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Local and Global Aspects of Separating Coordinates for the Klein-Gordon Equation*

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Abstract

Coordinate systems allowing for a separation of variables in the Klein - Gordon equation in n-dimensional manifolds are characterized by systems of n symmetric Killing tensors of order two including the metric tensor. In the case of separable coordinates in 1+1 - dimensional Minkowski space a simple geometric relation between the horizons (global boundaries) of their domains and the associated Killing tensors (local tensor fields) is established: Coordinate horizons are generated by vectors which are null both in the sense of the flat metric and of the other Killing tensor of the two-dimensional Stäckel system, when it is considered as an alternative, curved metric. These curved spaces display curvature singularities and signature changes.

1 Introduction

The subject of this paper are pseudoorthogonal coordinate systems in 1+1 - dimensional flat space-time according to which the Klein-Gordon equation is separable by a product ansatz

$$\Phi(x^0,x^1)=\Phi_0(x^0)\,\Phi_1(x^1) \tag{1}$$

into ordinary differential equations, so that its solution may be written as an integral over modes,

$$\Phi(x^0,x^1)=\int \mathrm{d}k\,c(k)\,\Phi_0^{(k)}(x^0)\,\Phi_1^{(k)}(x^1). \tag{2}$$

The physical motivation behind this work is the problem of constructing quantum states of fields on the background of general relativity. The approach by positive and negative frequency mode decompositions has been applied in some curved space-time examples, and also in flat space it has been carried out not only with respect to Minkowski time but also to non-inertial time variables, above all, the proper time of uniformly accelerated observers. This example results in the famous Unruh effect [1, 2]: In the framework of this particular quantum field theory the ordinary Minkowski vacuum displays thermal properties.

Quantization via mode decompositions of fields – provided by a separation of variables – is a local procedure at first sight, although the global nature of the resulting

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particle concept becomes evident at various instants. On the other hand, there are approaches by construction of local field algebras of given space-time domains (for example, the Rindler wedge [3]) which take into account global aspects, namely the extension and boundaries of these domains, from the beginning.

The investigation about global properties of separable coordinates is inspired by the question to what extent a positive and negative frequency mode decomposition may be generalized to non-inertial times and curved spaces to yield reasonable field quantizations.

2 Separable coordinate systems in 1+1 - dimensional Minkowski space

Including the Cartesian system there are 10 orthogonal coordinate systems in 1+1 - dimensional flat space such that the Klein-Gordon equation may be separated. In all of them coordinate lines are either straight lines or conic sections, the latter ones being arranged in one or two confocal families [4, 5]. In the following list curvilinear coordinates are denoted by μ, and ν.

1. Cartesian system: Coordinates t and x.

2. Elliptic system (E): Elliptic coordinates μ and ν are defined by

$$t^2 = \mu\nu, \quad x^2 = (1-\mu)(1-\nu); \quad 0 < \nu < \mu < 1. \tag{3}$$

μ and ν label ellipses of one and the same confocal family with mutually orthogonal intersections, given by the equations

$$\frac{t^2}{\mu} + \frac{x^2}{1-\mu} = 1. \tag{4}$$

This system is defined in the square $|t| + |x| < 1$. Like in the other four systems with only one family of coordinate lines each line contains both spacelike and timelike parts.

3. Hyperbolic system (H1): Defined by the same equations as E, but with $1 < \nu < \mu < \infty$, so that (4) describes one family of hyperbolas. H1 is the continuation of E to other domains in space-time, it may be defined in 4 wedges os space-time, $|t| - |x| > 1$ or $|x| - |t| > 1$. Two of its patches together with the finite domain of E are shown in the figure.

4. Hyperbolic system (H2): μ and ν are defined by

$$x^2 - t^2 = \mu + \nu, \quad tx = \mu\nu - \tfrac{1}{4}; \quad 0 < \nu < \mu < \infty. \tag{5}$$

There is one family of hyperbolas covering a half-space, $t + x > 1$,

$$\mu t^2 + tx - \mu x^2 = \mu^2 - \tfrac{1}{4}, \tag{6}$$

rotated against the t- and the x-axes in dependence of the parameter μ of the family. Figures of this and the following coordinate systems may be found in [4].

