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ISOMETRIC IMMERSIONS AND INDUCED GEOMETRIC STRUCTURES

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§0. Introduction. Let W be a smooth manifold with a fixed geometric structure h such as a Riemannian metric, a connection (in some bundle over W), a Pfaffian system on W, etc. Then a smooth map $f : V \to W$ induces a structure on V of the same type as h, say $g = f^*(h)$, which also can be written as

$$g = D(f) \tag{0.1}$$

where $D = D_h$ is a (non-linear differential) operator from the space M of maps $V \to W$ to the space G of pertinent structures on V, defined by $D_h(f) = f^*(h)$.

The global analytic and geometric study of D_h was started by J.Nash in his two fundamental papers $[\mathrm{Na}]_1$ and $[\mathrm{Na}]_2$ for the case of h being the standard (Euclidean) Riemannian metric $\sum_{i=1}^{q} dx_i^2$ on $W = \mathbb{R}^q$. Nash's work was motivated by the *isometric immersion problem* (cf. $1) asking for a solution f to the *inducing relation* (0.1) for a given Riemannian metric $g \in G$. (Recall that a C^∞-smooth map between Riemannian manifolds $(V, g) \to (W, h)$ is called an *isometric immersion* if it satisfies the *isometry* condition $f^*(h) = g$.)

The starting point of the general theory of induced geometric structures developed in [Gro] is a Nash-type implicit function theorem for a certain class of differential operators called infinitesimally invertible operators (see $2). This theorem combined with the theory of topological sheaves provides a general framework for the study of many problems related to the possibility of realizing given geometric structures of arbitrary type. We present here some of the results which are at the basis of the Nash-Gromov theory of isometric immersions and illustrate how the same results and ideas can be extended to other structures: connections in principal bundles, Carnot-Caratheodory (i.e. sub-riemannian) metrics on contact manifolds, Pfaffian systems and some others. Our exposition is based mainly on the material contained in [G-R] and [Gro] but also on work of the author ($[\mathrm{D'A}]_1$, $[\mathrm{D'A}]_2$, $[\mathrm{D'A}]_3$).

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This paper is in final form and no version of it will be submitted for publication elsewhere.

§1. The isometric immersion problem: historical introduction and basic results.

The basic example to have in mind when dealing with inducing geometric structures is, as the general title of the lectures suggests, the isometric immersion (imbedding)[1] problem. This can be formulated as follows: given a Riemannian manifold with metric g does there exist an immersion of V in some Euclidean space $\mathbb{R}^q$ such that the metric induced on V by this immersion equals g? Although the problem was considered by many different mathematicians in various specializations and under various conditions, the question of whether or not in general a Riemannian manifold can be isometrically immersed in Euclidean space remained open for a long time and the only results obtained for almost a century were restricted to quite special cases. What follows is a brief (and not at all complete) summary of the main results. For more details, see [G-R], [Gre], $[\mathrm{Na}]_1$, $[\mathrm{Na}]_2$.

1.1. In 1956, Nash ($[\mathrm{Na}]_2$) was able to solve the isometric immersion problem in global form and in an almost completely general situation: specifically, he showed that *every compact Riemannian manifold with C^r-metric, $r = 3, 4, \ldots, \infty$, can be isometrically embedded into an arbitrary small domain in the Euclidean space $\mathbb{R}^q$ for $q = \frac{3}{2}n(n+9)$.* Actually, it is worthwhile to introduce the *critical number* $s_n = \frac{1}{2}n(n+1)$ (this will be motivated by the discussion in 1.2 and 1.3. below) and write $q = 3s_n + 12n$. Also observe that the compactness assumption eventually turned out to be non-essential (as is seen in Gromov's theorem stated below), but the original result by Nash was significantly weaker for non-compact manifolds as it required much larger q, namely $q = 3(n+9)s_n$.

This value of the dimension of an Euclidean space isometrically containing all, possibly non-compact, n-dimensional manifolds was improved for $n \geq 4$ by R.Greene to $q = 24s_n + 22n + 14$ (see [Gre]).

