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WEYL ALGEBRA AND A REALIZATION OF THE UNITARY SYMMETRY *

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1 Introduction

The present paper is mainly expository, focusing on a presentation of the origins of the intrinsic unitary symmetry encountered in the study of bosonic systems with finite degrees of freedom and its relations with the fundamental structure of quantum mechanics, which is the algebra generated by the canonical commutation relations, called here the Weyl algebra. Its main source of inspiration was a highly original presentation of R. Howe in [11], who was aiming on elucidation of the role played in various physical theories by the concept of a dual pair. In distinction to Howe's article, we do not discuss here the dual pairs, but rather address ourselves the aim of explaining *raisons d'ètre* of the construction, usually atributed to Schwinger, of the representation theory of the $\mathbf{SU}(2)$ by the use of the formalism of bosonic creation and annihilation operators. The discussion here in some respects extends a similar account of the author given in [19], while entirely omitting the applications to special functions given there.

We have tried to make the paper self contained and readable, so that some of the computations (mostly matrix algebra) are omitted and some explanatory material is included.

To any given finite dimensional vector space V, complex or real, with a symplectic form $\omega : V \times V \to \mathbb{K}$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, (a nondegenerate skew symmetric bilinear form), one assigns a certain associative algebra $\mathcal{W}$ with unit $\mathbb{1}$, called the Weyl algebra, by the following construction, see [16] or [18].[1] $\mathcal{W}$ is the quotient of the tensor algebra $\mathcal{T}V = \bigoplus_{n=0}^{\infty} \mathcal{T}^n V$, where $\mathcal{T}^n V = \underbrace{V \otimes \ldots \otimes V}_{n-\text{times}}$, for $n \geq 1$, $\mathcal{T}^0 V = \mathbb{K} \cdot \mathbb{1}$, by the twosided ideal $\mathcal{J}$ generated by the elements of the form $v \otimes w - w \otimes v - \omega(v, w)\, \mathbb{1}$.

Recall that $(v_1, \ldots, v_d, w_1, \ldots, w_d)$ is said to be a symplectic basis of V if

$$\omega(v_i, w_k) = \delta_{ik} = -\omega(w_i, v_k), \quad \text{for } i, k = 1, 2, \ldots, d$$

*This paper is in final form and no version of it will be submitted for publication elsewhere.

[1]Note however that in the latter reference the description of the symplectic Lie algebra given in the Übung 16, page 218, is incorrect, cf. below.

$$\omega(v_i, v_k) = \omega(w_i, w_k) = 0, \quad \text{for } i, k = 1, 2, \ldots, d,$$

here δ_{jk} denotes the Kronecker delta. In other words the matrix of ω with respect to the basis $(v_1, \ldots, v_d, w_1, \ldots, w_d)$ has the standard form of the symplectic structure

$$\mathrm{J} = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix} \tag{1.1}$$

where I is the $d \times d$ identity matrix. Assuming $(v_1, \ldots, v_d, w_1, \ldots, w_d)$ is such a basis for V one can show without difficulty that $\mathcal{W}$ is the associative algebra with unit $1\!\!1$ generated by elements $\{v_1, \ldots, v_d, w_1, \ldots, w_d\}$ and relations

$$[v_j, w_k] := v_j w_k - w_k v_j = \delta_{jk} 1\!\!1, \quad [v_j, v_k] = 0 = [w_j, w_k], \quad j, k = 1, 2, \ldots, d. \tag{1.2}$$

The following extention property for symplectic automorphisms immediately follows from the universality of the above construction: *if $g : V \to V$ is a symplectic automorphism, i.e. $\omega(gv, gw) = \omega(v, w)$ for all $v, w \in V$, then g extends uniquely to an automorphism of the Weyl algebra $\mathcal{W}$.*

The significance of this purely algebraic structure derives from its abundant applications and variety of analytic realizations — see [9] for a review of those. However, we shall not pursue the study of the Weyl algebra basing on purely algebraic premises, but rather adopt a reverse viewpoint and start with an easily understood analytic realization of the Weyl algebra in the form of the algebra of partial differential operators with polynomial coefficients and deduce the underlying symplectic structure and some of its consequences from this model, familiar to mathematicians by the study of the euclidean harmonic analysis and to physicists from the Schrödinger representation of quantum mechanics.