5. Hyperbolic system (H3): These coordinates are defined by

$$t^2 = -\mu\nu, \quad x^2 = -(\mu+1)(\nu+1); \quad -\infty < \nu < -1,\ 0 < \mu < \infty. \tag{7}$$

There are two families of hyperbolas. μ labels the spacelike one,

$$\frac{t^2}{\mu} - \frac{x^2}{\mu+1} = 1, \tag{8}$$

the timelike one is obtained by inserting the (negative) variable ν into (8). This coordinate system covers the whole Minkowski space.

6. Hyperbolic system (H4): The definition of the next two systems of rotated hyperbolas are very similar, here it is

$$t - x = \sqrt{\mu\nu}, \quad t^2 - x^2 = \tfrac{1}{2}(\mu+\nu); \quad 0 < \nu < \mu < \infty. \tag{9}$$

The hyperbolas of this one family covering a wedge $t + x > 1$, $t > x$ are given by

$$(2\mu - 1)t^2 + 2tx - (2\mu+1)x^2 = \mu^2. \tag{10}$$

7. Hyperbolic system (H5): The difference to H4 is the range of ν, $-\infty < \nu < 0 < \mu < \infty$, therefore $t - x = \sqrt{-\mu\nu}$. Now μ and ν describe 2 different families of hyperbolas given by

$$(2\mu+1)t^2 - 2tx - (2\mu-1)x^2 = \mu^2 \tag{11}$$

and the same equation with μ repaced by ν. Their domain is a half-space $t + x > 0$.

8. Hyperbolic system (H6) – the Rindler coordinate system: These coordinates are more traditionally denoted by τ and r,

$$t = r \sinh\tau, \quad x = r\cosh\tau; \quad 0 < r < \infty, \quad -\infty < \tau < \infty. \tag{12}$$

The lines $\tau = \text{const.}$ are straight spacelike lines, $r = \text{const.}$ denotes timelike hyperbolas, the coordinate domain is one of the "Rindler wedges", usually $x > |t|$ is chosen.

9. Parabolic system (P1): This system is defined by

$$x - t = 2(\mu+\nu), \quad x + t = -\tfrac{1}{4}(\mu-\nu)^2; \quad -\infty < \nu < \mu < \infty. \tag{13}$$

The coordinate lines are arranged in 1 family of parabolas with lightlike axes, translated against each other,

$$x + t = \frac{1}{4}\left(2\mu - \frac{x-t}{2}\right)^2. \tag{14}$$

This system covers a half-space $t + x > 0$.

10. Parabolic system (P2): The second parabolic system is defined by

$$t = \tfrac{1}{2}(\mu^2 + \nu^2), \quad x = \mu\nu; \quad -\infty < \nu < \mu < \infty. \tag{15}$$

There is one family of parabolas in a wedge like H6 with the equations

$$x^2 - 2\mu^2 t + \mu^4 = 0. \tag{16}$$

With one exception (H3) the curvilinear systems do not cover the whole Minkowski space, their domains are bounded by lightlike horizons, whose connection to the confocality of the coordinate lines were studied in [5]. An exceptional system is the Rindler system with its time translation symmetry, as the vector field $\partial/\partial\tau$ is a Killing vector field. From the point of view of inertial coordinates Rindler time translation is a boost transformation. Beside the Cartesian system the Rindler system is the only one where coordinate lines are Killing trajectories.

