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ON THE FLUX HOMOMORPHISM FOR REGULAR POISSON MANIFOLDS

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ABSTRACT. We introduce the concept of the flux homomorphism for regular Poisson manifolds. First we establish a one-to-one correspondence between Poisson diffeomorphisms close to id and closed foliated 1-forms close to 0. This allows to show that the group of Poisson automorphisms is locally contractible and to define the flux locally. Then by means of the foliated cohomology we extend this local homomorphism to a global one.
KEYWORDS. Poisson manifold, Lagrangian submanifold, locally contractible, flux homomorphism

1. INTRODUCTION

Let (M, Λ) be a compact regular Poisson manifold of rank $2k < n = dim(M)$, that is Λ is an antisymmetric $(2,0)$-tensor of rank $2k$ which satisfies the integrability condition $[\Lambda, \Lambda] = 0$ (cf. [4]). By

$$\sharp : \Omega^1(M) \ni \alpha \mapsto \alpha^\sharp \in \mathcal{X}(M), \text{ where } \beta(\alpha^\sharp) = \Lambda(\alpha, \beta)\ \forall \alpha, \beta \in \Omega^1(M),$$

we denote the associated bundle homomorphism. The image of $\sharp$ is an integrable distribution. The resulting $2k$-dimensional foliation is called symplectic and denoted by $\mathcal{F}(\Lambda)$. By $L(M, \Lambda)$ we denote the Lie algebra of all infinitesimal automorphisms of (M, Λ) which are tangent to $\mathcal{F}(\Lambda)$.

Let us recall that there is a bijective correspondence between smooth isotopies f_t in $Diff^\infty_c(M)$ satisfying $f_0 = id$ and and smooth families of compactly supported vector fields X_t. This correspondence is given by the equality

$$\frac{df_t}{dt} = X_t \circ f_t. \tag{1}$$

In particular, a time-independent vector field corresponds to its flow.

The symbol $G(M, \Lambda)$ stands for the group of all leaf preserving diffeomorphisms satisfying $f^*\Lambda = \Lambda$. An isotopy f_t with $f_0 = id$ is said to be Poisson iff $X_t \in L(M, \Lambda)$ for each t. By $G(M, \Lambda)_0$ we denote the group of all Poisson automorphisms f such that

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there is a Poisson isotopy f_t with $f_0 = id$ and $f_1 = f$. As $G(M,\Lambda)$ is locally arcwise connected (see below), $G(M,\Lambda)_0$ is its identity component in the C^∞-topology.

The aim of this note is to extend the flux homomorphism (cf. [2], [1], [7]) to the case of regular Poisson manifolds. The notion of the differential complex of foliated forms allows to do it similarly as in the symplectic case. As a by-product we get the local contractibility of the group $G(M,\Lambda)$, the property which is important for itself and is the starting point for the definition of the flux homomorphism.

2. Foliated forms and their cohomology

Let $(\Omega^*(M),d)$ be the DeRham complex with compact support of a smooth manifold M. Given a regular foliation $\mathcal{F}$ on M we define the subcomplex $\Omega^*(M,\mathcal{F})$ as follows: $\omega \in \Omega^r(M,\mathcal{F})$ if and only if $\omega \in \Omega^r(M)$ and

$$\bar{\omega}(X_1,\dots,X_r) = 0 \quad \forall X_1,\dots,X_r \text{ tangent to } \mathcal{F}.$$

We set

$$\Omega^*(\mathcal{F}) = \Omega^*(M)/\Omega^*(M,\mathcal{F}).$$

By $\bar{\omega} \in \Omega^r(\mathcal{F})$ we will denote the class of $\omega \in \Omega^r(M)$. By putting $\bar{d}\bar{\omega} = \overline{d\omega}$ we define a new differential complex $(\Omega^*(\mathcal{F}),\bar{d})$, the complex of foliated smooth forms. Indeed, it is straightforward that $\bar{d}$ is well defined and $\bar{d}^2 = 0$. By $H^*(\mathcal{F})$ we denote the cohomology of $\Omega^*(\mathcal{F})$, and by $[\bar{\omega}]$ the cohomology class of $\bar{\omega}$. Clearly $H^r(\mathcal{F}) = 0$ if $r > \dim \mathcal{F}$.