2 Preliminaries

For the following we fix a positive integer d (number of bosonic degrees of freedom) and employing the usual multi-index notation we set for any nonnegative integer k

$$\mathcal{W}^{(k)} = \Big\{ \sum_{|\alpha|+|\beta|\leq k} a_{\alpha\beta} x^\alpha \partial^\beta \mid a_{\alpha\beta} \in \mathbb{C} \Big\},$$

where the height $|\alpha|$ of the multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$ is defined as $|\alpha| = \sum_{j=1}^{d} \alpha_j$ — recall the indices α_i are nonnegative integers.

Spaces $\mathcal{W}^{(k)}$ form a strictly increasing sequence

$$\mathcal{W}^{(0)} \subset \mathcal{W}^{(1)} \subset \cdots \subset \mathcal{W}^{(k)} \subset \cdots \tag{2.1}$$

The sum $\mathcal{W} = \bigcup_{k=0}^{\infty} \mathcal{W}^{(k)}$ is an associative algebra with unit $1\!\!1 \in \mathcal{W}^{(0)}$ under usual vector space operations and composition of operators as the multiplication — we shall envisage elements of $\mathcal{W}$ as operators acting on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing $\mathcal{C}^\infty$ functions on $\mathbb{R}^d$. For $A \in \mathcal{W}$ the smallest k such that $A \in \mathcal{W}^{(k)}$ is said to be the total degree of A. Further note that the sequence $(\mathcal{W}^{(k)})$ defines an algebra filtration of $\mathcal{W}$, i. e.

$$\mathcal{W}^{(k)} \cdot \mathcal{W}^{(l)} \subset \mathcal{W}^{(k+l)}.$$

By slight abuse of language we shall refer to $\mathcal{W}$ as the Weyl algebra in d indeterminates and ocasionally use the notation $\mathcal{W} = \mathcal{W}(\mathbb{R}^d)$. Moreover $\mathcal{W}$ is closed under the operation $A \mapsto A^*$ of taking adjoints, defined by means of the formula

$$(Af, g) = (f, A^*g), \qquad f, g \in \mathcal{S}(\mathbb{R}^d), \tag{2.2}$$

where $(f, g) = \int_{\mathbb{R}^d} f(x)\overline{g(x)}\,dx$ is the usual inner product. An element $A \in \mathcal{W}$ will be called hermitean, antihermitean, respectively, if $A^* = A$, $A^* = -A$, resp.

The commutator associated with the multiplication in $\mathcal{W}$, $[A, B] = AB - BA$ for any elements $A, B \in \mathcal{W}$, will be used to define the Lie algebra structure in $\mathcal{W}$. The map $\mathcal{W} \ni X \mapsto [P, X] \in \mathcal{W}$ for $P \in \mathcal{W}$ will be denoted $\mathrm{ad}(P)$ and termed the adjoint of P while $P \mapsto \mathrm{ad}(P)$ — the adjoint representation of $\mathcal{W}$. By the general algebra $\mathrm{ad}(P)$ is a derivation of the associative as well as of the Lie structure of $\mathcal{W}$.

Following a long standing tradition we shall denote $q_j = ix_j$ and $p_j = \partial_j$ the standard generators of $\mathcal{W}$ — here $j = 1, \ldots, d$ and $i = \sqrt{-1}$ is the imaginary unit! The complex, respectively real, span of p's and q's will be denoted by $\mathcal{M}_{\mathbf{C}}$, respectively $\mathcal{M}_{\mathbf{R}}$. In view of the canonical commutation relations (CCR)

$$[p_j, q_k] = i\delta_{jk}\,\mathbb{1}, \qquad [p_j, p_k] = 0 = [q_j, q_k], \quad j, k = 1, 2, \ldots, d, \tag{2.3}$$

it is clear that the formula

$$[X, Y] = iB(X, Y)\,\mathbb{1}, \qquad X, Y \in \mathcal{M}_{\mathbf{C}} \tag{2.4}$$

defines a symplectic form B on $\mathcal{M}_{\mathbf{C}}$, which is real on $\mathcal{M}_{\mathbf{R}}$. The symplectic basis

$$(p_1, \ldots, p_d, q_1, \ldots, q_d) \tag{2.5}$$

will be used to coordinatize the space $\mathcal{M}_{\mathbf{K}}$.