Then the dimension q of the ambient space was reduced to $s_n + 3n + 5$ in the framework of a general isometric immersion theory (i.e the h-principle announced in [Gro]:cf.(vii)-(viii) in Sect.1.3. of these Lectures). This theory allowed both compact and non-compact $V = V^n$ and an arbitrary, not necessarily Euclidean, receiving Riemannian manifold W of dimension $q \geq s_n + 3n + 5$. The detailed proof of the existence theorem for isometric embeddings $V^n \to \mathbb{R}^{s_n+3n+5}$ (following from this theory) was presented in the survey article by Gromov and Rochlin (see [G-R]) and the complete result (i.e. the h-principle) appeared much later in [Gro] with some improvement of the bound on q, namely for $q \geq s_n + 2n + 3$.

Recently, M. Günther suggested (see [Gu]) an isometric embedding construction working for even smaller q, namely applicable in the presence of a free map and for $q \geq s_n+n+5$; secondly he obtains an isometric embedding for $q \geq \max(s_n+2n,\ s_n+n+5)$. Unfortunately, his argument is rather difficult and the author was unable to follow it in detail.

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[1] Recall that a mapping of one smooth manifold into another is said to be an imbedding if it is a diffeomorphism onto a submanifold and to be an immersion if it is locally an imbedding. It is easy to see that a C^k-map is an immersion, for $k \geq 0$, if its differential is everywhere non-singular. Throughout these lectures (unless otherwise specified), in using the term immersion we mean to include imbeddings too.

1.2. The local problem. The problem of isometric immersion admits an obvious analytic interpretation. Namely, if $g_{ij}(v)$, $v = (u_1, \ldots, u_n)$, are the components of the Riemannian metric g in the local coordinates $u_1, \ldots, u_n$ on V then the condition for an immersion $V \to W = \mathbb{R}^q$ to be isometric is expressed by the following system of non-linear P.D.E. of the first order

$$(1.2.1) \qquad < \frac{\partial f}{\partial u^i}, \frac{\partial f}{\partial u^j} >_h = g_{ij} \qquad 1 \leq i \leq j \leq n$$

The unknown vector function f has q components $f^1, \ldots, f^q$ and $<,>_h$ denotes the scalar product

$$\sum_{\alpha=1}^{q} \frac{\partial f^\alpha}{\partial u^i} \frac{\partial f^\alpha}{\partial u^j}.$$

The functions g_{ij} are the components of the metric tensor of V. That is, $g_{ij} = < \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} >_g$. In analytical terms, our problem is to solve the system (1.2.1) which consists of $s_n = \frac{n(n+1)}{2}$ (non linear) P.D.E. in q unknown functions for a given positive symmetric matrix of functions $g_{ij} = g_{ij}(v)$ on V.

1.2.A. Remark. In the general case where (W, h) is an arbitrary Riemannian manifold, the local system expressing the isometry condition (1.2.1) is written:

$$< \nabla_i f, \nabla_j f >_h = g_{ij} \qquad 1 \leq i \leq j \leq n$$

where $<,>_h$ is the scalar product defined on $T_w(W)$, $w = f(v)$, by the Riemannian metric h on W, and $\nabla_i f = Df(\frac{\partial}{\partial u^i})$. That is, the $\nabla_i f$ are vector fields in W along (the coordinate chart in) V, as $\nabla_i f(v) \in T_w(W)$, for $w = f(v)$.

1.3. Comments and remarks. (i) If $q = s_n = \frac{n(n+1)}{2}$ then the system (1.2.1) is determinate and so it seems reasonable to think that it has a solution, whatever the unknown functions $g_{ij}(v)$. This idea was apparently first expressed by Schlaefli ([Sc]) and later called "Schlafly's conjecture." In [Sc], which is the earliest publication (1873) on isometric immersions, Schlaefli asserted (without giving a proof) that any n-dimensional Riemannian manifold can be isometrically immersed in an Euclidean space of dimension $s_n = \frac{n(n+1)}{2}$. In 1926, Janet ([Ja]) gave a proof of the local form of this conjecture for analytic Riemannian manifolds and in 1927 E.Cartan ([Ca]) published another proof of the same theorem based on an application of his theory to Pfaffian forms. The Janet-Cartan theorem which can be considered the first general immersion theorem in Riemannian geometry says that in any n-dimensional Riemannian manifold with a distinguished point v there exists a neighbourhood of $v \in V$ that has an isometric analytic immersion in $\mathbb{R}^{s_n}$. In this theorem $\mathbb{R}^{s_n}$ cannot be replaced by $\mathbb{R}^{s_n - 1}$, so that the number s_n is the least possible. The C^∞-version of the Janet-Cartan theorem has still not been proved. However, Greene ([Gre]) established that a local immersion in the class C^∞ is possible in a Euclidean space of dimension $q = s_n + n$ and two years later Gromov reduced it to $q = s_n + n - 1$ (see [Gro]). More recently, Gromov indicated an approach to $q = s_n + n - [\frac{n}{2}]$, where [] refers to the integer part of a number (see [Gro], p.162).