It is visible that the exterior product $\wedge$ in $\Omega^*(M)$ descends to $\Omega^*(\mathcal{F})$. Next, it is easily seen that $\iota(X)\bar{\omega} = \overline{\iota(X)\omega}$ and $f^*\bar{\omega} = \overline{f^*\omega}$ are correct definitions whenever X is tangent to $\mathcal{F}$ and f is leaf preserving. The former enables us to have the Lie derivative $L_X\bar{\omega} = \iota(X)\bar{d}\bar{\omega} + \bar{d}\iota(X)\bar{\omega}$ for X tangent to $\mathcal{F}$.

Further, for a smooth family $\omega_t \in \Omega^r(M)$, $t \in I$, we define

$$\int_0^1 \bar{\omega}_t dt = \overline{\int_0^1 \omega_t dt} \quad \text{and} \quad \int_0^1 [\bar{\omega}_t] dt = \Big[\overline{\int_0^1 \omega_t dt}\Big].$$

Theorem 1 [3]. *Given a smooth manifold M there is a one-to-one correspondence between regular Poisson structures Λ on M with rank of $\Lambda = 2k$ and the pairs $(\mathcal{F},\bar{\sigma})$ where $\mathcal{F}$ is a foliation of dim $2k$, and $\bar{\sigma} \in \Omega^2(\mathcal{F})$ is a foliated symplectic form of $\mathcal{F}$. The word 'symplectic' means that $\bar{d}\bar{\sigma} = 0$ and $\wedge^k\bar{\sigma} \neq 0$.*

In fact, for any $f \in C^\infty(M)$ there is a unique X_f tangent to $\mathcal{F}$ such that $\iota(X_f)\bar{\sigma} = -\overline{df}$. Then we define

$$\Lambda(df,dg) = \bar{\sigma}(X_f,X_g).$$

Conversely, it is well-known that at any point there exists a canonical chart $(x_1,\dots,x_{2k},y_1,\dots,y_q)$ $(q = n-2k)$ such that $\mathcal{F}(\Lambda)$ is defined by $dy_i = 0$, $i = 1,\dots,q$, and $\omega_L = \sum_{i=1}^k dx_i \wedge dx_{i+k}$ where ω_L is the symplectic form living on a leaf L. By choosing a complement to $T\mathcal{F}(\Lambda)$ (see e.g. [8]) one can extend the forms ω_L, $L \in \mathcal{F}(\Lambda)$, to a 2-form σ on M such that $\bar{\sigma}$ is related to Λ.

Corollary 1. *A compactly supported leaf preserving diffeomorphism f belongs to $G(M,\Lambda)$ iff $f^*\bar{\sigma}=\bar{\sigma}$.*

Finally, it is important that Moser's lemma still holds for compact regular Poisson manifolds (cf.[3]).

Lemma 1. *If $\bar{\sigma}_t$ is a smooth family of foliated symplectic forms which are $\mathcal{F}(\Lambda)$-cohomologous then there is a flow f_t tangent to $\mathcal{F}(\Lambda)$ such that $\bar{\sigma}_0 = f_t^*\bar{\sigma}_t$, where Λ is related to $\bar{\sigma}_0$.*

3. The local contractibility of $G(M,\Lambda)$

A deep feature of the symplectic geometry has been revealed by A. Weinstein in [11], namely the group of all compactly supported symplectomorphisms is locally contractible. An analogous fact is known for contact transformations (cf.[6]). The Weinstein's result can be generalized for the case of regular Poisson manifolds by making use of the foliated forms and Lemma 1. In this section we give the proof of this generalization (partly sketched).