To do this we first establish notations, mostly standard, related with the symplectic structure on $\mathbb{K}^{2d}$. A vector $v \in \mathbb{K}^{2d}$ ($\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$) will be written in the form $v = (\alpha, \beta)$ with $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\beta = (\beta_1, \ldots, \beta_d)$ defined by $\alpha_i = v_i$ and $\beta_i = v_{d+i}$ for $i = 1, \ldots, d$ and the symplectic structure is introduced by setting

$$\omega(v, w) = v^t\mathbf{J}w = \alpha_v \cdot \beta_w - \beta_v \cdot \alpha_w,$$

where $\mathbf{J}$ is given by (1.1) and $v = (\alpha_v, \beta_v)$, $w = (\alpha_w, \beta_w)$.

By $\mathbf{Sp}(d, \mathbb{K})$ we shall denote the matrix Lie group[2] consisting of square $2d \times 2d$ matrices g with entries in $\mathbb{K}$ satisfying

$$g^t\mathbf{J}g = \mathbf{J}. \tag{2.6}$$

[2]Here we are following the notation of Helgason [8], where other authors, e.g. [1], [5], use in this context notation $\mathbf{Sp}(2d, \mathbb{K})$.

Corresponding to the splitting $v = (\alpha, \beta)$ of vectors in $\mathbb{K}^{2d}$ we shall consider splitting of square matrices of the size $2d \times 2d$ into square blocks of size $d \times d$,

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

what enables us to reformulate the condition (2.6) as

$$g \in \mathrm{Sp}(d, \mathbb{K}) \iff A^t C = C^t A, \quad B^t D = D^t B, \quad A^t D - C^t B = \mathrm{I}. \tag{2.7}$$

The Lie algebra of $\mathrm{Sp}(d, \mathbb{K})$ is denoted $\mathfrak{sp}\,(d, \mathbb{K})$ and consists of derivations of the symplectic structure, i. e.

$$L \in \mathfrak{sp}\,(d, \mathbb{C}) \iff L^t \mathrm{J} + \mathrm{J} L = 0 \tag{2.8}$$

or in terms of the block splitting

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{sp}\,(d, \mathbb{C}) \iff D = -A^t, \quad B = B^t, \quad C = C^t. \tag{2.9}$$

Now we set up a correspondence between $2d \times 2d$ matrices and endomorphisms of $\mathcal{M}_{\mathbb{K}}$ by means of the basis (2.5).

Letting $P = (p_1, \ldots, p_d)$, $Q = (q_1, \ldots, q_d)$ we can write vectors in $\mathcal{M}_{\mathbb{K}}$ uniquely in the form

$$X(v) = \alpha \cdot P + \beta \cdot Q = \sum_{i=1}^{d} \alpha_i p_i + \sum_{i=1}^{d} \beta_i q_i. \tag{2.10}$$

Thus to each square $2d \times 2d$ matrix g with coefficients in $\mathbb{K}$ there corresponds a unique endomorphism of $\mathcal{M}_{\mathbb{K}}$, denoted by $X(g)$, such that

$$X(g)X(v) = X(gv), \qquad v \in \mathbb{K}^{2d}.$$

Clearly if g is a symplectic matrix, then $X(g)$ leaves the form B invariant and consequently extends to an automorphism of $\mathcal{W}$. Similarly, if L is a derivation of the symplectic structure, then $X(L)$ is a derivation of B and thus satisfies

$$[X(L)X, Y] + [X, X(L)Y] = 0, \qquad X, Y \in \mathcal{M}_{\mathbb{K}}.$$

It follows that $X(L)$ extends uniquely from $\mathcal{M}_{\mathbb{C}}$ to a derivation of associative algebra $\mathcal{W}$.

From the commutation relations (2.3) it follows readily that $\mathcal{W}^{(2)}$ is a Lie algebra with respect to $[\cdot, \cdot]$ containing $\mathcal{W}^{(1)}$ as an ideal. Moreover $\mathcal{W}^{(2)}$ acting on $\mathcal{W}^{(1)}$ by the adjoint representation annihilates the center $\mathcal{W}^{(0)}$. Clearly $\mathcal{W}^{(1)}$ is isomorphic to the complex Heisenberg algebra and its real form is spanned over $\mathbb{R}$ by $\{p_1, \ldots, p_d, q_1, \ldots, q_d, i\,\mathbb{I}\}$. Actually the structure of $\mathcal{W}^{(2)}$ can be fully described as follows. We set

$$S = \mathrm{span}_{\mathbb{C}}\{X^2 \mid X \in \mathcal{M}_{\mathbb{C}}\} \tag{2.11}$$

and taking notice of the proposition to follow, we shall refer to S as the quadratic Lie subalgebra of $\mathcal{W}$.