(ii) The statement that not every n-dimensional Riemannian manifold can be locally isometrically immersed in $\mathbb{R}^q$ where $q < s_n$ can be found in Appendix 1 of [G-R] where it is proven that for $r \geq n(s_n - 1)$ and $r - 1 \leq k \leq_\infty$ the set of Riemannian metrics that are locally induced on a smooth n-dimensional manifold with a distinguished point by local C^r-immersions in $\mathbb{R}^q$ for $q < s_n$ is nowhere dense in the set of all C^k-riemannian metrics on V endowed with the usual C^k-topology. The proof of this depends on the fact that for $q < s_n$, the system (1.2.1) is overdetermined and so it has no solution for almost all choices of g (even if g is real analytic).

(iii) The isometric embedding theory has a topological counterpart where the starting point is the Whitney theorem claiming that every n-dimensional manifold can be smoothly immersed into $\mathbb{R}^{2n-1}$ and embedded into $\mathbb{R}^{2n}$. But the essential analytic aspects of the isometric embedding theory, e.g. the local *isometric* immersions have nothing comparable in differential topology where all *local* problems are trivial by definition.

(iv) Nash's results are true for metrics and immersions of class C^r, $r \geq 3$. The case $r = 2$ is still not successfully treated: it is still unknown whether C^2-riemannian manifolds can be locally isometrically C^2-immersed in Euclidean space.

(v) The results concerning the C^1-case are very different from the smoother ones. It was discovered by Nash that this case is easier to treat and that surprisingly low dimensional euclidean spaces can be used. He proved in $[\mathrm{Na}]_1$ that an arbitrary smooth immersion $V \to \mathbb{R}^{n+2}$ can be deformed to an isometric C^1-immersion. Kuiper ([Ku]) improved this result by showing that it is true when $q \geq n + 1$. To prove his C^1-theorem Nash uses a remarkable technique which consists in starting with a "short" C^1-immersion $f_0 : V \to \mathbb{R}^q$ and then deforming the submanifold $f_0(V) \subset \mathbb{R}^q$ by means of an infinite sequence of "twisting" perturbations, trying to get the necessary isometry property.

(vi) By combining the Nash-Kuiper C^1-result with Whitney's theorem for immersions (see 1.3. (ii)) one has that: *Every compact n-dimensional manifold V admits an isometric immersion of class C^1 in $\mathbb{R}^{2n-1}$*. Note that, since Whitney's theorem applies in all dimensions and the Nash-Kuiper result holds for all codimensions ≥ 1, this statement holds for all dimensions except for $n = 1$. (In other words, it is always true except for the case when V is the circle S^1.)

(vii) (Compare 3.1.A). The h-principle for smooth immersions $V^n \to \mathbb{R}^q$, $q > n$ (Hirsch,1959), asserts that *every smooth map $V^n \to \mathbb{R}^q$ is an immersion provided there is a fiberwise injective bundle map $T(V) \to T(\mathbb{R}^q)$*.

In particular, the following a) and b) are true for *stably parallelizable* manifolds V. (Recall that V^n is called parallelizable if its tangent bundle is trivial, $T(V) = V \times \mathbb{R}^n$. It is called stably parallelizable if $V \times R$ is parallelizable.)

a) *Every smooth parallelizable V^n admits an immersion into $\mathbb{R}^{n+1}$*. (This is an immediate corollary of Hirsch's theorem: See (B'_2) at p.8 in [Gro]).

b) *Every parallelizable V^n admits a C^1-isometric immersion into $\mathbb{R}^{n+1}$*. We achieve this by combining the C^1-Nash-Kuiper result with a).