3 Stäckel systems

As it was shown by Stäckel [6] for the Hamilton-Jacobi- and later by Eisenhart [7] for the Laplace equation, to any separable coordinate system in n-dimensional space there is associated a Stäckel system, that is a linear space of n symmetric Killing tensors of order two, including the metric tensor. Killing tensors are characterized by the vanishing of their symmetrized first covariant derivatives,

$$K_{(ij;l)} = 0. \tag{17}$$

For any pair of Killing tensors of a Stäckel system the Nijenhuis-Schouten bracket vanishes, in its contravariant version this is

$$[K, L]^{ijk} = K^{ij}{}_{,l} L^{lk} - L^{ij}{}_{,l} K^{lk} = 0. \tag{18}$$

This property is equivalent to the existence of n constants of motion in the case of physical systems with a separable Hamilton-Jacobi equation [8],

$$\frac{1}{2} g^{ik} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^k} + V(x^i) = \text{const.} \tag{19}$$

The connection between separability and Stäckel systems is the fact that the tangent vectors to separable coordinate lines are the common eigenvectors of n linearly independent Killing tensors.

Given the diagonal (contravariant) metric $g^{ii}(x)$ in terms of orthogonal separable coordinates there is a straightforward method to determine $n-1$ other Killing tensors, K^{ij}, which of course must be diagonal, too. This method can be found, for example, in [8], where extensive studies about Stäckel systems in connection with geodesic Hamilton-Jacobi equations are performed. In these equations the contravariant metric plays a preferred rôle. If the matrix elements of a desired Killing tensor K are written as

$$K^{ii} = \rho_i \, g^{ii} \tag{20}$$

ρ_i may be obtained as solutions of the following set of partial differential equations

$$\partial_i \, \rho_j = (\rho_i - \rho_j) \, \partial_i \ln |g^{jj}|. \tag{21}$$

In [9] it was shown that the solutions of these equations are "elementary symmetric functions" in the coordinate variables.

4 A relation between Killing tensors and horizons

Concerning the problems mentioned in the introdution, on a geometric level it was possible to establish a close connection between local (Killing tensor fields) and global (coordinate horizons) aspects of separable coordinate systems. The basic idea is the interchange of the rôles of the flat metric tensor g^{ik} and another Killing tensor k^{ik} of the associated Stäckel system.

One obtains a curved space with metric k^{ik}, or k_{ik}, respectively, where the wave equation is separable again. Both metrics g_{ik} and k_{ik} can be applied (locally) to the same underlying manifold, albeit the curved one may cause curvature singularities, so that the extensions of the two metric spaces may be different. The hope is that the completely artificial horizons of separating coordinate patches in flat space would be more natural in the curved space-time domains generated in this way. The rest of this work will be dealing with their causal structure (null geodesics and curvature singularities).

For establishing the above relations the metric and another Killing tensor are expressed explicitely in terms of the curvilinear separable coordinates. In the elliptic and the hyperbolic systems (with the exception of H6) the line element has the form

$$\mathrm{d}s^2 = \frac{\mu-\nu}{4}\left(\frac{\mathrm{d}\mu^2}{(\mu-a)(\mu-b)} - \frac{\mathrm{d}\nu^2}{(\nu-a)(\nu-b)}\right). \tag{22}$$

The coordinate systems are distinguished by particular values of a and b ($= 0$, ± 1, or complex conjugate). Horizons occur when $\mu = \nu$, that is when $\mathrm{d}s^2 = 0$ – the coordinate system becomes degenerate there.

A particular Killing tensor obtained by means of (21) is given by the formula

$$K^{ik} = \begin{pmatrix} \nu\, g^{00} & 0 \\ 0 & \mu\, g^{11} \end{pmatrix}, \tag{23}$$

arbitrary Killing tensors of the Stäckel systems under consideration are linear combinations of g and K, in the following

$$k^{ik} = K^{ik} - c\, g^{ik} \tag{24}$$

will be used.

Lightlike covectors v_i (in the sense of the flat metric) are characterized by

$$g^{00}\, v_0^2 + g^{11}\, v_1^2 = 0; \tag{25}$$

null covectors in the sense of the metric k^{ik} by

$$(\nu - c)\, g^{00}\, v_0^2 + (\mu - c)\, g^{11}\, v_1^2 = 0. \tag{26}$$

So nonzero null vectors in both senses occur only in the limit $\mu \to \nu$, that is, on the coordinate horizons.

