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ON THE COMPACTIFICATION OF CONFIGURATION SPACES*

Martin Markl[†] and James D. Stasheff

Configuration spaces and their compactifications play an important rôle in contemporary mathematics and mathematical physics. In mathematics they appear in connection with the problem of integral representations of Vassiliev link invariants; the spaces over which one should integrate are variants of configuration spaces of points on the circle [BT94, Thu95]. In mathematical physics they appear in closed string field theory as 'sphere with holes', or equivalence classes of higher-dimensional Feynman diagrams describing the interactions of closed strings [TKV95].

One of the basic results says that the compactification $\mathsf{F}_m(n)$ of the moduli space $\mathring{\mathsf{F}}_m(n)$ of configurations of n distinct points in the m-dimensional plane $\mathbf{R}^m$ modulo the affine group action (translations and dilatations) has the structure of an operad in the category of manifold with corners [GJ93]. An algebro-geometric counterpart of $\mathsf{F}_m(n)$ is the Mumford-Knudsen compactification of the moduli spaces of punctured Riemann spheres. This space also admits, by [GK94b], the structure of an operad in the category of algebraic manifolds.

The results mentioned above stimulated a resurgence of interest in operads, which themselves were introduced by P. May [May72] many years ago. One of the substantial achievements was the definition of the so-called Koszulness for operads (an analog of the Koszulness for commutative algebras), due to V. Ginzburg and M. Kapranov [GK94b]. This definition has far-reaching implications for the 'renaissance' of operads as we experience it now, including some new disclosures in such classical fields as universal and homological algebra [FM95, Mar96b].

J. Stasheff observed [Sta97] that the compactification $C_n(S^1)$ of the configuration space $C_n^0(S^1)$ of n distinct points on the circle (he called this space the *cyclohedron*)

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forms a right module over the operad $\mathsf{K} := \mathsf{F}_1$. A brief explanation is necessary here. The n-th piece K_n of the operad K, the compactification $\mathsf{F}_1(n)$, is identified with the polyhedron called the *associahedron* [Sta63]. By a *module over an operad* we mean a (right) module in the sense introduced in [Mar96c]. We generalized in [Mar96a] the above statement to the case of an arbitrary manifold:

Theorem 1. *For an m-dimensional manifold V, let $C_n^0(V)$ denote the space of configurations of n distinct points in V and let $C_n(V)$ denote its compactification. Then the collection $C(V) := \{C_n(V)\}_{n\geq 1}$ is a (right) module (in the category of manifolds with corners) over the operad $\mathsf{F}_m := \{\mathsf{F}_m(n)\}_{n\geq 1}$.*

In fact, the theorem as formulated above is true only for parallelizable manifolds. In the general situation we would replace $C_n^0(V)$, $\overset{\circ}{\mathsf{F}}_m(n)$ and their compactifications with appropriate framed analogs, but the simplified statement indicates the proper generalization.

1. Koszul modules over operads. According to [GK94b], an operad $\mathcal{P}$ in the category of vector spaces is *quadratic* if it is generated by $\mathcal{P}(2)$ and if the ideal of relations is generated by a subspace of $\mathcal{P}(3)$. For quadratic operads there exists a reasonable theory of Koszulness, with all expected cohomological implications. An important fact is that the associated homology operads of all the moduli space operads mentioned above are Koszul.

Let $M = \{M(n)\}_{n\geq 1}$ be a (right) $\mathcal{P}$-module in the category of vector spaces. An analysis of explicit examples shows that the *quadraticity* of M would mean that it is generated by $M(1)$ and the relations live in $M(1) \otimes \mathcal{P}(2)$. Notice that everything is shifted down by one when compared to operads. For this types of module, it makes sense to define Koszulness. All this was done by us in [Mar96d] and, independently, by V. Ginzburg and A.A. Voronov in their preprint [GV]. They conjectured (though, as far as we know, have not proven yet) that the homology of algebraic configuration spaces forms a Koszul module over the operad of local configurations.