Let $\mathcal{F}$ be a (regular) foliation on a manifold M with $dim(M)=n$, $dim(\mathcal{F})=p$, $q=n-p$. Then the cotangent bundle of $\mathcal{F}$, $T^*\mathcal{F}=\bigcup_{x\in M}T_x^*L_x$ (L_x is the leaf passing through x), possesses a canonical Poisson structure which is exact (cf.[5]). That is, one has

$$\bar{\sigma}_{\mathcal{F}} = -\bar{d}\bar{\lambda}_{\mathcal{F}},$$

where $\bar{\lambda}_{\mathcal{F}}\in\Omega^1(\tau^*\mathcal{F})$, $\bar{\sigma}_{\mathcal{F}}\in\Omega^2(\tau^*\mathcal{F})$ are canonical foliated forms, and $\tau^*\mathcal{F}$ is the $2p$ dimensional foliation on $T^*\mathcal{F}$ induced by the canonical projection $\tau: T^*\mathcal{F}\to M$. More precisely, for $v\in T_u(\tau^*\mathcal{F})$ with $u\in T^*\mathcal{F}$ one puts

$$\bar{\lambda}_{\mathcal{F}}(v) = u(\pi_{L*}v), \tag{2}$$

where $L=L_x$, $u\in T_x^*L$, and $\pi_L: T^*L\to L$ is the canonical projection.

Let $\Lambda_{\mathcal{F}}$ be the $(2,0)$-tensor related to $\bar{\sigma}_{\mathcal{F}}$ by Theorem 1. Then $\Lambda_{\mathcal{F}}$ is the image of Λ_M, the canonical symplectic structure on T^*M under the canonical projection $T^*M\to T^*\mathcal{F}$. Clearly $\mathcal{F}(\Lambda_{\mathcal{F}})=\tau^*\mathcal{F}$ and $codim(\mathcal{F}(\Lambda_{\mathcal{F}}))=codim(\mathcal{F})=q$.

Definition 1 [10]. *Given a Poisson manifold (M,Λ) a submanifold C is called coisotropic if*

$$\sharp(\mathrm{Ann}(T_xC))\subset T_xC,\ \forall x\in C.$$

Here $\mathrm{Ann}(T_xC)=\{\alpha\in T_x^*C:\alpha(X)=0,\ \forall X\in T_xC\}$. Further, C is called Lagrangian if for any $x\in C$

$$\sharp(\mathrm{Ann}(T_xC) = T_xC\cap T_x(\mathcal{F}(\Lambda)).$$

Observe that the 0-section in $T^*\mathcal{F}(\Lambda)$ is a Lagrangian submanifold. Observe as well that for any Lagrangian C in M $dim(C)=k+q$, where $dim(\mathcal{F}(\Lambda))=2k$, $q=n-2k$.

Proposition 1. *Given a regular (M,Λ) the sections of $T^*\mathcal{F}(\Lambda)$ which are sufficiently C^1-close to the 0-section are identified with the foliated 1-forms on (M,Λ). Moreover, a section of $T^*\mathcal{F}(\Lambda)$ is a Lagrangian submanifold if and only if the corresponding 1-form is closed.*

Proof. The first assertion follows by definition. Suppose now $\bar{\omega} \in \Omega^1(\mathcal{F}(\Lambda))$ and $\bar{\omega}(M)$ is the corresponding section. It suffices to show that the canonical foliated form $\bar{\sigma}_{\mathcal{F}(\Lambda)}$ vanishes on $\bar{\omega}(M)$. This is equivalent to $0 = \bar{\omega}^*(\bar{\sigma}_{\mathcal{F}(\Lambda)}) = \bar{\omega}^*(-\bar{d}\bar{\lambda}_{\mathcal{F}(\Lambda)}) = -\bar{d}(\bar{\omega}^*\lambda_{\mathcal{F}(\Lambda)}) = -\bar{d}\bar{\omega}$.□