(viii) The notion of the h-principle in the sense of [Gro] applies to solutions of an arbitrary differential equation and/or an equality (and also of a non-equality such as the one describing immersions $V^n \to \mathbb{R}^q$ by the non degeneracy of the differential). But no single case of this h-principle has been proven or disproven for C^∞-smooth isometric immersions for $n \geq 2$ and $q \geq 4$, as pointed out to the author by M.Gromov. In fact, the h-principle established by Gromov applies to *free* isometric immersions (see (A) at p.9 in [Gro]) which automatically limit the local dimension by $s_n + n$ (while the *global* construction in [Gro] needs $q \geq s_n + 2n + 3$). On the other hand, the Janet-Cartan theorem and the general philosophy behind the h-principle suggest, according to M.Gromov (private communication), that every C^∞-smooth n-dimensional Riemannian manifold admits an isometric (possibly non free) embedding to $\mathbb{R}^{s_n+n}$ where, recall, $s_n = \frac{n(n+1)}{2}$ is the "local dimension" of the Janet-Cartan theorem. This is confirmed for $n = 2$ by Gromov's theorem allowing isometric C^∞-embeddings of *compact* surfaces $V^2 \to \mathbb{R}^5$ (see p.298 in [Gro]), where actually Chern conjectured a long time ago that every surface can be isometrically immersed in $\mathbb{R}^4$. On the other hand, there are many geometric obstructions for isometric C^∞-immersions $V^2 \to \mathbb{R}^3$ but none was found so far for isometric C^∞-immersions $V^n \to \mathbb{R}^{s_n}$ for $n \geq 3$ as was pointed out to me by M.Gromov. Also notice that every compact 3-manifold is known to admit an isometric embedding to $\mathbb{R}^{13}$ (see p.305 in [Gro]), which is the best result available today.

§2. Nash-Gromov implicit function theorems.

Consider a non-linear operator $D : F \to G$ where F and G are some functional spaces. Propositions which reduce properties of D such as being open, existence of inverse, etc., to the corresponding properties of its linearization (differential) will be called *implicit function theorems*. In a series of cases, such theorems allow one to solve for $f \in F$ the equation $D(f_0 + f) = (g_0 + g)$ if a solution f_0 of $D(f_0) = g_0$ is known, and if g is small in some sense.

2.1. As mentioned in the introduction, the first relevant theorem of the implicit function type was given by Nash who applied it to the solution of the isometric immersion problem. One of the main steps in Nash's demonstration amounts to inverting algebraically the linearized equations (see [Na]$_2$) corresponding to the system (1.2.1) in the last section. It turns out that this idea applies to many other instances of non-linear P.D.E. systems ([Gro], Sect 2.3.1.). In all these cases one needs an appropriate infinite dimensional implicit function theorem allowing a passage from (solutions of) the linearized systems to (solutions of) the non-linear systems themselves.

2.2. Let us briefly go through the analytic part of Nash's proof of his isometric embedding theorem.

2.2.A. Metric inducing operator. The operator from maps $f : V \to \mathbb{R}^q$ to Riemannian metrics g sending f to the induced metrics $f^*(h)$ for the standard metric $h = \sum_{i=1}^{q} dx_i^2$ on $\mathbb{R}^q$ is a differential operator of the first order. In fact, the value of the scalar product defined by $g = f^*(h)$ equals, by the very definition of $f^*(h)$, the Euclidean scalar products of the images of these fields in $\mathbb{R}^q$ under the differential of the map f.

In particular, if we use some local coordinates $u_1, \dots, u_n$ (in some coordinate chart of) V and apply the above to the fields $\partial_i = \frac{\partial}{\partial u^i}$, $i = 1, \dots, n$, we get the following relations (compare (1.2.1)) for the components $g_{ij} \stackrel{\text{def}}{=} < \partial_i, \partial_j >_g$

$$g_{ij} = < \partial_i f, \partial_j f >_h \tag{2.2.1}$$

Thus, locally, we have a quadratic differential operator of the first order sending maps $f = \big(f_1(v), f_2(v), \dots, f_q(v)\big)$ to matrix valued function $\{g_{ij}(v)\}_{i=1,\dots,n}$, where

$$< \partial_i f, \partial_j f >_h = \sum_{\alpha=1}^{q} \frac{\partial f^\alpha(v)}{\partial u^i} \frac{\partial f^\alpha(v)}{\partial u^j}.$$

We cannot write such a formula for $g = f^*(h)$ globally. All we have to remember is an operator $f \mapsto g = f^*(h)$ defined everywhere over V, which is expressible by (2.2.1) in every local coordinate system.