In the exceptional case of Rindler coordinates the line element is

$$\mathrm{d}s^2 = r^2\,\mathrm{d}\tau^2 - \mathrm{d}r^2 \tag{27}$$

and a one-parameter family of Killing tensors $K^{ik} - c\,g^{ik}$ is given by

$$k^{ik} = \begin{pmatrix} 1 - \frac{c}{r^2} & 0 \\ 0 & c \end{pmatrix}, \qquad c \in \mathbf{R}. \tag{28}$$

(Here the tensor K^{ik} $(c = 0)$ corresponding to (23) is singular.) Again, common null vectors can appear only when k^{ik} becomes proportional to $g^{ik} = \mathrm{diag}(1/r^2, -1)$, this is the case in the limit $r \to 0$, on the horizon. Here in both metrics g^{00} becomes singular and null vectors approach $(0,1)$.

For the parabolic coordinate systems the metric is explicitely conformal to the Minkowski metric η_{ik},

$$\mathsf{P1}: \qquad g_{ik} = (\mu - \nu)\,\eta_{ik}, \qquad k^{ik} = \frac{1}{\mu - \nu}\begin{pmatrix} \nu - c & 0 \\ 0 & -\mu + c \end{pmatrix}; \tag{29}$$

$$\mathsf{P2}: \qquad g_{ik} = (\mu^2 - \nu^2)\,\eta_{ik}, \qquad k^{ik} = \frac{1}{\mu^2 - \nu^2}\begin{pmatrix} \nu^2 - c & 0 \\ 0 & -\mu^2 + c \end{pmatrix}. \tag{30}$$

Again common null covectors can occur when $\mu \to \nu$ (P1), respectively $\mu \to \pm\nu$ (P2), that is on the horizons in both cases.

Conclusion: Horizons of separable coordinate systems for the Klein-Gordon equation in two-dimensional flat space-time are generated by the common null vectors of the metric tensor and another Killing tensor of the associated Stäckel system.

A generalization of this fact to higher dimensions is very probable – some examples in 3 dimensions have already been studied [10].

5 Curved space-times generated by Killing tensors

For all the nine curvilinear coordinate systems considered here it may be seen by separating the geodesic Hamilton-Jacobi equation ((19) with $V = 0$) that the resulting kinds of ordinary differential equations are the same for g^{ik} and $k^{ik} - c$ enters only into the separation constants – so that one may conclude that in all the cases geodesics are the same curves as in flat space, namely straight lines in (t, x) - space.

This may be checked explicitely by calculating null directions for any k^{ik}. For the elliptic coordinate system this will be done here. The Killing metric

$$k^{ik} = \frac{4}{\mu - \nu}\begin{pmatrix} \mu(1-\mu)(\nu - c) & 0 \\ 0 & -\nu(1-\nu)(\mu - c) \end{pmatrix} \tag{31}$$

has the following null covectors

$$v_i = \left(\sqrt{\nu(1-\nu)(\mu - c)},\ \pm\sqrt{\mu(1-\mu)(\nu - c)}\right). \tag{32}$$

To compare null lines with those of flat space-time it is suitable to transform v to cartesian coordinates, which results in

$$v^i(t,x) = \left(t^2 - c,\ tx \pm \sqrt{t^2 - c(t^2 - x^2 + 1) + c^2}\right) \tag{33}$$

for the directions of null vector fields. For them the equation

$$v^i{}_{,j}\, v^j \propto v^i \tag{34}$$

holds, so their integral curves are straight lines.

Moreover, their enveloping curve is an ellipse of the family (4) of coordinate lines belonging to the value $\mu = c$, or $\nu = c$, respectively, provided c lies among the coordinate values, $0 < c < 1$.