Proposition 2 [12]. *Let (M_i,Λ_i) $(i=1,2)$ be any Poisson manifold. A mapping $f:(M_1,\Lambda_1)\to(M_2,\Lambda_2)$ is a Poisson morphism if and only if $graph(f)=\{(x,y): x\in M_1, f(x)=y\}$ is a coisotropic submanifold of $(M_1\times M_2, -\Lambda_1\oplus\Lambda_2)$ (the Poisson product, cf.[10]).*

For the group $G(M,\Lambda)$ we need a more precise result. Given a regular (M,Λ) one constructs a new regular Poisson manifold $((M\times M)^0,\Lambda^0)$. Here

$$(M\times M)^0 = \{(x,y)\in M\times M : x,y \text{ lie on the same leaf of } \mathcal{F}(\Lambda)\}$$

is a $(4k+q)$-dimensional manifold such that if $(x_1,\dots,x_{2k},y_1,\dots,y_q)$ is a canonical chart at x, $(x'_1,\dots,x'_{2k},y'_1,\dots,y'_q)$ is a canonical chart at y, then $y_j=y'_j$ and $(x_1,\dots,x_{2k},x'_1,\dots,x'_{2k},y_1,\dots,y_q)$ is a canonical chart at (x,y). Next, Λ^0 is of rank $4k$ and the leaves of $\mathcal{F}(\Lambda^0)$ are precisely of the form $(L\times L,-\sigma_L\oplus\sigma_L)$ where σ_L is the symplectic form living on the leaf L. That is, the form $\bar{\sigma}^0\in\Omega^2(\mathcal{F}(\Lambda^0))$ corresponding to Λ^0 is written in the above chart as

$$\bar{\sigma}^0 = \sum_{1=1}^{k} -dx_i\wedge dx_{i+k} + dx'_i\wedge dx'_{i+k}.$$

Then it is not hard to observe

Proposition 3. *Let f be a leaf preserving diffeomorphism. Then $f\in G(M,\Lambda)$ if and only if $graph(f)$ is a Lagrangian submanifold of $((M\times M)^0,\Lambda^0)$.*

Specifically, the diagonal $\Delta\subset(M\times M)^0$ is a Lagrangian submanifold corresponding to the identity.

Proposition 4. *Let (M,Λ) be any regular Poisson manifold. Then there is a section J of the vector bundle $gl(T\mathcal{F}(\Lambda^0))\to M$ which is a fiberwise almost complex structure on $T\mathcal{F}(\Lambda)$, i.e. for any $u\in(M\times M)^0$ one has $J_u\in gl(T_u\mathcal{F}(\Lambda^0))$ (where $gl(V)$ is the space of all linear mappings on V) satisfying*

$$J_u^2=-id,\quad J_u^T\bar{\sigma}_u^0 J_u=\bar{\sigma}_u^0,\quad \bar{\sigma}_u^0(X_u,J_uX_u)>0$$

for any $X_u\in T_u\mathcal{F}(\Lambda^0)$, $X_u\neq 0$.

Proof. We follow a standard argument. Let g be any metric on M. Then we have the fiberwise product metric on the vector bundle $T\mathcal{F}(\Lambda^0)$ by

$$(g\oplus g)_u((X_1,X_2),(Y_1,Y_2)) = g_x(X_1,Y_1)+g_y(X_2,Y_2),$$

where $u = (x, y)$. We define a section A of $gl(T\mathcal{F}(\Lambda^0))$ by the equality

$$\bar{\sigma}^0_u(X, Y) = (g \oplus g)_u(A_u X, Y)$$

for any $u \in (M \times M)^0$, $X, Y \in T_u(\mathcal{F}(\Lambda^0))$. Then A is skew-adjoint with respect to $g \oplus g$, and $B = A^*A$ is positive definite with respect to $g \oplus g$. Let C be the square root of B. Then one puts $J(g) = C^{-1}A$ and one checks that $J(g)$ verifies the claim. □

Lemma 2. *Let $C \subset M$ be a compact Lagrangian submanifold simultaneously in regular Poisson manifolds $(M, \Lambda_1), (M, \Lambda_2)$. Assume that $\Lambda_1 = \Lambda_2$ on C. Then there are open neighborhoods U_1 and U_2 of C and a Poisson diffeomorphism $\phi : U_1 \to U_2$ which equals the identity on C.*

The proof is a leaf-by-leaf version of that of Lemma 3.14 [7] combined with an application of Lemma 1.

