Jan Rataj

The iterated version of a translative integral formula for sets of positive reach

In: Jan Slovák (ed.): Proceedings of the 16th Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1997. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 46. pp. [129]–138.

Persistent URL: http://dml.cz/dmlcz/701602

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 46 (1997), pp. 129-138

The iterated version of a translative integral formula for sets of positive reach*

Jan Rataj

Abstract

The technique of rectifiable currents is used to prove an integral formula expressing the curvature measure of intersection of q sets of positive reach integrated over all translations of the sets. The formula involves s.c. mixed curvature measures of sets of positive reach.

1 Introduction

The translative version of the principal kinematic formula of integral geometry has been proved recently for pairs of sets X, Y of positive reach in $\mathbb{R}^d$ in the form

$$\int_{R^d} C_k\Big(X \cap (Y+z), \big(A \cap (B+z)\big) \times G\Big) dz = \sum_{\substack{0 \le r,s \le d \\ r+s=d+k}} C_{r,s}(X, Y; A \times B \times G),$$

(see [4]), where $C_k(Z, \cdot)$ is the (generalized) curvature measure of the set $Z \subseteq \mathbb{R}^d$ of order $0 \le k \le d-1$ (a locally finite signed Borel measure concentrated on the unit normal bundle $\operatorname{nor} Z$ of Z, see [7]) and $C_{r,s}(X, Y, \cdot)$ is the mixed curvature measure of the sets X, Y and order r, s. The formula was proved first for convex bodies and 'ordinary' curvature measures by Schneider & Weil [5]. Weil [6] has proved an iterative version of this formula for convex bodies; this formula considers the curvature measures of the intersection of a general finite number (q) of bodies. He has introduced mixed curvature measures for q-tuples of convex polyhedra and extended this notion by continuity w.r.t. Hausdorff metric to q-tuples of convex bodies.

In this paper we give a proof of the iterated version of the principal kinematic formula for q-tuples of sets of positive reach and generalized curvature measures. The mixed curvature measures are introduced by means of rectifiable currents supported by the 'joint unit normal bundle' of the sets considered and the proof is based on the technique of geometric measure theory. The formula is proved under un additional condition (4) requiring, roughly speaking, that the Lebesque measure of translations $(z_2, \dots, z_q)$ for which X_1 'touches' $(X_2+z_2) \cap \cdots \cap (X_q+z_q)$ is zero (known examples of sets of positive reach violating this condition - see [3] - are quite intricate).

*The paper is in final form and no version of it will be submitted elsewhere.

The research has been supported by the Grant Agency of the Czech Republic, Project No. 201/96/0226.

The mixed curvature measures can be also represented - similarly as in the case of two bodies ([4, Section 4]) - as integrals of principal curvatures over the product of unit normal bundles. This can provide a deeper insight into the structure of mixed curvatures and will be shown elsewhere.

It has been shown in [6] that the iterated version of the principal kinematic formula and its translative version have important applications in stochastic geometry, e.g. it gives new relations for stationary Poisson processes of particles.

2 Preliminaries

Throughout the paper, the notation of [2] will be used for the basic notions of geometric measure theory. In particular, $\bigwedge_k V$, $\bigwedge^k V$ is the space of k-vectors, k-covectors in an Euclidean space V, respectively, $\langle\tau,\phi\rangle$ denotes the bilinear pairing ($\tau\in\bigwedge_k V$ and $\phi\in\bigwedge^k V$), $\Omega=e'_1\wedge\cdots\wedge e'_d\in\bigwedge^d\mathbb{R}^d$ is the volume d-form, $\{e'_1,\dots e'_d\}$ being the dual basis to the canonical orthogonal basis $\{e_1,\dots e_d\}$ of $\mathbb{R}^d$, and Ω^p the corresponding volume dp-form in $(\mathbb{R}^d)^p$. The induced multilinear mapping $\bigwedge_k L$ of a linear mapping $L:\mathbb{R}^m\to\mathbb{R}^n$ is defined by $\left(\bigwedge_k L\right)(u_1\wedge\cdots\wedge u_k)=L(u_1)\wedge\cdots\wedge L(u_k)$. For an open subset $U\in V$, $\mathcal{D}^k(U)$ is the set of all (differential) k-forms and $\mathcal{D}_k(U)$ the set of all k-currents on U. H^k is the k-dimensional Hausdorff measure and L^m the m-dimensional Lebesgue measure. For any finite family of vectors $u_1,\dots,u_k$ in V, we denote by $\mathrm{Cone}(u_1,\dots,u_k)=\{c_1u_1+\cdots+c_ku_k : c_j>0\}$ the positive cone spanned by $u_1,\dots,u_k$.