In fact, this is how one *defines* the notion of a (non-linear) differential operator between spaces of sections of vector bundles over V, say from a space F to G . Each function space is given locally by systems of finitely many functions on V and an operator $D : F \to G$ is called differential of order r, if it is *local*, i.e. if the value $D(f)$ (v) is expressible in terms of partial derivatives of (the components of) f at v. (See [Gro], p.114 for the formal definition of such an operator.)

2.3. Linearized equations. Let us start with a map $f : V \to \mathbb{R}^q$ inducing some $g = f^*(h)$ written in local coordinates according to (2.2.1) as $g = < \partial_i f, \partial_j f >$ and see how g changes if we replace f by $f + \phi$ for a "small" map $\phi : V \to \mathbb{R}^q$. We expand the scalar bilinear product

$$< \partial_i(f + \phi), \partial_j(f + \phi) > \; = \; < \partial_i f + \partial_i \phi, \partial_j f + \partial_j \phi >$$

by bilinearity and collect terms of the same degree. Thus we get the new induced metric $g_\phi = (f + \phi)^*(h)$ decomposed into the sum of three terms

$$g_\phi = g + L(\phi) + Q(\phi) \tag{2.3.1}$$

where

$$L(\phi) \; = \; < \partial_i f, \partial_j \phi > + < \partial_j f, \partial_i \phi > \tag{2.3.2}$$

and

$$Q(\phi) \; = \; < \partial_i \phi, \partial_j \phi >,$$

and where, to simplify our notations, we neglect writing indices i and j on the left hand side of our equation.

We observe that $L(\phi)$ is a *linear* differential operator with respect to ϕ where the coefficients are some combinations of the derivatives of f. (This particular $L = L_f$ is also *linear* in f but this is rather irrelevant at the moment.) On the other hand, $Q(\phi)$ is *quadratic* in ϕ and if ϕ is "small" one thinks of $Q(\phi)$ as being something like $(\phi')^2$ (where the $'$ refers to the derivative) which is much smaller than ϕ (never mind the example of $\phi(t) = \varepsilon \sin \varepsilon^{-3} t$, where $(\phi')^2 = \varepsilon^{-1} \sin \varepsilon^{-3} t$, tends to infinity as $\varepsilon \to 0$).

2.3.A. Remark. Notice that the above decomposition is typical for all differential operators

$$D(f+\phi)=D(f)+L(\phi)+Q(\phi)$$

where $L=L_\phi$ is a linear differential operator and $Q=Q_f$ is quadratic in the derivatives of ϕ.

2.4. Solving the linearized equations. Let us neglect the "small" term $Q(\phi) = <\partial_i\phi,\partial_j\phi>$ in (2.3.1) and try to solve the equation $L(\phi)=\psi$ for some given (and "secretly" thought of as being "small") metric $\psi=\{\psi_{ij}\}$. To do this, we follow Nash and add an extra equation to (2.3.2), namely we require $\phi=\phi(v): V\to\mathbb{R}^q$ to be orthogonal at each $v\in V$ to the image of the differential of f at v, that is to $D_vf(T_v(V))\subset\mathbb{R}^q=T_{f(v)}\mathbb{R}^q$.

In other words we try to solve simultaneously the two groups of equations

$$<\phi,\partial_i f> \;=\; 0 \qquad i=1,\dots,n \qquad \text{(a)}$$

$$<\partial_i f,\partial_j\phi> \;+\; <\partial_j f,\partial_i\phi> \;=\; \psi_{ij} \qquad i,j=1,\dots,n \qquad \text{(b)}$$

This looks harder than just solving (b) but amazingly it is not quite so. Indeed, let us differentiate (a) and write the resulting equations as

$$\partial_j<\phi,\partial_i f> \;=\; <\partial_j f,\partial_i f> + <\phi,\partial_j f> \;=\; 0 \qquad \text{(a')}$$

where $\partial_{ij}\overset{\text{def}}{=}\frac{\partial^2}{\partial u_i\partial u_j}$. Now we interchange i and j and combining (a') with (b) we transform the system (a)+(b) to the *equivalent* system

$$<\phi,\partial_i f> \;=0 \qquad i=1,\dots,n \qquad \text{(a)}$$

$$<\phi,\partial_{ij} f> \;=\; -\frac{1}{2}\psi_{ij} \qquad \text{(b)}^*$$

The new system (a)+(b)* is *not* differential anymore, it is *linear algebraic* with respect to ϕ and so it admits a solution provided that this system is non-singular, i.e. if the vector functions $\partial_i f,\partial_{ij}f : V\to\mathbb{R}^q$ are linearly independent at each point $v\in V$. Notice that there are $n+s_n$ of these functions for $s_n=\frac{n(n+1)}{2}$ being the number of the second derivatives, and so to be independent they need "lots of room." Namely, one needs $q\geq s_n+n$ (cf. Sect.1.3.)