Proposition 5. *Let (M, Λ) be a compact regular Poisson manifold. There are a neighborhood U_1 of the zero section Z in $T^*\mathcal{F}(\Lambda)$, a neighborhood U_2 of Δ in $(M \times M)^0$, and a Poisson diffeomorphism $\psi : U_1 \to U_2$ such that $\psi(x, 0) = (x, x)$ $\forall x \in M$ (i.e. ψ canonically identifies Z with Δ).*

Proof. We wish to define a diffeomorphism $\phi : U_1 \to U_2$ such that $\phi(Z) = \Delta$ and $\phi^*\bar{\sigma}^0 = \bar{\sigma}_{\mathcal{F}(\Lambda)}$ on Z. Then the assertion will follow by Corollary 1 and Lemma 2.

By Proposition 4 for metric g on M there exists an almost complex structure J on $T\mathcal{F}(\Lambda^0)$ compatible with $\bar{\sigma}^0$ and $(1/2)(g \oplus g)$. By $\sharp^g : T^*\mathcal{F}(\Lambda) \to T\mathcal{F}(\Lambda)$ we denote a fiberwise isomorphism induced by g, i.e. $g(\sharp^g(v), X) = v(X)$. Then we set

$$\phi(x, v) = \exp_{(x,x)}(J_{(x,x)}(\sharp^g_x(v), \sharp^g_x(v)))$$

for $x \in M$, $v \in T^*_x\mathcal{F}(\Lambda)$. Here exp is induced by $(1/2)(g \oplus g)$. Clearly $\phi(x, 0) = (x, x)$.

Let $u = (x, 0) \in Z$. A vector $X \in T_u(\tau^*\mathcal{F}(\Lambda))$ can be written as $X = (Y, v) \in T_xL \oplus T^*_xL$, where L passes through x. We then get

$$d_u\phi(X) = ((Y, Y), J_{(x,x)}(\sharp^g_x(v), \sharp^g_x(v))).$$

Therefore for $X_i = (Y_i, v_i)$ $(i = 1, 2)$ we have

$$\begin{aligned}
&\phi^*\bar{\sigma}^0_u(X_1, X_2) = \\
&= \bar{\sigma}^0_u((Y_1, Y_1), J_{(x,x)}(\sharp^g_x(v_1), \sharp^g_x(v_1)), (Y_2, Y_2), J_{(x,x)}(\sharp^g_x(v_2), \sharp^g_x(v_2))) \\
&= \bar{\sigma}^0_u((Y_1, Y_1), J_{(x,x)}(\sharp^g_x(v_2), \sharp^g_x(v_2))) - \bar{\sigma}^0_u((Y_2, Y_2), J_{(x,x)}(\sharp^g_x(v_1), \sharp^g_x(v_1))) \\
&= (1/2)(g \oplus g)_u((Y_1, Y_1), (\sharp^g_x(v_2), \sharp^g(v_2))) \\
&\quad - (1/2)(g \oplus g)_u((Y_2, Y_2), (\sharp^g_x(v_1), \sharp^g_x(v_1))) \\
&= v_2(Y_1) - v_1(Y_2) \\
&= -d\bar{\lambda}_{\mathcal{F}(\Lambda)u}(X_1, X_2) = \bar{\sigma}^0_{\mathcal{F}(\Lambda)u}(X_1, X_2).
\end{aligned}$$