Problem 1. *For which manifolds V is the homology $H_*(C(V))$ a Koszul module over the operad $H_*(\mathsf{F}_m)$?*

A special case of this problem, with $m = 1$ and $V = S^1$, was solved in [Mar96d] where we proved that the homology module $Cycl := H_*(C(S^1))$ of the cyclohedron is Koszul over the operad $Ass := H_*(\mathsf{F}_1)$ for associative algebras.

There is another source of examples of quadratic modules. If $\mathcal{P}$ is a cyclic operad in the sense of [GK94a], then there is a natural (right) $\mathcal{P}$-module $M_{\mathcal{P}} = \{M_{\mathcal{P}}(n)\}_{n\geq 1}$ with

$M_{\mathcal{P}}(n) := \mathcal{P}(n+1)$, $n \geq 1$, and modular structure defined by a formula involving the cyclic structure of $\mathcal{P}$ [Mar96d]. For lack of a better terminology, we call this module a module *associated* to the cyclic operad $\mathcal{P}$. We believe that we can prove the following claim.

Claim 1. *Let $\mathcal{P}$ be a quadratic cyclic Koszul operad. Then $M_{\mathcal{P}}$ is a quadratic Koszul (right) module over $\mathcal{P}$.*

2. Geometric decompositions and the bar construction. Recall that V. Ginzburg and M.M. Kapranov defined, for each operad $\mathcal{P}$ in the category of (differential) graded vector spaces, the graded differential cooperad $\mathcal{B}(\mathcal{P}) = (\mathcal{B}(P), d_{\mathcal{B}})$ called the *bar construction* on the operad $\mathcal{P}$. In a similar manner, we, in [Mar96a] and, independently, V. Ginzburg and A.A. Voronov [GV] defined, for any (differential graded) right $\mathcal{P}$-module, the *bar construction* $\mathcal{B}(M,\mathcal{P}) = (\mathcal{B}(M,\mathcal{P}), d_{\mathcal{B}})$ on M, which is a graded differential right comodule over the bar construction $\mathcal{B}(\mathcal{P})$.

We already know (Theorem 1) that $C_n(V)$ is, for any m-dimensional smooth manifold V, a manifold with corners. The skeletal filtration induces a spectral sequence converging to the cohomology $H^*(C_n(V))$. In [Mar96a] we proved:

Theorem 2. *The first term of the above mentioned spectral sequence is isomorphic to the bar construction on the module $H_*(C(V))$ over the operad $H_*(\mathsf{F}_m)$.*

The theorem above should be interpreted as a 'modular' analog of the similar theorem for the operad F_m proved by Getzler and Jones in [GJ93]. It shows that the generators of the bar construction correspond to pieces of a certain geometric decomposition of the underlying space. It is most manifest in the case of the associahedron where the generators correspond to a cell decomposition [Mar96c]. A similar, very explicit correspondence was described in [Mar96d] also for the cyclohedron. We may formulate:

Problem 2. *Identify the generators of the bar construction of Theorem 2 with the pieces of a geometric decomposition of the compactification $C_n(V)$.*

In the special case $V =$ the m-dimensional torus $T^m := (S^1)^{\times m}$, one would expect a decomposition similar to the Fox-Neuwirth decomposition of $\mathsf{F}_m(n)$ [FN62, GJ93].

3. Link invariants. As we have already observed, integral representations of Vassiliev knot invariants involve integration over certain variants of the configuration space of points on the circle. Let us explain this more carefully.

