Ľubomíra Horanská

On sectioning multiples of the nontrivial line bundle over Grassmannians

In: Jan Slovák and Martin Čadek (eds.): Proceedings of the 17th Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1998. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 54. pp. [59]–64.

Persistent URL: http://dml.cz/dmlcz/701615

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 54 (1998), pp. 59-64

ON SECTIONING MULTIPLES OF THE NONTRIVIAL LINE BUNDLE OVER GRASSMANNIANS

ĽUBOMÍRA HORANSKÁ

1. INTRODUCTION

Let $G_{n,k}$ denote the Grassmann manifold of all k-dimensional vector subspaces in the real Euclidean space $\mathbb{R}^n (n > k \geqq 1)$. All oriented k-dimensional vector subspaces in $\mathbb{R}^n$ form the so called oriented Grassmann manifold $\tilde{G}_{n,k}$. One has the obvious double covering $p : \tilde{G}_{n,k} \to G_{n,k}$ (universal, if $n \geqq 3$). Identifying each pair $(x,t) \in \tilde{G}_{n,k} \times \mathbb{R}$ with $(x', -t)$ whenever x and x' are two distinct points such that $p(x) = p(x')$, one obtains the total space of a line bundle $\xi_{n,k}$ over $G_{n,k}$. Since, as is well known, isomorphism classes of line bundles over a CW-complex are in one-to-one correspondence with its first $\mathbb{Z}_2$-cohomology group and one has $H^1(G_{n,k}; \mathbb{Z}_2) \cong \mathbb{Z}_2$, the line bundle ξ is usually referred to as *the* nontrivial line bundle over the Grassmann manifold $G_{n,k}$. Without loss of generality, we shall suppose $n \geqq 2k$ in the sequel ($G_{n,k}$ is diffeomorphic to $G_{n,n-k}$).

Now fixing n and k and putting $s\xi_{n,k} := \underbrace{\xi_{n,k} \oplus \cdots \oplus \xi_{n,k}}_{s \text{ times}}$, one can naturally ask the following.

Question 1.1. *What is the least s such that the vector bundle $s\xi_{n,k}$ admits a nowhere vanishing section?*

Remark. Question 1.1 can easily be generalized to: What is the least integer s_r, for a given $r > 0$, such that $s_r\xi_{n,k}$ admits r everywhere linearly independent sections? But we will deal only with $r = 1$ in this note.

Denote $d_{n,k} := k(n-k) = \dim(G_{n,k})$; $d_{n,k}$ will be written simply d throughout (n and k will always be clear from the context). As an easy consequence of the Steenrod obstruction theory, one sees that $(d+1)\xi_{n,k}$ always has a nowhere vanishing section. Hence the solution to the above question must be less than or equal to $d+1$.

For the special case of $G_{n,1}$, which is nothing but the projective space $\mathbb{R}P^{n-1}$, Question 1.1. is readily answered. Indeed, by that what we said above, $n\xi_{n,1}$ possesses a nowhere zero section, but the value of any cross-section of $(n-1)\xi_{n,1}$ must be zero at some point, because the top Stiefel-Whitney class $w_{n-1}((n-1)\xi_{n,1}) = {w_1}^{n-1}(\xi_{n,1}) \in H^{n-1}(G_{n,1}; \mathbb{Z}_2)$ is non-zero.

This paper is in final form and no version of it will be submitted for publication elsewhere

But Question 1.1 can be considered also from a different point of view. Namely, e.g. by Gitler-Handel [6] (or see [9]), the vector bundle $s\xi_{n,k}$ has a nowhere vanishing section if and only if there exists a map $f : G_{n,k} \to G_{s,1} = \mathbb{R}P^{s-1}$ such that the pull-back $f^*(\xi_{s,1})$ is precisely $\xi_{n,k}$ or equivalently that $f^*(w_1(\xi_{s,1})) = w_1(\xi_{n,k})$. However (see e.g. [8]), this is equivalent to the existence of a map $\tilde{f} : \tilde{G}_{n,k} \to \tilde{G}_{s,1} = S^{s-1}$ such that the diagram

$$\begin{array}{ccc} \tilde{G}_{n,k} & \xrightarrow{\tilde{f}} & \tilde{G}_{s,1} \\ p\downarrow & & p\downarrow \\ G_{n,k} & \xrightarrow{f} & G_{s,1} \end{array}$$

commutes, hence to the existence of a map $\tilde{f}$ that is equivariant with respect to the obvious $\mathbb{Z}_2$-action on the oriented Grassmann manifolds.