To explain the second equality above we note that the subspace $T_{(x,x)}\Delta$ is Lagrangian and, due to the compatibility of $(1/2)(g \oplus g)$, J and $\bar{\sigma}^0$, the subspace $J_{(x,x)}T_{(x,x)}\Delta \subset T_{(x,x)}(L \times L)$ is Lagrangian as well. The third equality follows again by the compatibility, and the fourth by the definition of $\sharp^g$. Finally, the fifth equality is a consequence of (2). □

By summing-up the above propositions we get

Theorem 2. *Let (M,Λ) be a compact regular Poisson manifold. There exist C^1-neighborhood $\mathcal{U}_{id}$ of the identity in $G(M,\Lambda)$, C^1-neighborhood $\mathcal{V}_0$ in the vector space of all foliated closed 1-forms on $(M,\mathcal{F}(\Lambda))$, and a homeomorphism $\Psi:\mathcal{U}_{id}\to\mathcal{V}_0$.*

Indeed, we make use of the Poisson diffeomorphism $\psi : U_1 \to U_2$ defined above. If $f \in G(M,\Lambda)$ is close to the identity then so is the Lagrangian submanifold $\psi^*(graph(f))$ in $T^*\mathcal{F}(\Lambda)$ (Propositions 5 and 3). Thus $\Psi(f) = \psi^*(graph(f))$ is identified with a closed foliated 1-form by Proposition 1.

It is visible that the neighborhood $\mathcal{V}_0$ above can be chosen convex. Hence we have

Corollary 2. *The group $G(M,\Lambda)$ is locally contractible.*

Proceeding in analogy with the symplectic case cf.[7]) one then defines the local flux homomorphism as follows.

Definition 2. *Let $\Psi:\mathcal{U}_{id}\to\mathcal{V}_0$ be as above. It f_t is a Poisson isotopy such that $f_t\in\mathcal{U}_{id}$ for any t then*

$$\mathrm{Flux}(\{f_t\}) := -[\bar{\omega}_1] \in H^1(\mathcal{F}(\Lambda)),$$

where $\bar{\omega}_t = \Psi(f_t)$.

In the next section we extend this definition to the whole group and show that it is independent of Ψ.

4. The flux homomorphism

In view of Corollary 2 the group $G(M,\Lambda)_0$ is locally arcwise connected. Therefore $\widetilde{G(M,\Lambda)_0}$, the universal covering of $G(M,\Lambda)_0$, is the totality of pairs $(f,\{f_t\})$ where $f=f_1\in G(M,\Lambda)$ and $\{f_t\}$ is the homotopy rel. endpoints class of the isotopy f_t, $t\in I$. The multiplication in $\widetilde{G(M,\Lambda)}_0$ can be thought of either by the pointwise multiplication over I of representants or, equivalently, by the juxtaposition of representants. The latter means that $\{g_t\}.\{f_t\}=\{h_t\}$ where

$$\begin{aligned} h_t &= f_{2t} \quad \text{for} \quad 0\le t\le 1/2 \\ &= g_{2t-1}\circ f_1 \quad \text{for} \quad 1/2\le t\le 1. \end{aligned}$$

Our purpose is to generalize the concept of the flux homomorphism (first introduced by E.Calabi in [2] for symplectomorphisms) to regular Poisson manifolds. Let $\bar{\sigma}=\bar{\sigma}(\Lambda)$ be the corresponding foliated symplectic form. Given a Poisson isotopy f_t we let

$$\mathrm{Flux}(\{f_t\}) := \int_0^1 [\iota(X_t)\bar{\sigma}]dt \in H^1(\mathcal{F}(\Lambda))$$

where the family X_t is defined by (1).

Theorem 3. Flux : $\widetilde{G(M,\Lambda)}_0) \to H^1(\mathcal{F}(\Lambda))$ *is a well defined continuous epimorphism. Moreover,* Flux *extends Definition 2.*

As the proof follows that for symplectomorphisms (cf.[1,p.182-3]) we give only a sketch of it.