2.4.A. Remark. The maps $\partial_i f,\partial_{ij}f$ are defined with local coordinates on V. But, it is easy to show that if they are linearly independent for one local coordinate system they are independent for all of them. Thus one can make the following

2.4.B. Definition. A C^2-map $f: V\to\mathbb{R}^q$ is called *free* if the vectors $\partial_i f(v)$, $\partial_{ij}f(v)\in\mathbb{R}^q$ are linearly independent at all points $v\in V$.

2.4.C. Examples of free maps. (i) The simplest example is the map $\mathbb{R}^n\to\mathbb{R}^q$, for $q=s_n+n$, given by:

$$(u_1,\dots,u_n)\mapsto(u_1,\dots,u_n;\,u_1u_1,\dots,u_nu_n)\,,$$

(where all the other products $u_i u_j$ with $i \leq j$ are placed between $u_1 u_1$ and $u_n u_n$).

(ii) More interestingly, every sphere S^n admits a free map to $\mathbb{R}^{s_n+n}$ called the *Veronese map*. For example, when $n = 2$, this is the map

$$f : S^2 \subset \mathbb{R}^3 \to \mathbb{R}^6, \quad \text{given by:}$$
$$(x, y, z) \mapsto (x^2, y^2, z^2, \sqrt{2}\,xy, \sqrt{2}\,xz, \sqrt{2}\,yz)\,.$$

(The image $f(S^2)$ is diffeomorphic to $\mathbb{R}P^2$ lying in a hyperplane $H \approx \mathbb{R}^5 \subset \mathbb{R}^6$, since $x^2 + y^2 + z^2 = 1$ on $S^2 \subset \mathbb{R}^3$.)

(iii) A *generic* map $V^n \to \mathbb{R}^q$ is *free for* $q \geq s_n + 2n$. This means that free maps are *open* and *dense* among all C^2-maps for $q \geq s_n + 2n$. This is a theorem due to Nash which follows by a straightforward application of Thom's transversality theorem. (See [Na]$_2$ and see also [Gro], p.33)

2.4.D. Conclusion. To summarize, we can say that the linearized system

$$L_f(\phi) = \psi$$

is solvable for all ψ, provided the map f is free. In particular, it is solvable for generic f if $q \geq s_n + 2n$.

Thus amazingly, we solved the original P.D.E. system (a)+(b) by pure algebra. In fact, this solution ϕ can be easily expressed as $\phi = M(\psi)$ where $M = M_f$ is a linear differential operator with respect to ψ where the coefficients are rational functions in the derivatives of f, so that these functions *have no poles at free maps* f.

This motivates the following definition (cf.[Gro], p.115).

A (non-linear) differential operator is called *infinitesimally invertible* on some space F of functions where it applies, if its *linearization* $L = L_f$ is invertible at every $f \in F$ by a differential operator, say $M = M_f$ where invertible means $L_f \circ M_f = Id$ and where the coefficients of $M_f(\psi)$ are smooth functions which depend on the derivatives of f of some order.

So we can express the above by saying that *the metric inducing operator is infinitesimally invertible on* (the space of) *free maps*.

This fact may seem rather exceptional, but the opposite is true. In fact, a *generic* differential operator D acting from s-tuples of functions to q-tuples for $q > s$ is infinitesimally invertible on an open dense set $\Omega = \Omega(D)$ of q-tuples of functions (See p.156 in [Gro]). Furthermore, in many geometric cases the operator $M = L^{-1}$ can be constructed along the same lines it was done above for the metric inducing operator.