If T is a fixed point free involution on a topological space X, then the least integer q for which there exists an equivariant map from X into S^{q-1} is called the level of (X,T) by Dai and Lam ([4]), which is the same as the genus of (X,T) in the sense of Švarc [16], or up

to 1 the same as the co-index of (X,T) studied by Conner and Floyd [2]. Taking the $\mathbb{Z}_2$-action mentioned above in the role of T and denoting $s(X)$ the level of (X,T), we have that the least s such that $s\xi_{n,k}$ admits a nowhere zero section is nothing but $s(\tilde{G}_{n,k})$ and thus Question 1.1 is answered when one can solve the following.

Problem 1.1'. *For given n and k, find the level $s(\tilde{G}_{n,k})$.*

As we have seen above, $s(\tilde{G}_{n,1}) = s(S^{n-1}) = n$ (and co-index$(S^{n-1}) = n-1$). Hence we shall confine ourselves to $\tilde{G}_{n,k}$ with $k \geqq 2$.

For general n and $k \geqq 2$, Question 1.1 seems to be very difficult.

It is true that (see e.g. Conner-Floyd [2;(3.5)])

$$s(\tilde{G}_{n,k}) = 1 + \text{co-ind}(\tilde{G}_{n,k}) \leqq 1 + \text{cat}(\tilde{G}_{n,k}/\mathbb{Z}_2) = 1 + \text{cat}(G_{n,k}), \tag{1.2}$$

where $\text{cat}(G_{n,k})$ is the Lyusternik-Shnirel'man category of $G_{n,k}$. But unfortunately $\text{cat}(G_{n,k})$ seems to be known only in some special cases. On the other hand, there is no indication that the difference $1 + \text{cat}(G_{n,k}) - s(\tilde{G}_{n,k})$ must be small.

Using results of Korbaš and Sankaran [8;Theorem 4(i)], we can formulate the following.

Proposition 1.3. *(a) Let $l \geqq 2$. Then $s(\tilde{G}_{2^l+1,2}) = d+1 = 2^{l+1}-1$.*

(b) If $n \geqq 2k \geqq 4$ and $(n,k) \neq (2^l+1,2)$, then $s(\tilde{G}_{n,k}) \leqq d$.

Now let $\text{ht}(w_1) := \text{height}(w_1) = \sup\{m \mid {w_1}^m \neq 0\}$, where w_1 is the first Stiefel-Whitney class of $\xi_{n,k}$. The top Stiefel-Whitney class of $s\xi_{n,k}$, $w_s(s\xi_{n,k}) = {w_1}^s$, is non-zero for $s \leqq \text{ht}(w_1)$; the value of $\text{ht}(w_1)$ is known due to Stong [15]. Consequently, there is no nowhere zero section of $s\xi_{n,k}$ if $s \leqq \text{ht}(w_1)$, and we obtain the following lower bound for $s(\tilde{G}_{n,k})$.

Proposition 1.4. *If $n \geqq 2k \geqq 4$, then $s(\tilde{G}_{n,k}) > \mathrm{ht}(w_1)$.*

In addition to this, we are able to calculate $s(\tilde{G}_{n,k})$ in several low-dimensional cases.

Proposition 1.5. $s(\tilde{G}_{8,3}) = s(\tilde{G}_{6,3}) = s(\tilde{G}_{7,3}) = 8$.

In the situation of Proposition 1.3(b), it seems reasonable to try to decide whether or not $(d-1)\xi_{n,k}$ has a nowhere vanishing section. On this we prove in §2 the following result.

Proposition 1.6. *(a) Let n be even and $k \geqq 3$ be odd, $n \geqq 2k$. Then $(d-1)\xi_{n,k}$ has a nowhere vanishing section on the $(d-1)$-skeleton of $G_{n,k}$. Moreover, either the restriction to the $(d-2)$-skeleton of every such section can be extended to a non-vanishing section on $G_{n,k}$ or the restriction to the $(d-2)$-skeleton of no such section can be extended to a non-vanishing section on $G_{n,k}$.*

(b) For $G_{8,3}$ the restriction to the 13-skeleton of every nowhere zero section of $14\xi_{8,3}$ existing on the 14-skeleton extends to a nowhere zero section on $G_{8,3}$.