Let f_t, g_t be two Poisson isotopies such that $\{f_t\} = \{g_t\}$ and $f_1 = g_1$. Therefore there is $F_{s,t}$, a smooth 2-parameter family in $G(M,\Lambda)_0$ satisfying

$$F_{0,t} = f_t \quad F_{1,t} = g_t \quad F_{s,0} = id \quad F_{s,1} = f_1 = g_1 \quad \forall s,t \in I.$$

We let

$$X_{s,t} = \frac{\partial F_{s,t}}{\partial t} \circ F_{s,t}^{-1} \quad Y_{s,t} = \frac{\partial F_{s,t}}{\partial s} \circ F_{s,t}^{-1}.$$

In particular, $t \mapsto X_{s,t}$ corresponds by (1) to $t \mapsto F_{s,t}$. We have the equality

$$\frac{\partial X_{s,t}}{\partial s} = \frac{\partial Y_{s,t}}{\partial t} + [X_{s,t}, Y_{s,t}]. \tag{3}$$

Then

$$\begin{aligned}\frac{\partial}{\partial s}\int_0^1 \iota(X_{s,t})\bar{\sigma}dt &= \int_0^1 \iota\Big(\frac{\partial X_{s,t}}{\partial s}\Big)\bar{\sigma}dt \\ &= \int_0^1 \iota\Big(\frac{\partial Y_{s,t}}{\partial t}\Big)\bar{\sigma}dt + \int_0^1 \iota([X_{s,t}, Y_{s,t}])\bar{\sigma}dt \\ &= \int_0^1 \iota([X_{s,t}, Y_{s,t}])\bar{\sigma}dt = \bar{d}\Big(\int_0^1 \bar{\sigma}(Y_{s,t}, X_{s,t})dt\Big).\end{aligned}$$

Here the second equality follows (3), the third by $Y_{s,0} = Y_{s,1} = 0$, and the fourth by a direct computation. Thus $\mathrm{Flux}(\{g_t\}) - \mathrm{Flux}(\{f_t\}) = [\bar{d}\bar{\omega}] = 0$ where $\bar{\omega} = \int_{I\times I} \bar{\sigma}(Y_{s,t}, X_{s,t})dt \wedge ds$.

That Flux is a homomorphism follows from the fact that the multiplication in $\widetilde{G(M,\Lambda)_0}$ can be represented by the juxtaposition.

Next, let $[\bar{\omega}] \in H^1(\mathcal{F}(\Lambda))$. This means that $\bar{d}\bar{\omega} = 0$. The equality $\iota(X)\bar{\sigma} = \bar{\omega}$ defines uniquely $X \in L(M,\Lambda)$. It is visible that $\mathrm{Flux}(\{\phi_t\}) = [\bar{\omega}]$ where ϕ_t is the flow of X.

The compatibility with Definition 2 will be shown after the following

Proposition 6. *If $\bar{\sigma} = -\bar{d}\lambda$ (i.e. $\bar{\sigma}$ is exact [3]) and f_t is a Poisson isotopy then* $\mathrm{Flux}(\{f_t\}) = [\lambda - f_1^*\lambda]$.

Proof. Let X_t be related to f_t by (1). Then

$$[\iota(X_t)\bar{\sigma}] = [f_t^*\iota(X_t)\bar{\sigma}] = -[f_t^*\iota(X_t)\bar{d}\lambda] = -[f_t^*L_{X_t}\lambda] = -\frac{d}{dt}[f_t^*\lambda].$$

It remains to integrate both sides over $[0,1]$. □

To prove the second assertion of Theorem 3 we make use of the diffeomorphism ψ from Proposition 5. Let $g_t = \psi^{-1} \circ (id \times f_t) \circ \psi$ be a Poisson isotopy in $\mathcal{V}_0 \subset T^*\mathcal{F}(\Lambda)$ (Theorem 2). By an obvious argument $\mathrm{Flux}(\{f_t\}) = \iota^*\,\mathrm{Flux}(\{g_t\})$, where ι is the zero section of $T^*\mathcal{F}(\Lambda)$.

