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NATURAL OPERATORS TRANSFORMING PROJECTABLE VECTOR FIELDS TO PRODUCT PRESERVING BUNDLES

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ABSTRACT. We determine all natural operators transforming projectable vector fields on a fibered manifold Y into vector fields on FY, where F is any product preserving bundle functor on the category of all fibered manifolds.

In the present paper, we describe all natural operators transforming projectable vector fields on fibered manifolds to vector fields on product preserving bundles. We follow the basic teminology from [2] and heavily use the result of Mikulski, [3], who classified all product preserving bundle functors on the category $\mathcal{FM}$ of fibered manifolds by means of product preserving bundle functors on the category $\mathcal{M}f$ of smooth manifolds. We also essentially use another result of his, namely the description of all natural transformations between product preserving bundle functors on $\mathcal{FM}$.

1. A classical result by Michor and others, [2] reads that every product preserving bundle functor $F : \mathcal{M}f \to \mathcal{FM}$ is a Weil functor.

We remind the result of Mikulski, [3] in the slightly modified form. Let $F : \mathcal{FM} \to \mathcal{FM}$ be a bundle functor and $i, j : \mathcal{M}f \to \mathcal{FM}$ be bundle functors defined as follows. For a manifold M, we set $i(M) = id_M : M \to M$ and $j(M) = pt_M : M \to pt$, where pt denotes the one-point manifold. Let $t_M : i(M) \to j(M)$ be the identity natural transformation. We put $G^F = F \circ i$, $H^F = F \circ j$ and $\mu^F = Ft : G^F \to H^F$. If F preserves products, then so do G^F and H^F. It follows the existence of Weil algebras A, B satisfying $G^F = T^A$ and $H^F = T^B$. Moreover, we have the natural transformation $\mu^F : T^A \to T^B$ determined by the algebra homomorphism $\mu^F_{\mathbb{R}} : A \to B$. To simplify the notation, we write $\widetilde{\mu}$ instead μ^F and μ instead $\mu^F_{\mathbb{R}}$.

By Mikulski, we define the functor $T^\mu : \mathcal{FM} \to \mathcal{FM}$ in the following way. For a fibered manifold $p : Y \to M$, put $T^\mu Y = T^A M \times_{T^B M} T^B Y$, the pull-back of $T^A M$ and $T^B Y$ with respect to $\widetilde{\mu}_M : T^A M \to T^B M$ and $T^B p : T^B Y \to T^B M$.

For another product preserving bundle functor $\bar{F} : \mathcal{FM} \to \mathcal{FM}$ and a natural transformation $\eta : F \to \bar{F}$ we have two natural transformations $\widetilde{\nu} = \eta \circ i : T^A \to T^{\bar{A}}$ and $\widetilde{\rho} = \eta \circ j : T^B \to T^{\bar{B}}$ determined by homomorphisms $\nu : A \to \bar{A}$ and $\rho : B \to \bar{B}$.

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They commute with the natural transformations $\widetilde{\mu} : T^A \to T^B$, $\widetilde{\bar{\mu}} : T^{\bar{A}} \to T^{\bar{B}}$ in the sense $\widetilde{\bar{\mu}} \circ \widetilde{\nu} = \widetilde{\rho} \circ \widetilde{\mu}$ and consequently $\bar{\mu} \circ \nu = \rho \circ \mu$. Then the result of Mikulski reads as follows.

Proposition 1. *Every product preserving bundle functor $F : \mathcal{FM} \to \mathcal{FM}$ is naturally equivalent to T^μ for some Weil algebra homomorphism $\mu : A \to B$. Moreover, for another product preserving bundle functor $\bar{F} = T^{\bar{\mu}}$, every natural transformation $\eta : F \to \bar{F}$ is identified with the product $\widetilde{\nu} \times \widetilde{\rho}|_{T^\mu}$, where $\widetilde{\nu} = \nu \circ i$, $\widetilde{\rho} = \eta \circ j$.*

In particular, every product fibered manifold $M \times N \to N$ coincides with $i(M) \times j(N)$, the product being in $\mathcal{FM}$. This implies $T^\mu(M \times N \to M) = T^A M \times T^B N \to T^A M$.

Now we remind some basic concepts and properties of Weil algebras and Weil bundles and give their generalizations for the functors in question.

Let $\mathbb{D}$ be the algebra of dual numbers and A an arbitrary Weil algebra. Then $A \otimes \mathbb{D}$ is identified with $A \times A$ with the multiplication defined by

$$(a,b)(c,d) = (ac, ad + bc) \tag{1}$$

Furthermore, $T^A \circ T^B = T^{A \otimes B}$ for any Weil algebras A, B.