Now let ε denote a trivial line-bundle and $\mathrm{span}(\alpha)$ be the largest number of everywhere linearly independent sections of the vector bundle α. As a step towards deciding whether or not $\mathrm{span}((d-1)\xi_{n,k}) \geqq 1$, we can consider a "stable version" of the above problem, namely the question whether or not $\mathrm{span}((d-1)\xi_{n,k} \oplus 2\varepsilon) \geqq 3$. On this we prove in §2 the following theorem.

Theorem 1.7. *Let X be a finite CW-complex of dimension $m \equiv 1 \pmod 4$ and λ be any vector bundle of rank $m+1$ over X. Then $\mathrm{span}(\lambda) \geqq 3$ if and only if $w_{m-1}(\lambda) = 0$.*

Corollary 1.8. *Let $n \equiv 2 \pmod 4$ and k odd be such that $n \geqq 2k \geqq 4$. Then $\mathrm{span}((d-1)\xi_{n,k} \oplus 2\varepsilon) \geqq 3$.*

Remark 1.9. By Crabb [3; Prop. 2.4.] or Stolz [14], one knows that (for $d > 4$) $\mathrm{span}((d-1)\xi_{n,k}) \geqq 1$ if and only if the cohomotopy Euler class of $(d-1)\xi_{n,k}$ vanishes. However our efforts to compute this Euler class have failed.

2. Proofs of results

Proof of Proposition 1.5. By [8, Theorem 4(ii)] there exists a map $f : G_{8,3} \to G_{8,1}$ such that $f^*(\xi_{8,1}) \cong \xi_{8,3}$. Notice that the vector bundle $8\xi_{8,1}$ is trivial. Indeed, using the well-known description of the stable tangent bundle of the projective space and parallelizability of $\mathbb{R}P^7$, we obtain

$$8\xi_{8,1} \cong TG_{8,1} \oplus \varepsilon \cong T\mathbb{R}P^7 \oplus \varepsilon \cong 7\varepsilon \oplus \varepsilon \cong 8\varepsilon.$$

Therefore $8\xi_{8,3}$ is also trivial and it follows that $s(\tilde{G}_{8,3}) \leqq 8$.

On the other hand applying Proposition 1.4 and Stong's result [15], we obtain $s(\tilde{G}_{8,3}) > \mathrm{ht}(w_1) = 7$. This shows that $s(\tilde{G}_{8,3}) = 8$.

Each nowhere zero section of $t\xi_{8,3}$ induces a nowhere zero section of $t\xi_{6,3}$, because there exists an equivariant map $\tilde{G}_{6,3} \to \tilde{G}_{8,3}$ ([8]). Hence $s(\tilde{G}_{6,3}) \leqq s(\tilde{G}_{8,3}) = 8$. Also by [15] $\mathrm{ht}(w_1) = 7 < s(\tilde{G}_{6,3})$. Consequently $s(\tilde{G}_{6,3}) = 8$.

The proof for $G_{7,3}$ is similar.

The following proof is based on obstruction theory (Liao [10], Mahowald [11], Milnor and Stasheff [12]).

Proof of Proposition 1.6(a). The vector bundle $(d-1)\xi_{n,k}$ has a nowhere vanishing section on the $(d-1)$-skeleton of $G_{n,k}$ if and only if the primary obstruction class vanishes. This primary obstruction class is nothing but the Euler class of $(d-1)\xi_{n,k}$ considered with a fixed orientation.

We first show that the Euler class $e((d-1)\xi_{n,k}) \in H^{d-1}(G_{n,k};\mathbb{Z})$ vanishes.