p-product of multivectors. Let p be a natural number and $r_1,\dots,r_p$ integers with $0\le r_i\le d$ and $r_1+\cdots+r_p=(p-1)d$. Let $\alpha_i\in\bigwedge_{r_i}(\mathbb{R}^d)$, $i=1,\dots,p$ be unit simple multivectors. Suppose that $\alpha_i=+1$ if $r_i=d$. Clearly there exist positively oriented orthonormal bases $\{a^i_1,\dots,a^i_d\}$ of $\mathbb{R}^d$ such that

$$\alpha_i=a^i_1\wedge\cdots\wedge a^i_{r_i},\quad 1\le i\le p. \tag{1}$$

We define the p-product as

$$[\alpha_1,\dots,\alpha_p]=s_1\cdots s_p\left\langle\bigwedge_{i=1}^{p}\bigwedge_{j=r_i+1}^{d}a^i_j,\Omega\right\rangle$$

(note that the definition is correct, though the basis elements are not determined uniquely). The p-product can be extended by linearity in each component to general p multivectors with sum od multiplicities giving $(p-1)d$.

Denote by $L(\alpha)$ the linear subspace of $\mathbb{R}^d$ associated with a simple multivector α. For $i=1,\dots,p$ we have

$$\dim\left(L(\alpha_1)\cap\cdots\cap L(\alpha_i)\right)\ge k_i$$

with $k_i=r_1+\cdots+r_i-(i-1)d$. Hence, generating the bases elements a^i_j consecutively for the intersection spaces, we can ensure that

$$a^i_j=a^1_j \text{ for all } 1\le j\le k_i,\ 1\le i\le p \tag{2}$$

Lemma 1 *Under* (1) *and* (2) *we have*

$$[\alpha_1, \dots, \alpha_p] = \prod_{i=2}^{p} \left\langle \bigwedge_{j=1}^{k_{i-1}} a_j^1 \wedge \bigwedge_{j=k_i+1}^{r_i} a_j^i, \Omega \right\rangle .$$

Proof: The result will be proved by induction. If $p = 2$, the assertion follows from the well known identity for two positively oriented orthonormal bases

$$\left\langle \bigwedge_1^{r_1} a_j^1 \wedge \bigwedge_1^{d-r_1} a_j^2, \Omega \right\rangle = \left\langle \bigwedge_{r_1+1}^{d} a_j^1 \wedge \bigwedge_{d-r_1+1}^{d} a_j^2, \Omega \right\rangle .$$

Let now $\alpha_1, \dots, \alpha_p$ be multivectors satisfying (1) and (2) and denote $\beta = a_1^1 \wedge \cdots \wedge a_1^{k_2}$. We have by definition

$$[\beta, \alpha_3, \dots, \alpha_p] = \left\langle \bigwedge_{k_2+1}^{d} a_j^i \wedge \bigwedge_{i=3}^{p} \bigwedge_{j=r_i+1}^{d} a_j^i, \Omega \right\rangle .$$

Consider the identity

$$\bigwedge_{r_1+1}^{d} a_j^1 \wedge \bigwedge_{r_2+1}^{d} a_j^2 = s_2 \left\langle \bigwedge_1^{r_1} a_j^1 \wedge \bigwedge_{k_2+1}^{r_2} a_j^2, \Omega \right\rangle \bigwedge_{k_2+1}^{d} a_j^1 .$$

It follows from the fact that both simple multivectors $\bigwedge_{r_1+1}^d a_j^1 \wedge \bigwedge_{r_2+1}^d a_j^2$ and $\bigwedge_{k_2+1}^d a_j^1$ are associated with the orthogonal complement of $L(\alpha_1) \cap L(\alpha_2)$, hence the differ only by a real multiple which can be verified e.g. by the exterior multiplication of both of them by $\bigwedge_1^{k_2} a_j^1$. We thus obtain

$$[\alpha_1, \dots, \alpha_p] = \left\langle \bigwedge_{i=1}^{p} \bigwedge_{j=r_i+1}^{d} a_j^i, \Omega \right\rangle = \left\langle \bigwedge_1^{r_1} a_j^1 \wedge \bigwedge_{k_2+1}^{r_2} a_j^2, \Omega \right\rangle [\beta, \alpha_3, \dots, \alpha_p]$$

and, using the induction assumption, the assertion follows.