Koszul, [2], defined the action of a Weil algebra A on tangent vectors of T^A in the following way. Let $m : \mathbb{R} \times TM \to TM$ be the multiplication of tangent vectors on a manifold M by reals. Applying T^A, we construct $T^A m : T^A\mathbb{R} \times T^A TM \to T^A TM$. Since $T^A\mathbb{R} = A$ and $T^A TM = T^{A \otimes \mathbb{D}} M$, using the exchange isomorphism $A \otimes \mathbb{D} \to \mathbb{D} \otimes A$, we obtain a map $A \times TT^A M \to TT^A M$. Taking an element $c \in A$, we have a map

$$L(c)_M : TT^A M \to TT^A M.$$

The maps $L(c)_M$ form a natural affinor on $TT^A M$. Taking into account the identification $A \otimes \mathbb{D} \simeq A \times A$, we obtain the following coordinate expression of $L(c)_M$

$$c(a_1, \dots, a_m, b_1, \dots, b_m) = (a_1, \dots, a_m, cb_1, \dots, cb_m).$$

For every natural bundle F, we have the flow operator $\mathcal{F}$ defined on vector fields by $\mathcal{F}(X) = \frac{\partial}{\partial t}|_0 F(Fl_t^X)$, where Fl_t^X denotes the flow of a vector field X to be lifted. This is a natural operator and in the case $F = T^A$ its composition with $L(c)$ yields a set of natural operators $T \to TT^A$ given by $L(c) \circ \mathcal{T}^A$.

The absolute operators, [2] are constructed from derivations as follows. The Lie algebra $\mathcal{A}utA$ associated with the Lie group of all algebra automorphisms of a Weil algebra A is identified with the set of all derivations $\operatorname{Der} A$. For any $D \in \operatorname{Der} A$, its one-parameter subgroup $\delta(t)$ on $\operatorname{Aut} A$ determines the vector field $D_M = \frac{d}{dt}|_0 \widetilde{\delta(t)}_M$ on $T^A M$. We obtain a natural operator $\operatorname{op} D$, defined by $(\operatorname{op} D)_M X = D_M$.

Now we present the result of Kolář, [1], giving the classification of all natural operators $T \to TT^A$.

Proposition 2. *All natural operators $TM \to TT^A M$ are of the form $L(c)_M \circ \mathcal{T}^A + (\operatorname{op} D)_M$ for some $c \in A$, $D \in \operatorname{Der} A$.*

2. We are going to generalize this result for fibered manifolds and projectable vector fields. First we investigate the absolute operators.

Let $\mu : A \to B$, $\bar{\mu} : \bar{A} \to \bar{B}$ be homomorphisms of Weil algebras and T^{μ}, $T^{\bar{\mu}}$ the associated bundle functors. By Proposition 1, natural transformations $T^{\mu} \to T^{\bar{\mu}}$ correspond to $(\widetilde{\mu}, \widetilde{\bar{\mu}})$-related pairs of homomorphisms $\nu : A \to \bar{A}$, $\rho : B \to \bar{B}$ such that the following diagrams commute

$$
\begin{array}{ccc}
T^{A}M & \xrightarrow{\widetilde{\mu}_M} & T^{B}M \\
{\scriptstyle \widetilde{\nu}_M}\downarrow & & \downarrow{\scriptstyle \widetilde{\rho}_M} \\
T^{\bar{A}}M & \xrightarrow[\widetilde{\bar{\mu}}_M]{} & T^{\bar{B}}M
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{\mu} & B \\
{\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \rho} \\
\bar{A} & \xrightarrow[\bar{\mu}]{} & \bar{B}
\end{array}
$$

Let us investigate all natural equivalences of T^{μ}. We denote by $\mathrm{Aut}(A,\mu,B) \subset \mathrm{Aut}\, A \times \mathrm{Aut}\, B$ the set of all μ-related pairs of automorphisms $\nu : A \to A$, $\rho : B \to B$. Analogously we define $\mathrm{Der}(A, \mu, B) \subset \mathrm{Der}\, A \times \mathrm{Der}\, B$ as the set of all μ-related pairs of derivations $D_A : A \to A$, $D_B : B \to B$. The relation between $\mathrm{Aut}(A, \mu, B)$ and $\mathrm{Der}(A, \mu, B)$ is given by the following lemma