2.5. Solving the equation $D(f_0 + \phi) = D(f_0) + \psi$ for small ψ. If we can solve $L_{f_0}(f_0) = \psi$ we get $D(f_0 + f_0) = D(f_0) + \psi + Q(\psi)$ with a quadratic error $Q(\psi_1)$. Then we repeat this with $f_1 = f_0 + \phi_0$ and $\psi_1 = Q\,(\psi = \psi_0)$, then again with $f_2 = f_1 + \phi_1$ and $\psi_2 = Q(\psi_1)$, etc. As each next error ψ_i is quadratic in ψ_{i-1}, one may hope for convergence if ψ is "small." This is justified in the real analytic case but not in the smooth case. Yet, following Nash, one can combine the above with some smoothing operators and enforce the convergence. (See [Na]$_2$,[Mo],[Ha]). As a result of this we get the following general

2.5.A. Theorem. *([Gro], p.118) If $D : F \to G$ is an infinitesimally invertible operator then it is open with respect to the C^∞-topology in our space of functions.*

In particular, if our D is a structure inducing operator, then the inducible structures contain a non-empty C^∞-open subset.

2.6. Examples.

(A) *Isometric immersions.* The most well known example of a result such as Theorem 2.5.A is the (now classical)

Theorem of Nash. ([Na]$_2$) *If $q \geq s_n + 2n$, then the space of C^∞-metrics on $V = V^n$ inducible from $\mathbb{R}^q$ contains a non-empty open subset.*

(B) *Inducing connections.* (See 3.1.D) Another example is related to the construction of a universal principal bundle-with connection (cf.[N-R], [D'A]$_1$). Here the corresponding structure inducing problem can be formulated as follows.

Given two C^∞-smooth principal bundles $X \to V$ and $Y \to W$ with the same structure group G and with C^∞-connections Γ on X and ∇ on Y we ask when there exists a C^∞-smooth map $f : V \to W$ such that the induced bundle $f^*(Y)$ over V with the induced connection $f^*(\nabla)$ is isomorphic to (X, Γ).

It is shown in [D'A]$_1$, that under certain regularity conditions the (non-linear differential) operator $D = D_\nabla$ which assigns to each smooth map $f : V \to W$ the connection $f^*(\nabla)$ on X (for a fixed on Y) is an open map. Here is a specific result

(B') Theorem. *([D'A]$_1$, p.77). If $\dim W \geq n \dim G$ and $\dim W \geq n$, where $n = \dim V$, then the space of C^∞-connections on V inducible from (a fixed) C^∞-connection on Y contains a non-empty-open subset.*

(C) *Inducing pairs (metric, connection).* (See [D'A]$_2$). Let $X \to V$ and $Y \to W$ be given S^1-bundles with connections Γ on X and ∇ on Y, respectively, and Riemannian metrics g on V and h on W. We study immersions $V \to W$ inducing both the given bundle with connection and the given metric. The problem naturally generalizes to higher dimensional bundles and in fact the subject in [D'A]$_2$ was partially motivated by the Yang-Mills equations where one could hope to get some insight by inducing both from some universal object. (Recall that, for example, the $SU(2)$-instantons over S^4 can be induced by certain maps $S^4 \to \mathbb{H}P^q$ (cf.[At])).

Unfortunately, the general method based on the Nash-Gromov implicit function theorem does not cover the Yang-Mills case for the reason that the partial differential equations expressing the inducing relation become, for bundles of rank > 1, very degenerate and the corresponding differential operator is *not infinitesimally invertible.*

But, for the case when the $Y \to W = \mathbb{C}P^q$ is the Hopf bundle, our method leads to the following

(C') Theorem. *([D'A]$_2$, p.4). If $2q \geq \frac{1}{2}n(n+1) + 3n$, $n = \dim V$, then there exists a non-empty open subset in the space of pairs (g, Γ) on V which can be induced from (h, ∇) on $\mathbb{C}P^q$.*

§3. Topological sheaves, the h-principle and induced structures.

It is a long way from the implicit function theorem, which claims some map $D : F \to G$ to be open, to actually solving the equation $D(f) = g$, i.e. showing D is onto. Nash invented a very special approach for isometric immersions $V \to \mathbb{R}^q$ which has rather limited applicability. The general method is due to Gromov, and consists of an assemblying process of local solution of our equation, one solution $f_v(v')$ for each $v \in V$, and v' ranging in some small neighbourhood $U_v \subset V$. This needs a rather elaborate form of topological sheaves and the method eventually applies to equations of geometric origin as they have some auxiliary symmetry. Eventually one obtains in this way more than just solvability of the equation $D(f) = g$, but also a description of the homotopy type of the space of solutions in terms of some space of continous sections of some auxiliary (jet) fibration over V. When this is possible (and has been proven by now in many cases, see [Gro]), one says that our (structure inducing) problem and / or the space of maps (solving some differential relation) in question satisfy the *h-principle*.