Definition: For $q \in \mathbb{N}$ and integers $0 \le r_i \le d$, $1 \le i \le q$, we define the differential forms $\varphi_{r_1,\dots,r_q} \in \mathcal{D}^{qd-1}(\mathbb{R}^{(q+1)d})$ by

$$\begin{aligned} \left\langle \bigwedge_{j=1}^{qd-1} (a_j^1, \dots, a_j^{q+1}), \varphi_{r_1,\dots,r_q}(x_1, \dots, x_q, u) \right\rangle &= \mathcal{O}_{r_{q+1}}^{-1} (-1)^{(q-1)dR_q + \sum_{i=1}^{q} (d-r_i)(R_i - r_i)} \\ &\times \sum_{\sigma \in Sh(r_1,\dots,r_{q+1})} \operatorname{sgn} \sigma \left[\bigwedge_1^{R_1} a_{\sigma_j}^1, \bigwedge_{R_1+1}^{R_2} a_{\sigma_j}^2, \dots, \bigwedge_{R_{q-1}+1}^{R_q} a_{\sigma_j}^q, \bigwedge_{R_q+1}^{qd-1} a_{\sigma_j}^{q+1} \wedge u \right], \end{aligned}$$

where $r_{q+1} = qd-1-r_1-\cdots-r_q$, $R_i = r_1+\cdots+r_i$, $\mathcal{O}_m = H^m(\mathbb{S}^m)$ and $Sh(r_1, \dots, r_{q+1})$ is the set of all permutations of $\{1, \dots, qd-1\}$ which are increasing on the subsets $\{1, \dots, R_1\}$, $\{R_1+1, \dots, R_2\}$, ..., $\{R_q+1, \dots, qd-1\}$. Since $\varphi_{r_1,\dots,r_q}(x_1, \dots, x_q, u)$ depends only on the last vector component u, we shall use the notation $\varphi_{r_1,\dots,r_q}(u)$.

Note that for $q = 1$, $\varphi_k(u)$ is the k-th Lipschitz-Killing curvature form [7]. For $\varepsilon > 0$ we also define the form $\psi_\varepsilon^{(q)} \in \mathcal{D}^{qd-1}(\mathbb{R}^{(q+1)d})$ by

$$\psi_\varepsilon^{(q)}(u) = \sum_{\substack{0\le r_1,\dots,r_q\le d\\ r_1+\dots+r_q\ge(q-1)d}} \varepsilon^{r_{q+1}}\varphi_{r_1,\dots,r_q}(u).$$

In the sequel we shall use the linear mapping $G:\ (\mathbb{R}^d)^{q+1}\to(\mathbb{R}^d)^{q-1}$

$$G(x_1,\dots,x_q,u) = (x_1 - x_2,\dots,x_1 - x_q)$$

and the projection $\pi:\ (\mathbb{R}^d)^{q+1}\to(\mathbb{R}^d)^2$

$$\pi(x_1,\dots,x_q,u) = (x_1,u).$$

Lemma 2 *For any $q\ge 2$, $0\le k\le d-1$ and $\varepsilon>0$ we have*

$$G^\#\Omega^{q-1}\wedge\pi^\#\varphi_k = \sum_{\substack{0\le\rho_1,\dots,\rho_q\le d\\ \rho_1+\dots+\rho_q=(q-1)d+k}} \varphi_{\rho_1,\dots,\rho_q}$$

and

$$G^\#\Omega^{q-1}\wedge\pi^\#\psi_\varepsilon^{(1)} = \sum_{\substack{0\le\rho_1,\dots,\rho_q\le d\\ \rho_1+\dots+\rho_q=(q-1)d+k}} \psi_\varepsilon^{(q)}.$$

Proof: The second statement is a consequence of the first one, which is to be proved. First, note that the simple multivectors

$$\tau = \bigwedge_{i=1}^{q+1}\bigwedge_{j=1}^{r_i}(\underbrace{o,\dots,o}_{(i-1)\times},a_j^i,\underbrace{o,\dots,o}_{(q-i+1)\times})$$

with $1\le r_i\le d, r_1+\dots+r_{q+1}=qd-1$ and $a_j^i\in\mathbb{R}^d$ form a basis of $\bigwedge_{qd-1}(\mathbb{R}^{(q+1)d})$. Moreover, the vectors a_j^i can be taken from positively oriented orthonormal bases of $\mathbb{R}^d$ $\{a_1^i,\dots,a_d^i\}$ and we can assume that (2) holds. We shall show that

$$\langle\tau, G^\#\Omega^{q-1}\wedge\pi^\#\varphi_k\rangle = \left\langle\tau, \sum_{\substack{0\le\rho_1,\dots,\rho_q\le d\\ \rho_1+\dots+\rho_q=(q-1)d+k}} \varphi_{\rho_1,\dots,\rho_q}\right\rangle. \tag{3}$$