Proposition 1 *S is a Lie subalgebra of the Lie algebra $\mathcal{W}^{(2)}$ and the latter decomposes as the semi direct product*

$$\mathcal{W}^{(2)} = \mathcal{W}^{(1)} \oplus S. \tag{2.12}$$

Moreover for any $P \in S$ the map $\mathrm{ad}(P)$ leaves invariant the space $\mathcal{M}_{\mathbb{C}}$ and is a derivation of the symplectic form (2.4) and conversely, every derivation of this form can be obtained this way.

Proof (sketch). We have pointed out above that $\mathcal{W}^{(1)}$ is a nilpotent ideal of the Lie algebra $\mathcal{W}^{(2)}$. Using further the Jacobi identity we see that for $T_1, T_2 \in \mathcal{W}^{(1)}$ and $P \in \mathcal{W}^{(2)}$

$$[\mathrm{ad}(P)T_1, T_2] + [T_1, \mathrm{ad}(P)T_2] = 0. \tag{2.13}$$

If S is defined as above, then $S \cap \mathcal{W}^{(1)} = \{0\}$ and clearly S and $\mathcal{W}^{(1)}$ together span the Lie algebra $\mathcal{W}^{(2)}$. Using the fact that $\mathrm{ad}(P)$ for any $P \in \mathcal{W}$ is a derivation of the associative algebra $\mathcal{W}$ we find that for any $X \in \mathcal{M}_{\mathbb{C}}$ and $P \in S$ $\mathrm{ad}(P)X \in \mathcal{M}_{\mathbb{C}}$ and using it twice we see that for any $X, Y \in \mathcal{M}_{\mathbb{C}}$

$$[X^2, Y^2] = 2iB(X, Y)(XY + YX).$$

In view of the identity

$$XY + YX = (X + Y)^2 - X^2 - Y^2 \tag{2.14}$$

this implies that S is a Lie subalgebra of $\mathcal{W}^{(2)}$. What we have actually shown above can be rephrased now as saying that $\mathcal{M}_{\mathbb{C}}$ is invariant under the adjoint action of the Lie algebra S. Moreover from (2.13) it follows that for $P \in S$ the restriction $\mathrm{ad}(P)$ to $\mathcal{M}_{\mathbb{C}}$ is a derivation of the symplectic form $B(\cdot, \cdot)$ and so can be represented with respect to the basis (2.5) by a matrix from $\mathfrak{sp}\,(d, \mathbb{C})$.

By induction with respect to d one can deduce from the identity (2.14) that $\dim_{\mathbb{C}} S = d(2d + 1)$, what agrees with the well known expression for the dimension of the symplectic Lie algebra $\mathfrak{sp}\,(d, \mathbb{C})$. Therefore the proof of the proposition will be completed if we can show that the map $S \ni P \mapsto \mathrm{ad}(P)_{|\mathcal{M}_{\mathbb{C}}}$ is injective (since then bijectivity will follow by equality of dimensions) and this assertion will follow from the analysis given in the subsequent section (cf. also [14], Chapter 1). □

3 The unitary part of the oscillator representation

Our analysis of the structure of the Lie algebra S will employ in a decisive manner a particular second order element of the Weyl algebra, namely the operator

$$H = -\frac{1}{2}\sum_{j=1}^{d}(p_j^2 + q_j^2) = \frac{1}{2}(r^2 - \Delta), \tag{3.1}$$

usually called the Hermite operator, which in physics is better known as the quantum mechanical hamiltonian of the d-dimensional isotropic harmonic oscillator. Here $\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$ is the usual Laplace operator in $\mathbb{R}^d$ and $r^2 = \sum_{j=1}^{d} x_j^2$ is the (euclidean) square of the length function.

By direct verification using the basis (2.5) we find that $(\mathrm{ad}(H)_{|\mathcal{M}_{\mathbb{C}}})^2 = -\mathrm{Id}$, i. e. the restriction of $\mathrm{ad}(H)$ to $\mathcal{M}_{\mathbb{C}}$ is a complex structure. $\pm i$ eigenvectors of $\mathrm{ad}(H)$ obtained by means of the standard construction from the basis (2.5) are the familiar creation and annihilation operators

$$a_j^+ = \frac{1}{\sqrt{2}}(x_j - \partial_j), \qquad a_j = \frac{1}{\sqrt{2}}(x_j + \partial_j), \quad j = 1, 2, \ldots, d. \tag{3.2}$$