Here are some examples

3.1. (A) (Compare 1.3.-(vii)) *Differential immersions $V^n \to \mathbb{R}^q$ satisfy the h-principle if $q > n$.* (See [Gro], p.7),

In fact, the natural map given by the differential from the space $Imm(V \to W)$ to the space of fiberwise injective linear homomorphisms $T(V) \to T(W)$ is a homotopy equivalence. It follows in particular that the existence of such a map implies the existence of immersions (provided $q > n$ where the h-principle is valid). For example, every stably parallelizable manifold can be immersed into $\mathbb{R}^{n+1}$ (as we saw already in 1.3.-(vii)).

(B) *Free maps $V^n \to \mathbb{R}^q$ satisfy the h-principle for $q \geq s_n + 2n + 1$.* (See [Gro], p.9). For example, every stable parallelizable manifold admits a free map $V^n \to \mathbb{R}^q$ for $q = s_n + 2n + 1$.

(C) (Cf.1.3.-(viii)) *Free isometric C^k-immersions $V^n \to \mathbb{R}^q$, $k = 5, 6, \ldots, \infty$, satisfy the h-principle for $q \geq s_n + 2n + 3$.* (See [Gro], p.12).

(D) (Cf.2.6.(B)) The global existence result indicated below (whose proof makes an essential use of Gromov's h-principle) gives a complete solution to the problem of inducing orthogonal connections from the universal bundles.

Theorem. ([D'A]$_1$). *Every connection in an $O(k)$-bundle over an n-dimensional manifold V can be induced from the standard $O(m)$-invariant connection in the canonical bundle over the Grassmann manifold $Gr_k(\mathbb{R}^m)$, for $m = \frac{nk}{2} + c_1 n + c_2 k + c_3$.*

Remark. The constant c_i can be made more explicit in terms of the topology and the bundle in question. In any case (see [D'A]$_1$, p.68 for the precise statement and a related discussion), all three c_i are bounded by a universal constant, something like $c_i \leq \frac{3}{2}$. We remind the reader that the existence of a connection inducing map was first proven by Narasimhan and Ramanan ([N-R]) in 1961 for a very large $m \sim 2n^2k^3$, which was later on reduced to $m \approx nk$ by Gromov ([Gro], p.95). It was also known that one could not go below $\frac{nk}{2}$ and so the result in [D'A]$_1$ $m \approx \frac{nk}{2}$ is optimal up to a specification of c_i.

(E) *Inducing subbundles of given codimension.* (Cf.[D'A]$_3$). A similar approach to that used in [D'A]$_1$ for studying connection inducing maps applies to the following problem. Consider two manifolds X and Y and smooth subbundles $S \subset T(X)$ $T \subset$

$T(Y)$ (both of fixed codimension k) and study maps $f : X \to Y$ which induce S from T. By a parallel construction to that used for the inducing connection operator (compare 2.6.(B)) one can show that the operator $D_T : f \mapsto S = f^*(T)$ is an open map provided that certain non-degeneracy conditions are satisfied. Thus the Gromov-h-principle machinery outlined in the beginning of this section allows one to obtain, under certain topological restrictions on the bundle X, one more instance of the h-principle which can be seen in the form of a global existence theorem for bundle inducing maps $f : X \to Y$ (see Theorem 5.A., [D'A]$_3$, p.102). In particular, this gives an effective criterion (which we state in (D') below) for the existence of integral submanifolds $L \subset Y$ of a given dimension l. (Observe that here the inducing equation $D_T(f) = S$ generalizes the classical Pfaff system where the solutions are (integral) submanifolds $L \subset Y$ tangent to T (i.e. having $T(L) \subset T$)).

(D') (See [D'A]$_3$, p.103) *A generic smooth C^∞ -subbundle h of codimension k of the tangent bundle $T(Y)$ admits an integral submanifold $L \subset Y$ of dimension l, provided $2l(k+1) \leq \dim Y$.*

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