Indeed, $e(8\xi_{6,3}) = 0$, because (see [8]) the vector bundle $8\xi_{6,3}$ is trivial. Now take the remaining Grassmannians considered in 1.6(a). For them one readily verifies that for s such that $3 \leqq k \leqq 2^s < n \leqq 2^{s+1}$ we have $d-3 \geqq 2^{s+1}$. Since by Stong [15] $\mathrm{ht}(w_1) = 2^{s+1}-1$ in each of those cases, we have that the mod 2 reduction of $e((d-3)\xi_{n,k})$, which is precisely w_1^{d-3}, vanishes. Hence $e((d-3)\xi_{n,k}) = 2x$ for some $x \in H^{d-3}(G_{n,k};\mathbb{Z})$. Finally we have

$$e((d-1)\xi_{n,k}) = e((d-3)\xi_{n,k})e(2\xi_{n,k}) = x2e(2\xi_{n,k}) = 0$$

(for $2e(2\xi_{n,k}) = 0$ see [12; Problem 9.A]).

Now, the secondary obstructions for two non-vanishing sections of the vector bundle $(d-1)\xi_{n,k}$ on $(G_{n,k})_{(d-1)}$ differ by an element of the subgroup $(Sq^2 + w_2((d-1)\xi_{n,k}))H^{d-2}(G_{n,k};\mathbb{Z})$ in $H^d(G_{n,k};\mathbb{Z}_2)$. Hence if we show that

$$(Sq^2 + w_2((d-1)\xi_{n,k}))H^{d-2}(G_{n,k};\mathbb{Z}) = 0,$$

that will prove that either the secondary obstruction for any non-vanishing section of $(d-1)\xi_{n,k}$ on $(G_{n,k})_{(d-1)}$ is zero or the secondary obstruction for any non-vanishing section of $(d-1)\xi_{n,k}$ on $(G_{n,k})_{(d-1)}$ is non-zero, which will then complete the proof of 1.6(a).

To start, first observe that $H^{d-2}(G_{n,k};\mathbb{Z}) = \mathbb{Z}_2$ and $H^{d-2}(G_{n,k};\mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (see Fuchs [5]). For computing $Sq^2(H^{d-2}(G_{n,k};\mathbb{Z}))$ we need to recognize those cohomology classes in $H^{d-2}(G_{n,k};\mathbb{Z}_2)$ which lie in the image of the mod 2 reduction homomorphism $\rho_2 : H^{d-2}(G_{n,k};\mathbb{Z}) \to H^{d-2}(G_{n,k};\mathbb{Z}_2)$. This homomorphism appears in the exact sequence

$$\dots \xrightarrow{2\times} H^{d-2}(G_{n,k};\mathbb{Z}) \xrightarrow{\rho_2} H^{d-2}(G_{n,k};\mathbb{Z}_2) \xrightarrow{\delta} H^{d-1}(G_{n,k};\mathbb{Z}) \xrightarrow{2\times} \dots,$$

where δ is the Bockstein homomorphism associated with the short exact sequence of coefficients $0 \to \mathbb{Z} \xrightarrow{2\times} \mathbb{Z} \to \mathbb{Z}_2 \to 0$.

Since $H^{d-2}(G_{n,k};\mathbb{Z}) \cong \mathbb{Z}_2$, we see that $\rho_2 : H^{d-2}(G_{n,k};\mathbb{Z}) \to H^{d-2}(G_{n,k};\mathbb{Z}_2)$ is a monomorphism. That means that there exists a unique nonzero class $a \in H^{d-2}(G_{n,k};\mathbb{Z}_2)$ such that $a \in \mathrm{Im}(\rho_2) = \mathrm{Ker}(\delta)$. However $\rho_2 \circ \delta$ is nothing but the Steenrod square Sq^1, and therefore we have $Sq^1(a) = 0$.

The following lemma will also be useful.

Lemma 2.1. *Under the hypotheses of 1.6(a), let $y \in H^{d-2}(G_{n,k};\mathbb{Z}_2)$. Then $(Sq^2 + w_2((d-1)\xi_{n,k}))(y) = w_1{}^2(\xi_{n,k}).y$.*

Proof. It is known (Milnor and Stasheff [12]) that $Sq^2(y) = v_2.y$ for all $y \in H^{d-2}(G_{n,k};\mathbb{Z}_2)$, where $v_2 = w_1{}^2(G_{n,k}) + w_2(G_{n,k})$ is the second Wu class.

Now, if $n \equiv 0 \pmod 4$, then $v_2 = 0$ by [1] and $w_2((d-1)\xi_{n,k}) = \binom{d-1}{2} w_1{}^2(\xi_{n,k}) = w_1{}^2(\xi_{n,k})$. Hence $(Sq^2 + w_2((d-1)\xi_{n,k}))(y) = w_1{}^2(\xi_{n,k}).y$.