Lemma 3. *$\mathrm{Aut}(A, \mu, B)$ is a Lie subgroup in $\mathrm{Aut}\, A \times \mathrm{Aut}\, B$ and its Lie algebra $\mathcal{A}ut(A, \mu, B)$ coincides with $\mathrm{Der}(A, \mu, B)$.*

Proof. Clearly, $\mathrm{Aut}(A, \mu, B)$ is an algebraic subgroup in $\mathrm{Aut}\, A \times \mathrm{Aut}\, B$. We prove it is closed. Define $f_1 : \mathrm{Aut}\, A \to \mathrm{Hom}(A, B)$, $\varphi \mapsto \mu \circ \varphi$ and $f_2 : \mathrm{Aut}\, B \to \mathrm{Hom}(A, B)$, $\psi \mapsto \psi \circ \mu$. Then $\mathrm{Aut}(A, \mu, B) = \{(\varphi, \psi) \in \mathrm{Aut}\, A \times \mathrm{Aut}\, B; f_1(\varphi) = f_2(\psi)\}$, which is closed in $\mathrm{Aut}\, A \times \mathrm{Aut}\, B$.

Let $D \in \mathcal{A}ut(A, \mu, B)$ and $(\nu(t), \rho(t))$ be its one-parameter subgroup on $\mathrm{Aut}\, A \times \mathrm{Aut}\, B$. It holds $\mu \circ D_A = \mu \circ \frac{d}{dt}|_0 \nu(t) = \frac{d}{dt}|_0(\mu \circ \nu(t))$, since μ,ν are linear maps. This is equal to $\frac{d}{dt}|_0(\rho(t) \circ \mu) = \frac{d}{dt}|_0(\rho(t)) \circ \mu = D_B \circ \mu$.

Conversely, let $D = (D_A, D_B) \in \mathrm{Der}(A, \mu, B)$. Put $\nu(t) = \exp(tD_A)$, $\rho(t) = \exp(tD_B)$. We have $\mu \circ \nu(t)(a) = \mu(\sum_{n=0}^{\infty} \frac{t^n D_A^n}{n!}(a)) = \sum_{n=0}^{\infty} \frac{t^n (\mu \circ D_A^n)(a)}{n!}$ for any $a \in A$. Using repeatedly $\mu \circ D_A = D_B \circ \mu$, we obtain $\sum_{n=0}^{\infty} \frac{t^n (D_B^n \circ \mu)(a)}{n!}$, which is $\rho(t) \circ \mu(a)$. This completes the proof. □

Another important property of functors T^{μ} is the iteration one analogous to Weil functors T^{A}. It holds

$$T^{\bar{\mu}} \circ T^{\mu} = T^{\bar{\mu} \otimes \mu} \tag{2}$$

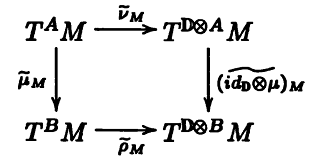
for any Weil algebra homomorphisms $\mu : A \to B$, $\bar{\mu} : \bar{A} \to \bar{B}$.

Let $Y \to M$ be any fibered manifold, $\mu : A \to B$ a homomorphism between Weil algebras A, B and $D \in \mathrm{Der}(A, \mu, B)$. If $\delta(t)$ is its one-parameter subgroup, then $D_Y = \frac{d}{dt}|_0 \widetilde{\delta(t)}$ determines a vector field D_Y on $T^{\mu}Y$ and for a projectable vector field X on Y we obtain the absolute natural operator $(\mathrm{op}\, D)_Y$ defined by $(\mathrm{op}\, D)_Y X = D_Y$.

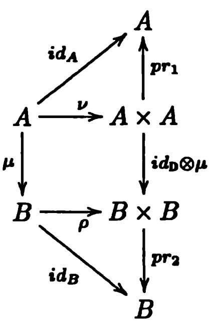
3. All absolute natural operators $T_{proj}Y \to TT^{\mu}Y$ are given by the following assertion

Proposition 4. *Every absolute natural operator $A_Y : TY \to TT^{\mu}Y$ is of the form $(\operatorname{op} D)_Y$ for some $D \in \operatorname{Der}(A, \mu, B)$.*

Proof. One can immediately verify, that $\operatorname{op} D$ is a natural transformation $T^{\mu} \to TT^{\mu}$. We have $T^A = T^{id_A}$ for the identity homomorphism on A and consequently $T = T^{id_{\mathbb{D}}}$. Applying (2) we obtain $TT^{\mu} = T^{id_{\mathbb{D}} \otimes \mu}$. We use the decomposition of A_Y in the sense of the second part of Proposition 1 into the couple of natural transformations $\widetilde{\nu}$, $\widetilde{\rho}$ satisfying the commutativity of the following diagram