Clearly these form another set of generators of $\mathcal{W}$, this one subject to the relations

$$[a_j, a_k^+] = \delta_{jk}\,\mathbb{1}, \qquad [a_j, a_k] = 0 = [a_j^+, a_k^+], \quad j, k = 1, 2, \ldots, d \tag{3.3}$$

and satisfying moreover

$$(a_j)^* = a_j^+, \quad j = 1, 2, \ldots, d.$$

Now, $\mathrm{ad}(H)$ leaves S invariant and is a derivation of $\mathcal{W}$. It follows that its eigenvalues on S are $\pm 2i$ and 0 and in fact one has the decomposition

$$S = S_{+2i} \oplus S_0 \oplus S_{-2i}, \tag{3.4}$$

where

$$\begin{aligned} S_{+2i} &= \mathrm{span}_{\mathbb{C}}\{a_j^+ a_k^+ \mid 1 \le j \le k \le d\}, \\ S_0 &= \mathrm{span}_{\mathbb{C}}\{\tfrac{1}{2}(a_k a_j^+ + a_j^+ a_k) \mid 1 \le k, j \le d\}, \\ S_{-2i} &= \mathrm{span}_{\mathbb{C}}\{a_j a_k \mid 1 \le j \le k \le d\} \end{aligned}$$

are the corresponding eigenspaces. The validity of this decomposition can be established by counting dimensions.

Now it is a trivial exercise in linear algebra to compute for P belonging to each of the summands in (3.4) the matrix of $\mathrm{ad}(P)_{|\mathcal{M}_{\mathbb{C}}}$ with respect to the basis

$$(\mathbf{a}^+, \mathbf{a}) = (a_1^+, \ldots, a_d^+, a_1, \ldots, a_d). \tag{3.5}$$

There is no need nor place to reproduce here these computations in the full extent — we shall need their results only for the case of $P \in S_0$ — but it is worth to point out that they imply the asserted injectivity of the map $P \mapsto \mathrm{ad}(P)_{|\mathcal{M}_{\mathbb{C}}}$, what then completes the proof of the Proposition 1.

We have seen before that the derivations of B are of the form $X(g)$ for $g \in \mathfrak{sp}\,(d, \mathbb{C})$ and we have just checked that every derivation of B is necessarily of the form $\mathrm{ad}(P)_{|\mathcal{M}_{\mathbb{C}}}$ for a unique $P \in S$. Combining these two conclusions we arrive at a specific parametrization of the Lie algebra S, which is nothing else but an infinitesimal (i. e. Lie algebraic) version of the so called oscillator (or metaplectic) representation.

Definition 1 *Given $g \in \mathfrak{sp}\,(d, \mathbb{C})$ let $\omega_{\mathbb{C}}(g) \in S$ be the unique element such that*

$$X(g) = \mathrm{ad}(\omega_{\mathbb{C}}(g))_{|\mathcal{M}_{\mathbb{C}}}. \tag{3.6}$$

Then the map

$$\mathfrak{sp}\,(d, \mathbb{C}) \ni g \mapsto \omega_{\mathbb{C}}(g) \in S \subset \mathcal{W}$$

is a faithful representation of the Lie algebra $\mathfrak{sp}\,(d, \mathbb{C})$ by differential operators belonging to the quadratic Lie algebra $S \subset \mathcal{W}(\mathbb{R}^d)$ called the oscillator representation of $\mathfrak{sp}\,(d, \mathbb{C})$.

Remark. Although it is of no direct concern to our subject, we recall that the restriction of the oscillator representation to the real Lie algebra $\mathfrak{sp}\,(d,\mathbb{R})$, denoted $g \mapsto \omega(g)$, maps the latter upon antihermitean differential operators and can be exponentiated to the double covering group — the metaplectic group $\mathbf{Mp}(d,\mathbb{R})$ — of $\mathbf{Sp}(d,\mathbb{R})$. This is a classical theorem of Shale – Weil. Moreover R. Howe in [12] have shown that one can exponentiate a certain cone in $S \oplus \mathbb{C}$ whose closure contains $\omega(\mathfrak{sp}\,(d,\mathbb{R}))$, the image of the real symplectic algebra, to a contraction semigroup acting on the Hilbert space $L^2(\mathbb{R}^d)$.