If $r_1+\ldots+r_q\ne(q-1)d+k$, both sides of (3) vanish. We thus limit ourselves to the case $r_1+\ldots+r_q=(q-1)d+k$. Then the right hand side of (3) equals just $\langle\tau,\varphi_{r_1,\dots,r_q}\rangle$ (all other summands vanish). In the following computations we make use of the fact that the sign of the permutation changing the mutual position of two neighbour blocs of i and j elements is ij.

$$
\begin{aligned}
&\langle \tau, G^{\#}\Omega^{q-1} \wedge \pi^{\#}\varphi_k\rangle \\
&= (-1)^{k(q-1)d} \left\langle \bigwedge_{k_q+1}^{r_1} (a_j^1, o, \dots, o) \wedge \bigwedge_{i=2}^{q} \bigwedge_{j=1}^{r_i} (o, \dots, a_j^i, \dots, o), G^{\#}\Omega^{q-1} \right\rangle \\
&\qquad \times \left\langle \bigwedge_1^k (a_j^1, o, \dots, o) \wedge \bigwedge_{j=1}^{r_{q+1}} (o, \dots, o, a_j^{q+1}), \pi^{\#}\varphi_k \right\rangle \\
&= (-1)^{k(q-1)d} \left\langle \bigwedge\nolimits_{(q-1)d}(DG) \left(\bigwedge_{k_q+1}^{r_1} (a_j^1, o, \dots, o) \wedge \bigwedge_{i=2}^{q} \bigwedge_{j=1}^{r_i} (o, \dots, a_j^i, \dots, o) \right), \Omega^{q-1} \right\rangle \\
&\qquad \times \left\langle \bigwedge_1^k (a_j^1, o) \wedge \bigwedge_{j=1}^{r_{q+1}} (o, a_j^{q+1}), \varphi_k \right\rangle \\
&= (-1)^{k(q-1)d} \left\langle \bigwedge_{k+1}^{r_1} (a_j^1, \dots, a_j^1) \wedge \bigwedge_{i=2}^{q} \bigwedge_{j=1}^{r_i} (\underbrace{o, \dots, o}_{(i-2)\times}, -a_j^i, \underbrace{o, \dots, o}_{(q-i)\times}), \Omega^{q-1} \right\rangle \\
&\qquad \times \left\langle \bigwedge_1^k a_j^1 \wedge \bigwedge_{j=1}^{r_{q+1}} a_j^{q+1}, \Omega \right\rangle \\
&= (-1)^{k(q-1)d}(-1)^{r_2+\dots+r_q}(-1)^{\sum_{i=2}^{q}(i-2)d(d-r_i)} \\
&\qquad \times \prod_{i=2}^{q-1} \left\langle \bigwedge_{j=k_i+1}^{k_{i-1}} a_j^1 \wedge \bigwedge_{j=1}^{r_i} a_j^i, \Omega \right\rangle \left\langle \bigwedge_{j=1}^{k} a_j^1 \wedge \bigwedge_{j=1}^{r_{q+1}} a_j^{q+1} \wedge u, \Omega \right\rangle \\
&= (-1)^{k(q-1)d}(-1)^{r_2+\dots+r_q}(-1)^{\sum_{i=2}^{q}(i-2)d(d-r_i)}(-1)^{\sum_{i=2}^{q} k_i(d-r_i)} \\
&\qquad \times \prod_{i=2}^{q-1} \left\langle \bigwedge_{j=1}^{k_{i-1}} a_j^1 \wedge \bigwedge_{j=k_i+1}^{r_i} a_j^i, \Omega \right\rangle \left\langle \bigwedge_{j=1}^{k} a_j^1 \wedge \bigwedge_{j=1}^{r_{q+1}} a_j^{q+1} \wedge u, \Omega \right\rangle .
\end{aligned}
$$

Equation (3) follows now from Lemma 1, since

$$
\begin{aligned}
(-1)^{k(q-1)d+\sum_{i=2}^{q}(k_i(d-r_i)+(i-2)d(d-r_i)+r_i)} &= (-1)^{(k-1)(q-1)d+\sum_{i=2}^{q}(d-r_i)(k_i-(i-2)d-1)} \\
&= (-1)^{(q-1)dR_q+\sum_{i=1}^{q}(d-r_i)(R_i-r_i)}.
\end{aligned}
$$

