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Abstract

We will give an introduction to the Skyrme model from a mathematical point of view. Hereby, we show that it is difficult to solve the field equation even by means of the classical ansatz, the so-called hedgehog ansatz. Our main result is an extended existence proof for solutions of the field equation in the hedgehog ansatz.

1 Introduction

About 1960 the British physicist Tony Hilton Royle Skyrme (1922 - 1987) introduced a field theory which is now known as the **Skyrme model** [6, 7]. This is a non-linear field theory of low energy hadron physics, which identifies baryons as topological solitons of a field theory for pions.

A **Skyrme field** is a smooth map

$$U: \mathbb{R}^3 \times \mathbb{R} \longrightarrow SU(2)$$

with boundary condition $U(x) \to 1$ for $|x|_{\mathbb{R}^3} \to \infty$. Here $\mathbb{R}^3 \times \mathbb{R}$ is physical space-time and $SU(2)$ is the group of special unitary 2×2 matrices. In the literature (e. g. [2]) the structure of this map is often described by

$$U(x) = \sigma(x) + i\pi(x) \cdot \tau, \quad \text{with} \quad \sigma^2 + \pi^2 = 1 \ ,$$

with pion field $\pi = (\pi_1, \pi_2, \pi_3)$, scalar σ-field σ and Pauli-matrices $\tau = (\tau_1, \tau_2, \tau_3)$ [2]. The space of all these maps U is called **configuration space** $\mathcal{C}$. Using the indices $\mu, \nu = 0, 1, 2, 3$, we define the components of the Maurer-Cartan form L by $L_\mu = U^{-1}\partial_\mu U$. Indices will be raised using the Minkowski metric $(+,-,-,-)$. The model is characterized by its **Lagrangian**

$$\mathcal{L} = -\frac{F_\pi^2}{16} Tr(L_\mu L^\mu) + \frac{1}{32e^2} Tr([L_\mu, L_\nu][L^\mu, L^\nu]) \ , \tag{1}$$

*This paper is in final form and no version of it will be submitted elsewhere.

where the first term is the Lagrangian of the σ-model and the second is the so-called Skyrme term, which ensures the existence of soliton solutions. The constants are the pion decay constant F_π and a unit-free parameter e introduced by Skyrme to stabilize the solitons [1]. Without loss of generality, we will fix $F_\pi \cdot e = 1$. Because of the Skyrme term the model is no longer renormalizable.
We will only look for time-independent critical points of the energy function. Therefore, we consider the **energy function** $\mathcal{E}$ given by:

$$\mathcal{E} = -\int_{\mathbb{R}^3} \mathcal{L}$$

A variation of the energy function yields the **field equation** for the Skyrme model $(i,j=1,2,3)$[2]:

$$\partial_i \left(L^i + \frac{1}{4}\left[L_j, [L^j, L^i]\right]\right) = 0 \tag{2}$$

Solutions of this differential equation, called **Skyrmions**, are characterized by a topological invariant, called **baryon number** B of the particle. If U is extended to the compactification S^3 of $\mathbb{R}^3$, which is possible due to the boundary condition, the invariant B is given by the degree of the field $U: S^3 \to S^3$. It has the form

$$B(U) := \deg(U) = \int_{S^3} U^*\omega \Big/ \int_{S^3} \omega = \frac{1}{24\pi^2} \int_{S^3} \varepsilon_{ijk} Tr(L_i L_j L_k)\, dx^3 \in \mathbb{Z} \tag{3}$$

for the canonical volume form ω on S^3.

2 The Hedgehog Ansatz

For solving the differential equation (2) we need an ansatz. Skyrme used the so-called **hedgehog ansatz**, which has rotational symmetry in all three space directions [6]. Such a Skyrme field has the form

$$U: \mathbb{R}^3 \longrightarrow SU(2)$$

$$U(x) = \exp\left(f(r)\frac{x}{r}\cdot\tau\right) = \left(\frac{x}{r}\sin f(r)\cdot\tau, \cos f(r)\right) \tag{4}$$

with $x=(x_1,x_2,x_3)$, $r=|x|$ and a C^2-function $f: [0,\infty] \to \mathbb{R}$, satisfying $f(r) \xrightarrow{r\to\infty} 0$. Although we restrict the functional to a subspace and look for critical points there, the theorem of Coleman-Palais [5] guarantees that critical points of the energy function in the hedgehog ansatz are already critical points in the whole configuration space $\mathcal{C}$. Using this ansatz a straightforward calculation gives the Lagrangian

$$\mathcal{L}(r) = -\frac{1}{2}\left(\frac{df(r)}{dr}\right)^2 - \frac{1}{r^2}\sin^2 f(r) - \frac{1}{r^2}\left(\frac{df(r)}{dr}\right)^2 \sin^2 f(r) - \frac{1}{2r^4}\sin^4 f(r) \tag{5}$$

and, using the Euler-Lagrange equation, we obtain the field equation

$$2r\frac{df(r)}{dr} + \frac{d^2 f(r)}{dr^2}\left(r^2 + 2\sin^2 f(r)\right) + \left(\frac{df(r)}{dr}\right)^2 \sin 2f(r) - \tag{6}$$