Coming back to the computations alluded to above we should point out that they are refered to the basis (3.5) rather then the originaly chosen symplectic basis (2.5) of $\mathcal{M}_{\mathbb{C}}$. In effect we get another parametrization of the Lie algebra S, $\omega'_{\mathbb{C}} : \mathfrak{sp}\,(d,\mathbb{C}) \to S$, related to the previous one by

$$\omega'_{\mathbb{C}}(g) = \omega_{\mathbb{C}}(WgW^{-1}), \qquad W = \frac{1}{\sqrt{2}}\begin{pmatrix} -\mathrm{I} & \mathrm{I} \\ -i\mathrm{I} & -i\mathrm{I} \end{pmatrix},$$

where W is the matrix of the base change.

Recall that the subspaces

$$\mathcal{M}^+_{\mathbb{C}} = \mathrm{span}_{\mathbb{C}}\{a_1^+, \ldots, a_d^+\}, \qquad \mathcal{M}^-_{\mathbb{C}} = \mathrm{span}_{\mathbb{C}}\{a_1, \ldots, a_d\}$$

are the eigenspaces of $\mathrm{ad}(H)_{|\mathcal{M}_{\mathbb{C}}}$ corresponding to the eigenvalues i and $-i$ respectively, and so the decomposition

$$\mathcal{M}_{\mathbb{C}} = \mathcal{M}^+_{\mathbb{C}} \oplus \mathcal{M}^-_{\mathbb{C}}. \tag{3.7}$$

is preserved by the operator $\mathrm{ad}(P)_{|\mathcal{M}_{\mathbb{C}}}$ for each $P \in S_0$, since P commutes with H, and hence its matrix has a block form compatible with this decomposition. To be more precise let us consider the following elements of S_0,

$$r_{jk} = \frac{1}{2}(a_j a_k^+ + a_k^+ a_j) = a_k^+ a_j + \frac{1}{2}\delta_{jk}, \qquad j,\, k = 1,\, 2,\, \ldots,\, d$$

and let $R = (r_{jk})$ denote the $d \times d$ matrix with entries r_{jk}. Then a routine calculation proves the following.

Proposition 2 *The elements r_{jk}, $j, k = 1, 2, \ldots, d$, are linearly independent and span over $\mathbb{C}$ a Lie subalgebra $S_0 \subset S$ isomorphic to the Lie algebra $\mathfrak{gl}\,(d,\mathbb{C})$ via the map*

$$\eta : \mathfrak{gl}\,(d,\mathbb{C}) \ni g \mapsto \mathrm{tr}(gQ) = \sum_{j,k=1}^{d} g_{jk} r_{kj} \in \mathfrak{g}, \qquad g = (g_{jk}). \tag{3.8}$$

Moreover

$$\mathrm{ad}(\eta(g))a_j^+ = \sum_{k=1}^{d} g_{kj} a_k^+, \qquad \mathrm{ad}(\eta(g))a_j = -\sum_{k=1}^{d} g_{jk} a_k, \qquad j = 1,\, 2,\, \ldots,\, d. \tag{3.9}$$

In other words, by taking matrices with respect to the basis $(\mathbf{a}^+, \mathbf{a})$, the map $P \mapsto \mathrm{ad}(P)$ provides a Lie algebra isomorphism of S_0 with the Lie algebra $\mathfrak{sp}_0(d,\mathbb{C})$ consisting of matrices $\begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix}$, $A \in \mathfrak{gl}\,(d,\mathbb{C})$.

We also note that the centre of S_0 is one dimensional and consists of scalar multiples of

$$H = \frac{1}{2}\sum_{j=1}^{d}(a_j a_j^+ + a_j^+ a_j) = \frac{1}{2}(r^2 - \Delta) \tag{3.10}$$

This clarifies to some extent the role played by that operator in the above developments.

To better orient the reader we write down the matricial counterparts of the other summands of the decomposition (3.4), which can be obtained by analogous arguments. We have

$$\mathfrak{sp}_{+2i}(d,\mathbb{C}) = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \mid B^t = B \right\},$$

$$\mathfrak{sp}_{-2i}(d,\mathbb{C}) = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \mid C^t = C \right\}.$$

These are the $\pm 2i$ eigenspaces of the map $L \mapsto [Z, L]$, where $Z = \mathrm{diag}(i\mathrm{I}, -i\mathrm{I})$ is the matrix of $\mathrm{ad}(H)$ with respect to $(\mathbf{a}^+, \mathbf{a})$. Finally

$$\mathfrak{sp}\,(d,\mathbb{C}) = \mathfrak{sp}_{+2i}(d,\mathbb{C}) \oplus \mathfrak{sp}_0(d,\mathbb{C}) \oplus \mathfrak{sp}_{-2i}(d,\mathbb{C}). \tag{3.11}$$

From the formulas (3.9) it is clear that the representation $g \mapsto \mathrm{ad}(\eta(g))$ of the Lie algebra $\mathfrak{gl}\,(d,\mathbb{C})$ can be exponentiated to a representation of the full linear group $\mathrm{GL}(d,\mathbb{C})$.