Let $K \subset \mathbf{R}^3$ be a knot. Consider the configuration space $C^0_{k,l}(\mathbf{R}^3; K)$ of $k+l$ points in $\mathbf{R}^3$ such that the last l points belong to the knot $K \subset \mathbf{R}^3$. Let $C_{k,l}(\mathbf{R}^3; K)$ be the compactification of this space. The invariant is then given as a linear combination (with certain correction terms) of integrals of the form

$$\int_{C_{k,l}(\mathbf{R}^3;K)} \pi_*(\omega),$$

where $\pi_*(\omega)$ is a differential form which we need not specify here. In order that this number be indeed a knot invariant, the contributions of the form

$$\int_{\partial C_{k,l}(\mathbf{R}^3;K)} (\partial\pi)_*(\omega) \tag{1}$$

must cancel. Here again $(\partial\pi)_*(\omega)$ is a certain form which we will not specify now, and $\partial C_{k,l}(\mathbf{R}^3; K)$ is the boundary of the manifold with corners $C_{k,l}(\mathbf{R}^3; K)$, i.e. the closure of codimension one faces.

Among them, there are the so-called *principal faces* for which the cancellation follows from an easy combinatorial argument, and the so-called *hidden faces* (we use the terminology of [BT94]), for which the vanishing is sometimes very delicate. For the so-called *anomalous faces*, which are special cases of the hidden faces, even a correction term must be added, see [Thu95] for details. We suggest the following approach.

Let $L \subset \mathbf{R}^3$ be the line $\{(x, y, z) \in \mathbf{R}^3;\ x = y = 0\}$. Denote by $C^0_{k,l}(\mathbf{R}^3; L)$ the 'local model' for $C^0_{k,l}(\mathbf{R}^3; K)$, i.e. the configuration space of $k+l$ distinct points in $\mathbf{R}^3$ such that the last l points belong to the line L. We believe that we may show, using methods similar to those in [Mar96a], that the space $C_{k,l}(\mathbf{R}^3; K)$ admits an 'action' of the compactification $\mathsf{F}_{k,l}(3)$ of the moduli space $\mathring{\mathsf{F}}_{k,l}(3) := C^0_{k,l}(\mathbf{R}^3; L)/\mathrm{Aff}(\mathbf{R}^3; L)$, where $\mathrm{Aff}(\mathbf{R}^3; L)$ is the group of dilatations and translations of $\mathbf{R}^3$ that preserve L.

Problem 3. *Try to understand the algebraic nature of the 'action' of $\mathsf{F}_{k,l}(3)$ on the space $C_{k,l}(\mathbf{R}^3; K)$ mentioned above. Is there a related, sensible notion of Koszulness?*

There is an obvious generalization of the situation above. For a submanifold L of the manifold V, we have the space $C_{k,l}(V; L)$ with an action of $\mathsf{F}_{k,l}(m)$. It is natural to ask if there are meaningful examples of this situation. In such a case it would be interesting to develop a general theory for these objects.

Let us go back to the initial question. It is clear that the faces of $\partial C_{k,l}(\mathbf{R}^3; K)$ generate $C_{k,l}(\mathbf{R}^3; K)$ under the action in Problem 3. Some indications suggest that the form $(\partial\pi)_*(\omega)$ behaves well under the pull-backs of the action. We believe that this may contribute to the demystifying of the problem of hidden faces:

Problem 4. *Try to understand the 'hidden faces problem' conceptually in terms of the action of Problem 3.*

4. Traces, approximations, and delooping machines. Operads were designed to describe varieties of algebras in symmetric monoidal categories. To be more precise, for an object X of a symmetric monoidal category $\mathcal{C}$, there is the so-called endomorphism operad End_X with $End_X(n) = \mathrm{Hom}_{\mathcal{C}}(X^{\times n}, X)$ [May72]. A $\mathcal{P}$-algebra structure on X is then, by definition, an operad map $a : \mathcal{P} \to End_X$. We also sometimes say that X is a *$\mathcal{P}$-space.*

Similarly, for two arbitrary objects $X, Y \in \mathcal{C}$, the collection $End_{X,Y}$ with $End_{X,Y}(n) = \mathrm{Hom}_{\mathcal{C}}(X^{\times n}, Y)$ is a natural right module over the endomorphism operad End_X. This means that, if X itself has a $\mathcal{P}$-algebra structure, then there is an induced right $\mathcal{P}$-module structure on the collection $End_{X,Y}$. In [Mar96d] we defined, for a right $\mathcal{P}$-module M, an *M-trace on a $\mathcal{P}$-algebra X* as a homomorphism $t : M \to End_{X,Y}$.