Concerning curvature, the Ricci scalar of $k_{ik} := (k^{ik})^{-1}$ is

$$R = 2\frac{(\mu\nu)^2 - 2(1-c)\mu\nu(\mu+\nu) - c(\mu+\nu)^2 + 4c^2(\mu+\nu) + 2c(3-4c)\mu\nu - 6c^3 + 3c^4}{(\mu - c)^2\,(\nu - c)^2}, \tag{35}$$

it has a singularity at $\mu = c$ or $\nu = c$. From the determinant of the metric,

$$\det(k_{ik}) = -\frac{(\mu - \nu)^2}{16\mu\nu(1-\mu)(1-\nu)(\mu - c)(\nu - c)} \tag{36}$$

it is seen that the metric is indefinite either for both μ and ν being less or greater than c, and definite for $\nu < c < \mu$. So, when k_{ik} is employed as metric tensor, the ellipse labelled by the paramter value c divides the domain covered by elliptic coordinates into four portions with Lorentzian metrics separated by one with a Euclidean one – the interior of this ellipse (see figure). The tangents of this ellipse are the null geodesics of the pseudoriemannian spaces – like the triangle (A, B, C) – between it and the horizons, which are tangent to all ellipses and hyperbolas and distinguished by dashed lines. For other values of c, $c < 0$ or $c > 1$, the singularity appears along one of the hyperbolas of H1.

Concerning the other coordinate systems the situation is analogous: When the parameter c lies in the range of the coordinate values, there appears one coordinate line dividing the space into a part "inside" with a Euclidean metric, and parts with Lorentzian metrics bounded by this line and the horizons. So horizons may be characterized also as enveloping surfaces of curvature singularities of one-parameter families of metrics with varying c's defined on the same manifold. They are reminiscent of black hole horizons; the main difference is that the singularities have null tangents. For a physical interpretation of the curved spaces further investigations will be necessary; at least for the example associated with H6 such an interpretation in the framework of two-dimensional gravity [11] seems to be available quite straightforwardly.

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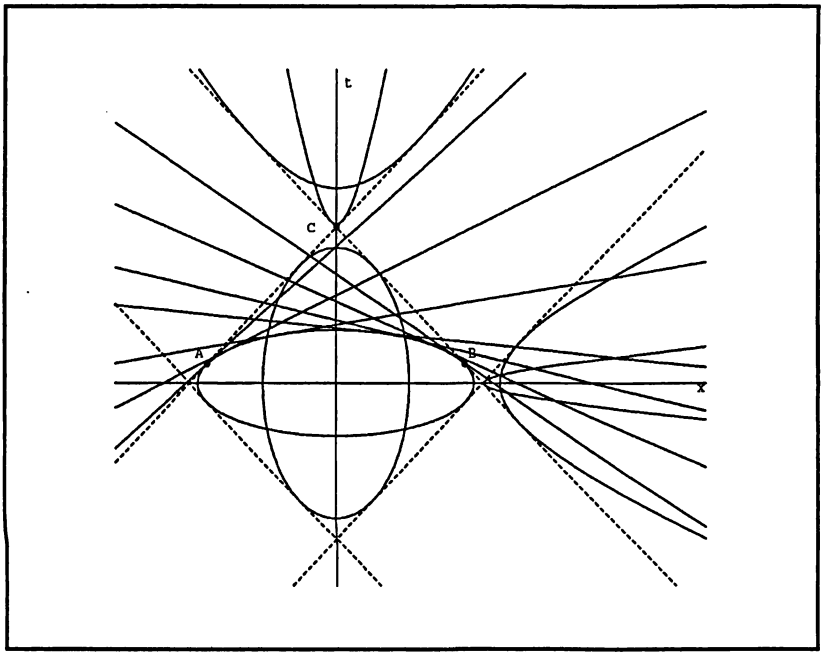

Figure: Domains of E with two ellipses and H1 with two hyperbolas each in two wedges. The straight lines are null geodesics of a curved space generated from E. The interior of their enveloping ellipse has a positive-definite metric.

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