Corollary 1 *The mapping $g \mapsto \mathrm{ad}(\eta(g))$ is the differential $d\rho$ of the representation $\rho(g) : \mathcal{M}_\mathbb{C} \to \mathcal{M}_\mathbb{C}$ defined in the following way. For $g = (g_{jk}) \in \mathrm{GL}(d,\mathbb{C})$ we let $g^{-1} = (G_{jk})$ to denote the inverse matrix to g and set*

$$\rho(g)a_j^+ = \sum_{k=1}^{d} g_{kj}a_k^+, \qquad \rho(g)a_j = \sum_{k=1}^{d} G_{jk}a_k, \qquad j = 1, 2, \ldots, d. \tag{3.12}$$

Finally for each $g \in \mathrm{GL}(d,\mathbb{C})$ the map $\rho(g)$ is a symplectic automorphism of $\mathcal{M}_\mathbb{C}$, hence extending to an automorphism of $\mathcal{W}$, which we denote also by $\rho(g)$. Thus we get a representation $g \mapsto \rho(g)$ of the group $\mathrm{GL}(d,\mathbb{C})$ by automorphisms of the Weyl algebra $\mathcal{W}(\mathbb{R}^d)$.

The second part of the Corollary follows immediately from the Proposition 2, in view of the universality used for defining ρ.

4 Some polynomial algebra

It follows from CCR (3.3) in particular, that the subalgebras $\mathcal{P}[a_1^+, \ldots, a_d^+]$ and $\mathcal{P}[a_1, \ldots, a_d]$ generated by the creation operators $a_1^+, \ldots, a_d^+$, resp. annihilation operators $a_1, \ldots, a_d$, are naturally isomorphic with the algebra $\mathcal{P}(\mathbb{R}^d)$ of polynomial functions on $\mathbb{R}^d$ by means of the substitution map, viz. $P(x) = \sum_{|\alpha|\le l} p_\alpha x^\alpha \mapsto P(a^+) = \sum_{|\alpha|\le l} p_\alpha (a^+)^\alpha$, and similarly for the case of $\mathcal{P}[a_1, \ldots, a_d]$.

Moreover, these both subalgebras are invariant under the representation ρ of $\mathrm{GL}(d,\mathbb{C})$ on $\mathcal{W}$ and it follows from (3.12) that its restriction to $P[a_1^+, \ldots, a_d^+]$ is equivalent to the standard action of $\mathbf{GL}(d,\mathbb{C})$ on $\mathcal{P}(\mathbb{R}^d)$, while that on $\mathcal{P}[a_1, \ldots, a_d]$ is its contragredient.

From the well known facts concerning the action of the general linear group on the polynomial algebra we infer the following. The subspaces of homogeneous elements of degree l, viz. $\mathcal{P}^l[a_1^+, \ldots, a_d^+]$, resp. $\mathcal{P}^l[a_1, \ldots, a_d]$, are invariant and irreducible under the action of $\mathbf{GL}(d,\mathbb{C})$. They are also inequivalent for $l \neq l'$, so the decompositions $\mathcal{P}[a_1^+, \ldots, a_d^+] = \bigoplus_{l=0}^{\infty} \mathcal{P}^l[a_1^+, \ldots, a_d^+]$, resp. $\mathcal{P}[a_1, \ldots, a_d] = \bigoplus_{l=0}^{\infty} \mathcal{P}^l[a_1, \ldots, a_d]$, into homogeneous subspaces give the decomposition into irreducible $\mathbf{GL}(d,\mathbb{C})$-modules, which remain irreducible after restriction to $\mathbf{U}(d)$, cf. [2].

A standard ordering argument (of the Poincaré–Birkhoff–Witt type) shows that the multiplication map

$$\mathcal{P}[a_1^+, \ldots, a_d^+] \otimes \mathcal{P}[a_1, \ldots, a_d] \ni P \otimes Q \mapsto PQ \in \mathcal{W} \tag{4.1}$$

is an isomorphism of vector spaces commuting with the above action of $\mathbf{GL}(d,\mathbb{C})$, i. e. is an isomorphism of $\mathbf{GL}(d,\mathbb{C})$-modules.