3 Mixed curvature measures

Definitions. Let $q \in \mathbb{N}$ and $X_1, \dots, X_q \in \mathbb{R}^d$ be sets of positive reach. We define the *joint unit normal bundle* $\operatorname{nor}(X_1, \dots, X_q)$ by

$$
\begin{aligned}
&\operatorname{nor}(X_1, \dots, X_q) \\
&= \{(x_1, \dots, x_q, u) \in \mathbb{R}^{qd} \times \mathbb{S}^{d-1} : \exists (x_i, m_i) \in \operatorname{nor} X_i, u \in \operatorname{Cone}(m_1, \dots, m_q)\}.
\end{aligned}
$$

The set $\operatorname{nor}(X_1, \dots, X_q)$ is countably H^{qd-1}-rectifiable, since it is a Lipschitz image of $\operatorname{nor} X_1 \times \dots \times \operatorname{nor} X_q \times \mathbb{S}^{d-1}$. Hence, we can introduce the rectifiable current

$$
N_{X_1,\dots,X_q} = \left(H^{qd-1} \llcorner \operatorname{nor}(X_1, \dots, X_q)\right) \wedge a_{X_1,\dots,X_q},
$$

where $a_{X_1,\dots,X_q}$ is the unit simple $(qd-1)$-vectorfield associated with $\mathrm{nor}(X_1,\dots,X_q)$ with orientation given by

$$\langle a_{X_1,\dots,X_q}, \psi_\varepsilon^{(q)}\rangle > 0 \text{ for } \varepsilon < \min_i \operatorname{reach} X_i.$$

Given integers $0 \le r_1,\dots,r_q \le d-1$ with $r_1+\dots+r_q \ge (q-1)d$, we define the *mixed curvature measure* of $X_1,\dots,X_q$ and order $r_1,\dots,r_q$ as

$$C_{r_1,\dots,r_q}(X_1,\dots,X_q;A) = N_{X_1,\dots,X_q}(\mathbf{1}_A\varphi_{r_1,\dots,r_q}).$$

Proposition 1 *The mixed curvature measures have the following properties:*

(a) $C_{r_1,\dots,r_q}(X_1,\dots,X_q;\cdot)$ *is a signed Radon measure on* $\mathbb{R}^{(q+1)d}$ *supported by* $\partial X_1 \times \partial X_q \times \mathbb{S}^{d-1}$.*;*

(b) *homogeneity: for* $c_1,\dots c_q > 0$,

$$C_{r_1,\dots,r_q}(c_1X_1,\dots,c_qX_q;c_1A_1\times\dots\times c_qA_q\times B) = c_1^{r_1}\cdots c_q^{r_q}C_{r_1,\dots,r_q}(X_1,\dots,X_q;A_1\times\dots\times A_q\times B);$$

(c) *symmetry: for any permutation* σ *of* $\{1,\dots,q\}$,

$$C_{r_{\sigma(1)},\dots,r_{\sigma(q)}}(X_{\sigma(1)},\dots,X_{\sigma(q)};A_{\sigma(1)}\times\dots\times A_{\sigma(q)}\times B) = C_{r_1,\dots,r_q}(X_1,\dots,X_q;A_1\times\dots\times A_q\times B).$$

Proof: Statements (a) and (b) are obvious, let shall show (c). Denoting by ω the mapping

$$(x_1,\dots,x_q,u)\mapsto(x_{\sigma(1)},\dots,x_{\sigma(q)},u),$$

we have $\mathrm{nor}(X_{\sigma(1)},\dots,X_{\sigma(q)}) = \omega(\mathrm{nor}(X_1,\dots,X_q))$, and there is a sign $s=\pm1$ with $\omega^\#\varphi_{r_{\sigma(1)},\dots,r_{\sigma(q)}} = s\omega^\#\varphi_{r_1,\dots,r_q}$. With the same sign it holds

$$a_{X_{\sigma(1)},\dots,X_{\sigma(q)}} = s\Big(\bigwedge\nolimits_{qd-1}\omega\Big)a_{X_1,\dots,X_q},$$

thus

$$N_{X_{\sigma(1)},\dots,X_{\sigma(q)}}(\mathbf{1}_{\omega A}\varphi_{r_{\sigma(1)},\dots,r_{\sigma(q)}}) = N_{X_1,\dots,X_q}(\mathbf{1}_A\varphi_{r_1,\dots,r_q}).$$

It is convenient to extend the definition of mixed curvature measures for order factors $r_i \le d$. It can be simply done by setting

$$C_{d,\dots,d,r_{k+1},\dots,r_q}(X_1,\dots,X_q;\cdot) = L_d\llcorner X_1\otimes\dots\otimes L^d\llcorner X_k\otimes C_{r_{k+1},\dots,r_q}(X_{k+1},\dots,X_q;\cdot)$$

for any $k=1,\dots,d$ and by using the symmetry property (c) from Proposition 1. It is clear that all statements of Proposition 1 remain valid, with the exception that $C_{r_1,\dots,r_q}(X_1,\dots,X_q;\cdot)$ is supported only by $X_1\times\dots\times X_q\times\mathbb{S}^{d-1}$. Note that in the case $q=1$, $C_k(X,\cdot)$ is the common (generalized) k-th curvature measure of X.