For any topological operad $\mathcal{P}$ and any topological space X, there exists the free topological $\mathcal{P}$-algebra $\mathcal{F}_{\mathcal{P}}(X)$ (or $\mathcal{P}X$ in the original notation of P. May) on X, constructed as a certain quotient space of the disjoint union $\coprod_{n\geq 1} \mathcal{P}(n) \times_{\Sigma_n} X^{\times n}$ [May72]. One of the central statements of the homotopy theory of Hopf spaces is the following *approximation theorem* [May72]:

Theorem 3. *Let $\mathcal{C}_n$ be the little n-disks operad. Then the space $\mathcal{F}_{\mathcal{C}_n}(X)$ has the homotopy type of the n-fold loop space on the n-fold suspension on X,*

$$\mathcal{F}_{\mathcal{C}_n}(X) \sim \Omega^n S^n X.$$

In the same vein, it is easy to show that, given a (topological) operad $\mathcal{P}$ and a (topological) $\mathcal{P}$-algebra A, for each $\mathcal{P}$-module M there exists the *free M-trace* $T_M(A)$ on the algebra A, i.e. a (topological) space $T_M(A)$ together with a homomorphism $t : M \to End_{T_M(A),Y}$ having the obvious universal property. We may formulate the following problem (using the same notation as in Theorem 1):

Problem 5. *Let A be an F_m-space and V an m-dimensional Riemannian manifold. Describe the homotopy type of the free trace $T_{C(V)}(A)$.*

Even an answer to the special case $n = 1$ and $V = S^1$, i.e. a description of the homotopy type of

$$T_{C(S^1)}(\Omega X),$$

where ΩX is the loop space on X considered as a K-algebra [Sta63], would be very interesting. Let us recall the following beautiful classical theorem [Sta63],[May72].

Theorem 4. *A connected topological space X is a K-algebra if an only if it has the homotopy type of a loop space.*

This immediately leads us to the formulation of:

Problem 6. *Is there an analogous statement as Theorem 4 also for traces? Is there a corresponding 'delooping machine'?*

5. Cyclic homology. Let $\mathcal{P}$ be a cyclic operad in the sense of [GK94a]. Let A be a $\mathcal{P}$-algebra. Recall that an invariant bilinear form on A is, by definition, a vector space W together with a bilinear map $B : A \otimes A \to W$, which satisfies an invariance condition [GK94a]. Recall also that there exist the *universal invariant bilinear form* $\lambda : A \otimes A \to \lambda(\mathcal{P}, A)$. V. Getzler and M.M. Kapranov then defined the *cyclic homology* of the algebra A as the (left) nonabelian derived functor of the functor $\lambda(\mathcal{P}, -) : A \mapsto \lambda(\mathcal{P}, A)$, a generalization of the cyclic homology of an associative algebra [LQ84].

Let now $\mathcal{P}$ be an *arbitrary*, not necessary cyclic, operad, and let M be a (right) $\mathcal{P}$-module. Consider, as in 5.3, the free trace functor $T_M(-) : A \mapsto T_M(A)$. We have the following theorem.

Theorem 5. *If the operad $\mathcal{P}$ is cyclic and $M = M_{\mathcal{P}}$ is, as in 5.1, the module associated to $\mathcal{P}$, then the left derived functor of the functor $T_M(-)$ is the cyclic homology in the sense of [GK94a].*

The theorem follows from the fact that traces over the associated module $M_{\mathcal{P}}$ are in one-to-one correspondence with invariant bilinear forms, which we proved in [Mar96d]. Thus we may formulate:

Problem 7. *For any, not necessary cyclic, operad $\mathcal{P}$, study the nonabelian left derived functor of the free trace functor $T_M(-) : \mathcal{P}$-algebras $\to M$-traces, as a natural generalization of the cyclic homology.*

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