$$-\sin 2f(r) - \frac{1}{r^2}\sin^2 f(r)\sin 2f(r) = 0\ .$$

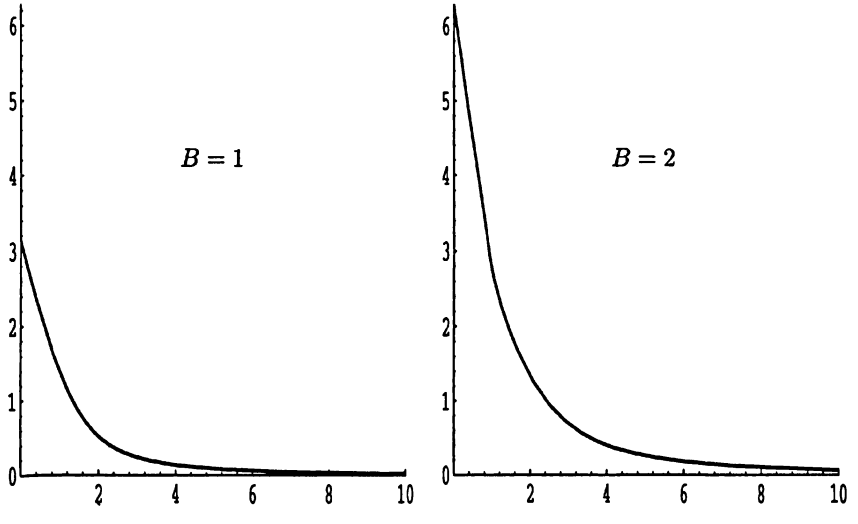
For this non-linear second order differential equation formulated by Skyrme in 1961 no-one has found an explicit solution until now.

Another straightforward calculation using definition (3) shows that

$$B = f(0)/\pi \, .$$

So the subspace of all rotationally symmetric fields in $\mathcal{C}$ is presented by the space $\mathcal{F}$ of C^2-functions f with boundary conditions $f(r) \overset{r\to\infty}{\longrightarrow} 0$ and $f(0) = \pi B$, $B \in \mathbb{Z}$.

E. g. for $B = 0$ the vacuum-field $f \equiv 0$ is the only solution with energy $\mathcal{E} = 0$. For $B = 1$ or $B = 2$ a numerical calculation indicates the shape of the corresponding functions f:



3 Existence of Skyrmions

Since it seems to be impossible to solve the field equation (5), we want to present here a concept to prove the existence of solutions for this equation and therefore the existence of Skyrmions. There are already two different proofs of this done for the hedgehog ansatz with invariant $B = 1$ by [4] and a general existence proof by [3]. We want to change it slightly to prove the existence of solutions in the hedgehog-ansatz for any baryon number $B \in \mathbb{Z}$.

The main idea is to consider a minimizing sequence of functions $(f_m)_{m\in\mathbb{N}}$ on $\mathbb{R}^+$ for the energy function $\mathcal{E}$, i.e. $\lim_{m\to\infty} \mathcal{E}(f_m) = \inf_{f\in\mathcal{F}} \mathcal{E}(f)$ and to show that it converges to a function $f_0 \in \mathcal{F}$ with $\mathcal{E}(f_0) = \inf_{f\in\mathcal{F}} \mathcal{E}(f)$.
Because the energy $\mathcal{E}$ is bounded, it is possible to show that the first Sobolev norm of every function in the sequence $(f_m)_{m\in\mathbb{N}}$ is bounded on any closed interval $I \subset \mathbb{R}^+\backslash 0$. By Sobolev's embedding theorem, there is a function f_0 on $\mathbb{R}^+$ and a subsequence of $(f_n)_{n\in\mathbb{N}}$ converging uniformly to $f_0 \in C^0(I)$ and weakly in $H^1(I)$. So we get a function $f_0 \in H^1(\mathbb{R}^+)$, such that the inequality

$$0 \leq \mathcal{E}(f_0|_I) \leq \underline{\lim}_{m\to\infty} \mathcal{E}(f_m|_I) \leq C$$

holds for every interval $I \subset \mathbb{R}^+\backslash 0$. Hence, we obtain

$$\mathcal{E}(f_0) = \inf_{f\in\mathcal{F}} \mathcal{E}(f) \ .$$

The last step of the proof is to show that f_0 satisfies the boundary conditions:
(i) $f_0(r) \to 0$ for $r \to \infty$:
Every term of the energy density $e = -\mathcal{L}$ is positive. Hence, the integral of those terms that do not contain derivatives of f is less than the total energy. Using this fact and the Hölder inequality we finally obtain

$$0 \leq f_m(r) \leq 2\mathcal{E}(f_m(r))^{\frac{1}{2}} \cdot r^{-\frac{1}{2}} \leq C \cdot r^{-\frac{1}{2}} \ .$$

For $m \to \infty$, this results in

$$0 \leq f_0(r) \leq \mathcal{E}(f_0(r))^{\frac{1}{2}} \cdot r^{-\frac{1}{2}} \leq \mathcal{E}(f_m(r))^{\frac{1}{2}} \cdot r^{-\frac{1}{2}} \leq C \cdot r^{-\frac{1}{2}}$$

and so

$$f_0(r) \overset{r\to\infty}{\longrightarrow} 0 \ .$$

(ii) $f_0(0) = \pi B$:
We have:

$$f_0(0) = \pi B \iff \lim_{r\to 0} \frac{1}{r} \int_0^r |f_0(\rho) - \pi B|^2 d\rho = 0$$

Assume that $f_0(0) \neq \pi B$, so

$$\lim_{r\to 0} \frac{1}{r} \int_0^r |f_0(\rho) - \pi B|^2 d\rho > 0.$$

Hence, there exists an $\alpha > 0$, such that for all $\varepsilon > 0$:

$$\|f_0 - \pi B\|_{L^\infty([0,\varepsilon])} = \|f_0 - f_m(0)\|_{L^\infty([0,\varepsilon])} \geq \alpha$$

Since we have seen that $f_m \in \mathcal{F}$ converges uniformly to f_0, we get a contradiction. □

By linearization of the field equation (6) we can see that the solutions $f \in \mathcal{F}$ of (6) are isolated. So a single hedgehog-type solution, i.e. a solution of type (4), leads via translation and rotation by $SO(3)$ to a 6-dimensional manifold of solutions.

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