If $n \equiv 2 \pmod 4$, then we have $v_2 = w_1{}^2(\xi_{n,k})$ by [1] and $w_2((d-1)\xi_{n,k}) = \binom{d-1}{2} w_1{}^2(\xi_{n,k}) = 0$. So again $(Sq^2 + w_2((d-1)\xi_{n,k}))(y) = w_1{}^2(\xi_{n,k}).y$.

To complete the proof of Proposition 1.6(a), first observe that by Jaworowski [7] $w_{k-2}w_k{}^{n-k-1}, w_{k-1}{}^2w_k{}^{n-k-2}$ and $w_{k-2}w_k{}^{n-k-1} + w_{k-1}{}^2w_k{}^{n-k-2}$ can be taken as the three non-zero elements in $H^{d-2}(G_{n,k};\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Now analyse the following four possibilities in $H^d(G_{n,k};\mathbb{Z}_2) \cong \mathbb{Z}_2$.

(1) $w_1{}^2w_{k-2}w_k{}^{n-k-1} \neq 0, \quad w_1{}^2w_{k-1}{}^2w_k{}^{n-k-2} \neq 0$;
(2) $w_1{}^2w_{k-2}w_k{}^{n-k-1} \neq 0, \quad w_1{}^2w_{k-1}{}^2w_k{}^{n-k-2} = 0$;
(3) $w_1{}^2w_{k-2}w_k{}^{n-k-1} = 0, \quad w_1{}^2w_{k-1}{}^2w_k{}^{n-k-2} \neq 0$;
(4) $w_1{}^2w_{k-2}w_k{}^{n-k-1} = 0, \quad w_1{}^2w_{k-1}{}^2w_k{}^{n-k-2} = 0$.

The Cartan and Wu formulae give that in the situation under consideration

$$Sq^1(w_{k-2}w_k{}^{n-k-1}) = w_1w_{k-2}w_k{}^{n-k-1}, \; q^1(w_{k-1}{}^2w_k{}^{n-k-2}) = w_1w_{k-1}{}^2w_k{}^{n-k-2}.$$

Hence in case (1), $w_{k-2}w_k{}^{n-k-1} + w_{k-1}{}^2w_k{}^{n-k-2}$ is the unique nonzero element in $\mathrm{Im}(\rho_2)$, and by Lemma 2.1 we have that $(Sq^2 + w_2((d-1)\xi_{n,k}))H^{d-2}(G_{n,k};\mathbb{Z})$ is generated by $w_1{}^2(w_{k-2}w_k{}^{n-k-1} + w_{k-1}{}^2w_k{}^{n-k-2}) = 0$, and therefore the subgroup in question is trivial.

Similarly one shows in cases (2) and (3) that $(Sq^2 + w_2((d-1)\xi_{n,k}))H^{d-2}(G_{n,k};\mathbb{Z})$ is trivial. Finally, in case (4) the unique nonzero element in $\mathrm{Im}(\rho_2)$ is one of the elements $w_{k-2}w_k{}^{n-k-1}, w_{k-1}{}^2w_k{}^{n-k-2}, w_{k-2}w_k{}^{n-k-1} + w_{k-1}{}^2w_k{}^{n-k-2}$. But since the $(Sq^2 + w_2((d-1)\xi_{n,k}))$-image of each of them is zero (see Lemma 2.1), we have that the subgroup $(Sq^2 + w_2((d-1)\xi_{n,k}))H^{d-2}(G_{n,k};\mathbb{Z})$ is trivial also in this case. This closes the proof of Proposition 1.6(a).

Proof of Proposition 1.6(b). By Proposition 1.5 $\mathrm{span}(8\xi_{8,3}) \geqq 1$. Then of course also $14\xi_{8,3}$ has a nowhere vanishing section whose restriction to $(G_{8,3})_{(14)}$ has its secondary obstruction zero. But then, as we know from the proof of 1.6(a), the secondary obstructions for all nowhere vanishing sections of $14\xi_{8,3}$ on $(G_{8,3})_{(14)}$ vanish. This completes the proof.