4 The translative formula

Definition: We shall say that the sets $X_1, \dots, X_q \subseteq \mathbb{R}^d$ of positive reach *lie in general position*, if there do not exist vectors $x, m_1, \dots, m_q$ with $(x, m_i) \in \operatorname{nor} X_i$, $1 \le i \le q$ and $o \in \operatorname{Cone}(m_1, \dots, m_q)$.

Lemma 3 *Let $X_1, \dots, X_q$ be sets of positive reach in $\mathbb{R}^d$.*

(a) If $(x_i, m_i) \in \operatorname{nor} X_i$ for $1 \le i \le q$ and $u \in \operatorname{Cone}(m_1, \dots, m_q) \cap \mathbb{S}^{d-1}$, then $(x, u) \in \operatorname{nor}(X_1 \cap \dots \cap X_q)$.

(b) *Suppose that $X_1, \dots, X_q \subseteq \mathbb{R}^d$ lie in general position. Then* $\operatorname{reach}(X_1 \cap \dots \cap X_q) > 0$ *and for any $x \in \partial X_1 \cap \dots \cap \partial X_q$,*

$$(x, u) \in \operatorname{nor}(X_1 \cap \dots \cap X_q) \textit{ iff } \exists (x, m_i) \in \operatorname{nor} X_i,\ u \in \operatorname{Cone}(m_1, \dots, m_q) \cap \mathbb{S}^{d-1}.$$

Proof: For $q = 2$ we can use directly [1, Theorem 4.10.(3)]. For general q, the result follows by induction, since, under the assumptions of the Lemma, the sets $X_1 \cap \dots \cap X_{q-1}$ and X_q satisfy the assumptions of [1, Theorem 4.10.].

For a given $(q-1)$-tuple of translations $z = (z_2, \dots, z_q) \in \mathbb{R}^{(q-1)d}$, we shall denote for brevity

$$\bar{X}(z) = X_1 \cap (X_2 + z_2) \cap \dots \cap (X_q + z_q),$$

$$\operatorname{nor}^*(z) = \{(x, u) \in \operatorname{nor} \bar{X}(z) : x \in \partial X_1 \cap \partial(X_2 + z_2) \cap \dots \cap \partial(X_q + z_q)\}$$

and

$$N_z^* = N_{\bar{X}(z)} \llcorner \mathbf{1}_{\operatorname{nor}^*(z)}.$$

Let us further introduce the mapping

$$\Gamma(x_1, \dots, x_q, u) = (x_1, x_1 - x_2, \dots, x_1 - x_q, u).$$

Lemma 4 *Suppose that for some $q \le d$ the sets $X_1, \dots, X_q \in \mathbb{R}^d$ of positive reach satisfy*

$$L^{(q-1)d}(\{z : X_1, X_2 + z_2, \dots, X_q + z_q \text{ do not lie in general position}\}) = 0. \quad (4)$$

Then for any $0 \le k \le q-1$ and for any nonnegative Borel measurable function h on $\mathbb{R} \times \mathbb{R}^{(q-1)d} \times \mathbb{R}$ with compact support we have

$$\int \left(N_z^* \llcorner h(\cdot, z, \cdot)\right)(\varphi_k) dz = \sum_{\substack{0 \le r_1, \dots, r_q \le d-1 \\ r_1 + \dots + r_q = (q-1)d + k}} \left(N_{X_1, \dots, X_q} \llcorner h \circ \Gamma\right) \varphi_{r_1, \dots, r_q}.$$

Proof: Slicing the current $N_{X_1,\dots,X_q}\llcorner h\circ\Gamma$ by the mapping G (see [2, §4.3.8] we obtain

$$\begin{aligned}\Big(N_{X_1,\dots,X_q}\llcorner h\circ\Gamma\Big)&(G^{\#}\Omega^{q-1}\wedge\pi^{\#}\varphi_k)\\ &= \int\langle N_{X_1,\dots,X_q}\llcorner h\circ\Gamma, G, z\rangle(\pi^{\#}\varphi_k)dz\\ &= \int\pi_{\#}\langle N_{X_1,\dots,X_q}\llcorner h\circ\Gamma, G, z\rangle(\varphi_k)dz.\end{aligned}$$