One can factorize $g_t \circ \iota = \bar{\omega}_t \circ h_t$, where h_t is a diffeomorphism family on M and $\bar{\omega}_t$ from Definition 2 can be regarded as a family of sections of $T^*\mathcal{F}(\Lambda)$. Thanks to $\bar{\sigma}_{\mathcal{F}(\Lambda)} = -d\bar{\lambda}_{\mathcal{F}(\Lambda)}$ and to Proposition 6 we have

$$\mathrm{Flux}(\{g_t\}) = [\bar{\lambda}_{\mathcal{F}(\Lambda)} - g_1^*\bar{\lambda}_{\mathcal{F}(\Lambda)}].$$

In view of the equality $\bar{\omega}_1^*\bar{\lambda}_{\mathcal{F}(\Lambda)} = \bar{\omega}_1$ we then have

$$\iota^*(\mathrm{Flux}(\{g_t\})) = -[\iota^* g_1^*\bar{\lambda}_{\mathcal{F}(\Lambda)}] = -[h_1^*\bar{\omega}_1^*\bar{\lambda}_{\mathcal{F}(\Lambda)}] = -[\bar{\omega}_1].$$

This completes the proof.

In view of the proof of Theorem 3 we have

Corollary 3. *If ϕ_t is a flow of $X \in L(M,\Lambda)$ then* $\mathrm{Flux}(\{\phi_t\}) = [\iota(X)\bar{\sigma}]$.

5. Final comments

(1) Since $G(M,\Lambda)_0 = \widetilde{G(M,\Lambda)}_0/\pi(G(M,\Lambda)_0)$ the Flux descends to a homomorphism $S: G(M,\Lambda)_0 \to H^1(\mathcal{F}(\Lambda))/\Xi$, where Ξ is the image under Flux of the first homotopy group $\pi(G(M,\Lambda)_0)$. It can be shown that $Ker(S)$ coincides with the group of all Hamiltonian diffeomorphisms. These are homomorphisms generated by the elements of flows of compactly supported Hamiltonian vector fields of (M,Λ). Recall that $X \in L(M,\Lambda)$ is Hamiltonian [4] if $X = (du)^\sharp$ with $u \in C^\infty(M)$.

(2) If (M,Ω) is an open symplectic manifold the second Calabi homomorphism is relevant (cf.[2]). We consider the universal covering group $\widetilde{Ker(S)}$ of $Ker(S)$. As $Ker(S)$ is locally arcwise connected then $\widetilde{Ker(S)}$ consists of homotopy rel. endpoints classes of Hamiltonian isotopies [1]. If $(f,\{f_t\}) \in \widetilde{Ker(S)}$ then $X_t = (du_t)^\sharp$ where a compactly supported smooth function u_t is uniquely defined (M open!), and X_t corresponds to f_t by (1). Then the homomorphism is expressed by

$$\tilde{R}: \widetilde{Ker(S)} \ni \{f_t\} \longmapsto \int_0^1 \Big(\int_M u_t\eta\Big)dt \in \mathbb{R}$$

where η is the symplectic volume form. $\tilde{R}$ is indispensable to the proof (and even to the formulation) the perfectness theorems for symplectic structures (cf.[1]). It has been shown in [9] that the group of Poisson-Hamiltonian diffeomorphisms of the torus is perfect. Further steps are limited by the lack of analogs of $\tilde{R}$.

(3) Denote by $\hat{G}(M,\Lambda)$ the subgroup of $G(M,\Lambda)$ generated by all $\exp(X)$ where $X \in L(M,\Lambda)$. Clearly $\hat{G}(M,\Lambda)_0 = \hat{G}(M,\Lambda)$. The question is whether $G(M,\Lambda)_0 = \hat{G}(M,\Lambda)$. It can be deduced from results of [1] that this is the case for symplectic manifolds.

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