Proof of Theorem 1.7. The existence of three sections of λ is equivalent to the existence of a section for the associated bundle $V_3(\lambda)$ whose fiber is the Stiefel manifold of orthonormal 3-frames in the fiber of λ. This manifold is $(m-3)$-connected, and therefore $V_3(\lambda)$ has a section over the $(m-2)$-skeleton of X.

Then the primary obstruction to extending the above section to the $(m-1)$-skeleton is nothing but the Stiefel-Whitney class $w_{m-1}(\lambda)$ (note that we have $\pi_{m-2}(V_3(\mathbb{R}^{m+1})) = \mathbb{Z}_2$).

It is clear that $\mathrm{span}(\lambda) \geqq 3$ implies $w_{m-1}(\lambda) = 0$. On the other hand, $w_{m-1}(\lambda) = 0$ implies that we have a section of $V_3(\lambda)$ on the $(m-1)$-skeleton of X. Hence we certainly have a section of $V_3(\lambda)$ on X with a finite singularity set. But this set can be removed, since the homotopy group $\pi_{m-1}(V_3(\mathbb{R}^{m+1}))$ is trivial (see Paechter [13]) in our situation. This closes the proof.

Proof of Corollary 1.8. Using Stong's result on the height of w_1 we compute $w_{d-1}((d-1)\xi_{n,k} \oplus 2\varepsilon) = {w_1}^{d-1}(\xi_{n,k}) = 0$. Thus we can apply Theorem 1.7 in this case.

Acknowledgements. I would like to thank Július Korbaš for his guidance and many helpful suggestions and also Peter Zvengrowski for useful comments.

References

1. V. Bartík, J. Korbaš, *Stiefel-Whitney characteristic classes and parallelizability of Grassmann manifolds*, Rend. Circ. Mat. Palermo, Suppl. **6** (1984), 19-29.
2. P. Conner, E. Floyd, *Fixed point free involutions and equivariant maps*, Trans. Amer. Math. Soc. **105** (1962), 222-228.
3. M. C. Crabb, *$\mathbb{Z}_2$-Homotopy Theory*, London Math. Soc. Lecture Note Series **44**, Cambridge Univ. Press, Cambridge, 1980.
4. Z. D. Dai, T. Y. Lam, *Levels in algebra and topology*, Comment. Math. Helvetici **59** (1984), 376-424.
5. D. B. Fuchs, *Classical manifolds (Russian)*, Current Problems in Mathematics. Fundamental Directions (Russian), Itogi Nauki i Tekhniki, Akad. Nauk. SSSR **12**, Vsesoyuz. Inst. Nauch. i Tekhn. Inform., Moscow, 1986, 253-314.
6. S. Gitler, D. Handel, *The projective Stiefel manifolds-I*, Topology **7** (1968), 39-46.
7. J. Jaworowski, *An additive basis for the cohomology of real Grassmannians*, Lecture Notes in Math. 1474, Springer-Verlag, Berlin, 1991, 231-234.
8. J. Korbaš, P. Sankaran, *On continuous maps between Grassmann manifolds*, Proc. Indian Acad. Sci. (Math. Sci.) **101** (1991), 111-120.
9. J. Korbaš, P. Zvengrowski, *The vector field problem: A survey with emphasis on specific manifolds*, Exposition. Math. **12** (1994), 3-30.
10. S. D. Liao, *On the obstructions of fiber bundles*, Annals of Math. **60** (1954), 146-191.
11. M. Mahowald, *On obstruction theory in orientable fiber bundles*, Trans. Amer. Math. Soc. (1964), 315-349.
12. J. W. Milnor, J. D. Stasheff, *Characteristic Classes*, Annals of Math. Studies 76, Princeton Univ. Press, N.J., 1974.
13. G. Paechter, *The groups $\pi_r(V_{n,m})$ (I)*, Quart. J. Math. Oxford **(2) 7** (1956), 249-268.
14. S. Stolz, *The level of real projective spaces*, Comment. Math. Helvetici **64** (1989), 661-674.
15. R. E. Stong, *Cup products in Grassmannians*, Topology Appl. **13** (1982), 103-113.
16. A. S. Švarc, *The genus of a fibre space (Russian)*, Trudy Moskov. Mat. Obščestva **11** (1962), 99-126.

Department of Mathematics
Faculty of Chemical Technology
Slovak Technical University
Radlinského 9
812 37 Bratislava
Slovakia
horanska@cvt.stuba.sk