+/-}$ are consequences of the corresponding relations in the Yang-Baxter algebra. The Serre conditions (24,25) expressed with the given Yang-Baxter matrix map to the equation $\prod_{k=0}^{-C_{a,b}}(1 - q_{a,a}^k q_{a,b}^2) = 0$. This is always valid, since $C_{a,b} = 2(\lambda_a, \lambda_b)/(\lambda_a, \lambda_a)$. ♣

5 Combinatorics in the braid ring

Finally we collect the definitions and combinatorial results on the braid group and it's integral ring which have been used in the third section. Due to lack of space, the proofs have been omitted. All of them proceed by induction or simply by inspection but nevertheless some of them are quite intricate. The 'classical' overviews on the braid group are [1], [5].

The braid group B_n is the group generated by the set $\{\tau_i; i \in \{1,\ldots,n-1\}\}$ modulo the relations of Artin $\tau_i\tau_j = \tau_j\tau_i$, if $\mid i-j \mid\geq 2$, and $\tau_i\tau_{1+i}\tau_i = \tau_{1+i}\tau_i\tau_{1+i}$. We will use the abbreviations ($i<j$, $i,j,k,l \in \{1,\ldots,n\}$) $\tau_{i,j} = \tau_i\tau_{1+i}\ldots\tau_{j-2}\tau_{j-1}$, $\tau_{j,i} = \tau_{j-1}\tau_{j-2}\ldots\tau_{1+i}\tau_i$, $\mu_{k,l} = \tau_{k,l}\tau_{l,k}$.

We define the braid factorial taking values in the integral ring of the braid group, as braid!(0) = 1, braid!(1 + n) = braid!(n) $\sum_{i=1}^{1+n}\tau_{1+n,i}$. We also need a braid generalization of the binomial coefficient,

$$\text{braid}(n,0) = 1, \quad \text{braid}(n,k) = \sum_{1\leq i_1<\ldots<i_k\leq n}\prod_{j=0}^{k-1}\tau_{i_{k-j},n-j}.$$

11 *Let there be maps $D_n : B_n \to B_n$, $D_n(\tau_i) = \tau_{n-i}$, $D_n(\alpha\beta) = D_n(\beta)D_n(\alpha)$ (a rotation of the braid graph about an angle π with an axis perpendicular to the drawing plane), $R_n \in Aut(B_n)$, $R_n(\tau_i) = \tau_{n-i}$, $R_n(\alpha\beta) = R_n(\alpha)R_n(\beta)$ (a rotation of the braid graph about an angle π with an axis in the drawing plane parallel to the orientation of the graph), and let $T_l^n \in Hom(B_n, B_{l+n-1})$ be the translation $T_l^n(\tau_i) = \tau_{l+i-1}$. Then*

$$\begin{aligned}
braid(n,k) &= R_n(braid(n,n-k)), && (81)\\
braid(1+n,k) &= braid(n,k-1) + braid(n,k)\tau_{1+n,1+n-k}, && (82)\\
braid!(n) &= braid(n,k)T_{n-k+1}^k(braid!(k))braid!(n-k) && (83)\\
&= T_{1+k}^{n-k}(braid!(n-k))braid!(k)\cdot && (84)\\
&\quad \cdot D_n(braid(n,k)), && (85)\\
\prod_{k=1}^m(1-\tau_{1+m-k,m}\tau_m^2) &= \sum_{l=0}^m(-)^l braid(m,l)\prod_{n=1}^l(\tau_{1+m-n,m}\tau_m^2), && (86)
\end{aligned}$$

$$\prod_{k=1}^{n-1}(1-\tau_{n-k,n}) = \sum_{l=0}^{n-1}(-)^l braid(n-1,l)\prod_{k=1}^{l}\tau_{n-k,n}. \tag{87}$$

Due to the following, the braid factorial and binomial are true generalizations of the familiar quantities.

12 *The homomorphism of groups* $B_n \to \mathbb{C}\backslash\{0\}$, $\tau_i \mapsto q$ *maps* $braid!(n) \mapsto \prod_{i=1}^{n}\frac{1-q^i}{1-q} = n!_q \xrightarrow{q\to 1} n!$, $braid(n,k) \mapsto \frac{[n]!_q}{[k]!_q[n-k]!_q} = \begin{bmatrix} n \\ k \end{bmatrix}_q \xrightarrow{q\to 1} \begin{pmatrix} n \\ k \end{pmatrix}$.

Now we turn to the important elements z_n of the integral braid ring, recursively defined as $z_1 = 1$, $z_{1+n} = T_2^{1+n}(z_n)\sum_{i=1}^{n}\tau_{1,i}(1-\mu_{i,1+n})$. It should be noticed that this definition is similar to the recursive definition of the elements $\theta_{1+n} = T_2^{1+n}(\theta_n)\mu_{1,1+n}$, $\theta_1 = 1$. It is known that θ_n generates the center of B_n, [5]. z_n can be regarded as the result of 'cutting' the element θ_n into pieces by a particular type of Fox-derivative. I do not yet know the relevance of this observation.

13 $z_n = \sum_{i=1}^{n-1}(1-\mu_{i,n})\tau_{i,1}T_2(z_{n-1}) = (\prod_{i=1}^{n-1}(1-\tau_{n-i,n-1}\tau_{n-1}^2))braid!(n-1) = braid!(n)\prod_{j=1}^{n-1}(1-\tau_{n,j})$, $(1-\tau_{l-1,l})(1-\tau_{l-2,l})\dots(1-\tau_{1,l})z_{l+m} = z_l r$, *for* $n > 1$, $m \geq 1$, $l \geq 2$ *and some* r *in the braid ring.*

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