From Lemma 2 it follows that

$$\Big(N_{X_1,\dots,X_q}\llcorner h\circ\Gamma\Big)(G^{\#}\Omega^{q-1}\wedge\pi^{\#}\varphi_k) = \sum_{\substack{0\le r_1,\dots,r_q\le d-1\\ r_1+\dots+r_q=(q-1)d+k}}\Big(N_{X_1,\dots,X_q}\llcorner h\circ\Gamma\Big)\varphi_{r_1,\dots,r_q}$$

(remark that $\Big(N_{X_1,\dots,X_q}\llcorner h\circ\Gamma\Big)(\varphi_{r_1,\dots,r_q})=0$ if $r_i=d$ for some i). It is thus sufficient to show that

$$\pi_{\#}\langle N_{X_1,\dots,X_q}\llcorner h\circ\Gamma, G, z\rangle = N_z^*\llcorner h(\cdot,z,\cdot)$$

for $L^{(q-1)d}$-almost all z. Recall that

$$N_{X_1,\dots,X_q}\llcorner h\circ\Gamma = \Big(H^{qd-1}\llcorner\operatorname{nor}(X_1,\dots,X_q)\Big)\wedge(h\circ\Gamma)a_{X_1,\dots,X_q}.$$

Denoting the restriction $g=G\mid\operatorname{nor}(X_1,\dots,X_q)$ and using [2, §4.3.8], we get

$$\langle N_{X_1,\dots,X_q}\llcorner h\circ\Gamma, G, z\rangle = \Big(H^{d-1}\llcorner g^{-1}(z)\Big)\wedge(h\circ\Gamma)\tilde{a}_z,$$

where

$$\tilde{a}_z = \frac{a_{X_1,\dots,X_q}\lrcorner G^{\#}\Omega^{q-1}}{\operatorname{ap}J_{d-1}g}$$

is a unit simple $(d-1)$-vectorfield associated with $g^{-1}(z)$. Suppose now that z is such that $X_1, X_2+z_2,\dots,X_q+z_q$ lie in general position. Then, by Lemma 3 (b) we have $\pi(g^{-1}(z))=\operatorname{nor}^*(z)$ and using the 'area formula' for currents [2, §4.1.30], we obtain

$$\pi_{\#}\langle N_{X_1,\dots,X_q}\llcorner h\circ\Gamma, G, z\rangle = \Big(H^{d-1}\llcorner\operatorname{nor}^*(z)\Big)\wedge h(\cdot,z,\cdot)\hat{a}_z,$$

where

$$\hat{a}_z = \frac{\Big(\bigwedge_{d-1}\pi\Big)\tilde{a}_z}{\operatorname{ap}J_{d-1}\Big(\pi\mid g^{-1}(z)\Big)}\circ\Big(\pi\mid g^{-1}(z)\Big)^{-1}$$

is again a unit simple $(d-1)$-vectorfield associated with nor^*z (we use the simple fact that $\pi\mid g^{-1}(z)$ is one-to-one). According to the definition of N_z^*, it is sufficient to show that $a_{\bar{X}(z)}\mid\operatorname{nor}^*(z)=\hat{a}_z$. Since both are unit simple $(d-1)$-vectorfields associated with $\operatorname{nor}^*(z)$, they can differ only by sign. But, using the relations above we have

$$\begin{aligned}\langle\hat{a}_z,\psi_\varepsilon^{(1)}\rangle &= c_1\Big\langle\Big(\bigwedge_{d-1}\pi\Big)\tilde{a}_z,\psi_\varepsilon^{(1)}\Big\rangle\\ &= c_1\langle\tilde{a}_z,\pi^{\#}\psi_\varepsilon^{(1)}\rangle\\ &= c_2\langle a_{X_1,\dots,X_q}\lrcorner G^{\#}\Omega^{q-1},\pi^{\#}\psi_\varepsilon^{(1)}\rangle\\ &= c_2\langle a_{X_1,\dots,X_q}, G^{\#}\Omega^{q-1}\wedge\pi^{\#}\psi_\varepsilon^{(1)}\rangle\end{aligned}$$

with positive factors c_1, c_2. But $G^\# \Omega^{q-1} \wedge \pi^\# \psi_\varepsilon^{(1)} = \psi_\varepsilon^{(q)}$ by Lemma 2, hence the last expression is positive for small ε, which means that $a_{\bar{X}(z)} \mid \mathrm{nor}^*(z) = \hat{a}_z$ and the proof of the Lemma is complete.