For each pair (r, s) of nonnegative integers, we shall denote by $\mathcal{W}^{(r,s)}$ the image of $\mathcal{P}^r[a_1^+, \ldots, a_d^+] \otimes \mathcal{P}^s[a_1, \ldots, a_d]$ under the map (4.1) — it is the subspace spanned by the products PQ, with $P = P(a^+) \in \mathcal{P}^r[a_1^+, \ldots, a_d^+]$ and $Q = Q(a) \in \mathcal{P}^s[a_1, \ldots, a_d]$.

Proposition 3 *For any $R \in \mathcal{W}^{(r,s)}$ the following commutation relation holds*

$$[H, R] = (r - s)R. \tag{4.2}$$

A simple direct proof of this result is obtained by observing that the normaly ordered monomials $(a^+)^\alpha a^\beta = a_1^{+\alpha_1} \cdot a_2^{+\alpha_2} \cdots a_d^{+\alpha_d} \cdot a_1^{\beta_1} \cdot a_2^{\beta_2} \cdots a_d^{\beta_d}$ with $|\alpha| = r$, $|\beta| = s$ form a basis of $\mathcal{W}^{(r,s)}$ and applying to them the differentiation property of $\mathrm{ad}(H)$ together with the fact that a_j, a_j^+ are $\mathrm{ad}(H)$ eigenvectors

$$[H, a_j^+] = ia_j^+, \qquad [H, a_j] = -ia_j, \quad j = 1, 2, \ldots, d.$$

The final proposition given below is the fact we were searching for, namely it brings a description of the factorial decomposition of $\mathcal{W}$. However it will be more convenient to state the result for the restriction of the $\mathbf{GL}(d,\mathbb{C})$-action to the compact real form $\mathbf{U}(d)$, since in this case it is somewhat more transparent.

Let us consider the action of $\mathbf{U}(d)$ on $\mathcal{W}$ obtained by restricting the matrices in (3.12) to be unitary. Since now $g^{-1} = \overline{g}^t$, the transformation formula for $\{a_j\}$ reads $\rho(g)a_j = \sum_{k=1}^d \overline{g}_{kj} a_k$, $j = 1, 2, \ldots, d$. Now let $\mathcal{P}[z_1, \ldots, z_d, \overline{z}_1, \ldots, \overline{z}_d]$ be the polynomial algebra in $z_1, \ldots, z_d, \overline{z}_1, \ldots, \overline{z}_d$ regarded in a natural fashion as a $\mathbf{U}(d)$-module by setting for $P \in \mathcal{P}[z_1, \ldots, z_d, \overline{z}_1, \ldots, \overline{z}_d]$

$$(gP)(z,\overline{z}) = P(zg, \overline{zg}), \qquad g \in \mathbf{U}(d),$$

where $z = (z_1, \ldots, z_d)$, $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_d)$ are row vectors. $\mathcal{P}^l[z_1, \ldots, z_d, \overline{z}_1, \ldots, \overline{z}_d]$ denotes as usual the subspace of positive homogeneous polynomials of degree l. Consider for any pair of nonnegative integers (r, s) such that $r + s = l$ the subspace

$\mathcal{P}^{(r,s)} \subset \mathcal{P}^l[z_1, \ldots, z_d, \overline{z}_1, \ldots, \overline{z}_d]$ consisting of polynomials homogeneous of bidegree (r, s), i. e. such that $P(\lambda z, \overline{\lambda z}) = \lambda^r \overline{\lambda}^s P(z, \overline{z})$. Finally, let $\mathcal{H}^{(r,s)} \subset \mathcal{P}^{(r,s)}$ be the subspace of harmonic polynomials. The spaces $\mathcal{H}^{(r,s)}$ are invariant and irreducible under the action of $\mathbf{U}(d)$ and comprise what is known as the spherical (class one) representations of $\mathbf{U}(d)$. In the most familiar to physicists case of $\mathbf{SU}(2)$ these representations exhaust the whole dual space of the group.

Proposition 4 *The following is an equivalence of* $\mathrm{U}(d)$*-modules*

$$\mathcal{W} \simeq \bigoplus_{r\geq 0, s\geq 0} \infty \cdot \mathcal{H}^{(r,s)}.$$

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