Theorem 1 *Let $X_1, \ldots, X_q$ be a sequence sets of positive reach in $\mathbb{R}^d$ such that any its subsequence $X_{i_1}, \ldots, X_{i_p}$ of $p \le d$ sets fulfils the condition (4). Then for any $0 \le k \le d-1$ and for any nonnegative Borel measurable function h on $\mathbb{R}^{(q+1)d}$ with compact support we have*

$$\int \int h(x,z,u)\, C_k\Big(\bar{X}(z); d(x,u)\Big) dz$$
$$= \sum_{\substack{0 \le r_1, \ldots, r_q \le d \\ r_1 + \cdots + r_q = (q-1)d + k}} \int h(x, x - z_2, \ldots, x - z_q, u)\, C_{r_1, \ldots, r_q}\Big(X_1, \ldots, X_q; d(x_1, \ldots, x_q, u)\Big).$$

Proof: Consider the partition

$$\mathrm{nor}\, \bar{X}(z) = \bigcup_{I \subseteq \{1, \ldots, q\}} \mathrm{nor}_I \bar{X}(z),$$

where

$$\mathrm{nor}_I \bar{X}(z) = \mathrm{nor}\, \bar{X}(z) \cap \left(\bigcap_{i \in I} \partial(X_i + z_i) \times \mathbb{S}^{d-1} \right) \cap \left(\bigcap_{i \notin I} \mathrm{int}(X_i + z_i) \times \mathbb{S}^{d-1} \right)$$

(we set $z_1 = 0$ here). Note that

$$\mathrm{nor}_I \bar{X}(z) = \left(\bigcap_{i \notin I} \mathrm{int}(X_i + z_i) \times \mathbb{S}^{d-1} \right) \cap \mathrm{nor} \left(\bigcap_{i \in I} (X_i + z_i) \right)$$

and, consequently,

$$\int \int_{\mathrm{nor}_I \bar{X}(z)} h(z,x,u)\, C_k\Big(\bar{X}(z); d(x,u)\Big) dz$$
$$= \int \int_{\mathrm{nor}^*(z_I))} \int_{\bigcap_{i \notin I}(X_i + z_i)} h(x,z,u)\, dz_{I^C}\, C_k\Big(\bigcap_{i \notin I} (X_i + z_i), d(x,u) \Big) dz_I,$$

where $z_I = (z_i : i \in I)$, $z_{I^C} = (z_i : z \in I^C)$ and

$$\mathrm{nor}^*(z_I) = \left\{ (x,u) \in \mathrm{nor} \left(\bigcap_{i \in I} (X_i + z_i) \right) : x \in \bigcap_{i \in I} \partial(X_i + z_i) \right\}.$$

The proof is completed by applying Lemma 4 to the sets $(X_i : i \in I)$ and function $(x, z_I, u) \mapsto \int_{\bigcap_{i \notin I}(X_i + z_i)} h(x,z,u)\, dz_{I^C}$.

Remark. Similarly as in [4, p. 269] it can be shown that (4) is satisfied if all X_i's are convex bodies or if the boundaries ∂X_i are C^{d-1}-smooth. From [1, Theorem 6.11] it follows that for $X_1, \ldots, X_q$ of positive reach, $X_1, \theta_2 X_2, \ldots, \theta_q X_q$ satisfy (4) for almost all rotations $(\theta_2, \ldots, \theta_q) \in \big(SO(d)\big)^{q-1}$.

References

[1] Federer H.: Curvature measures. *Trans. Amer. Math. Soc.* **93**(1959), 418-491

[2] Federer H.: *Geometric Measure Theory.* Springer Verlag, Berlin 1969

[3] Rataj J.: Remarks to a translative formula for sets of positive reach. *Geom. Dedicata* (to appear)

[4] Rataj J., Zähle M.: Mixed curvature measures for sets of positive reach and a translative integral formula. *Geom. Dedicata* **57**(1995), 259-283

[5] Schneider R., Weil W.: Translative and kinematic integral formulae for curvature measures. *Math. Nachrichten* **129**(1986), 67-80

[6] Weil W.: Iterations of translative integral formulae and non-isotropic Poisson processes of particles. *Math. Z.* **205**(1990), 531-549

[7] Zähle M.: Integral and current representation of Federer's curvature measures. *Arch. Math.* **46**(1986), 557-567

Matematický ústav Univerzity Karlovy
Sokolovská 83
186 00 Praha 8
Czech Republic