Taking into account the fact that A_Y covers the identity on $T^{\mu}Y$, the identification (1), the correspondence between natural transformations on Weil bundles and homomorphisms of their associated Weil algebras, we reduce the above diagram to the following commutative one



Put $D_A = pr_2 \circ \nu$, $D_B = pr_2 \circ \rho$. We prove that $D_A \in \operatorname{Der} A$, $D_B \in \operatorname{Der} B$. We have $\nu(ab) = (ab, D_A(ab))$. On the other hand $\nu(a)\nu(b) = (ab, aD_A(b) + bD_A(a))$, using (1) again. It follows that $D_A \in \operatorname{Der} A$ and analogously $D_B \in \operatorname{Der} B$. Since $id_{\mathbb{D}} \otimes \mu \simeq (\mu, \mu)$ in (1), one immediately verifies that $(D_A, D_B) \in \operatorname{Der}(A, \mu, B)$. This proves our assertion. □

4. In the next step, we generalize the Koszul action to a functor T^{μ} and find all natural affinors on T^{μ}. Let $\mu : A \to B$ be a Weil algebra homomorphism, $c \in A$ and $(y, z) \in T^{\mu}Y = T^M \times_{T^B M} T^B Y$. We define a map $L(c)_Y : TT^{\mu}Y \to TT^{\mu}Y$ as follows

$$L(c)_Y(y, z) = (L(c)_M(y), L(\mu(c))_Y(z)) \tag{3}$$

where $L(c)_M(y)$ and $L(\mu(c))_Y(z)$ are the classical natural affinors by Koszul on Weil bundles $T^A M, T^B Y$.

Proposition 5. *Let $\mu : A \to B$ be a Weil algebra homomorphism. Then every natural affinor on T^μ is of the form $L(c)$, $c \in A$.*

Proof. We are searching for all natural transformations on TT^μ over the identity on T^μ, which are linear on fibers of $TT^\mu \to T^\mu$. Similarly as in the proof of the previous assertion we deduce that every natural affinor in question is determined by the couple of $T\mu$-related natural transformations $\widetilde{\nu}_M : TT^AM \to TT^AM$ and $\widetilde{\rho}_M : TT^BY \to TT^BY$, which are clearly natural affinors in the classical sense. By [2], there is $c \in A$ and $d \in B$ such that $\widetilde{\nu}_M(y) = L(c)_M(y)$ and $\widetilde{\rho}_M(z) = L(d)_Y(z)$ for any $(y,z) \in T^\mu Y \simeq T^AM \times_{T^BM} T^BY$. Using (1) and the coordinate form of the Koszul action, we directly verify, that $d = \mu(c)$, which completes the proof. □

5. The following theorem states our main result. We recall that $\mathcal{T}^\mu$ is the flow operator of T^μ.

Theorem 6. *Let $\mu : A \to B$ be a homomorphism of Weil algebras and X be a projectable vector field on a fibered manifold Y. Then every natural operator $A_Y : T_{proj}Y \to T^\mu Y$ is of the form*

$$A_Y X = L(c)_Y \circ \mathcal{T}^\mu + (\mathrm{op}\, D)_Y,$$

$c \in A$, $D \in \mathrm{Der}(A, \mu, B)$.

Proof. By Lemma 44.2 of [2], all natural operators $A_Y : T_{proj}Y \to TT^\mu Y$ are of a finite order r given by Weil algebras A and B. By Chapter X of [2] they are with bijection with natural transformations

$$J^r_{proj}(Y, TY) \times_Y T^\mu Y \to TT^\mu Y \tag{4}$$

over the identity on $T^\mu Y$.

Let $V^r_{m,n}$, S, F be the standard fibers of $J^r_{proj}(Y, TY)$, $T^\mu Y$, $TT^\mu Y$ for $Y = \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, respectively. Further, let V_0 denote the set of all r-jets of constant vector fields on Y. Clearly, it holds $S = N_1^m \times N_2^n$, $F = N_1^m \times N_2^n \times \mathbb{R}^m \times N_1^m \times \mathbb{R}^n \times N_2^n$, where N_1, N_2 are the nilpotent ideals of Weil algebras A, B. Let us denote by $V_{0,m}$ the set of all r-jets of constant vector fields on Y, the fiber components of which are zeros. By the general theory, [2], if for natural operators A_1, A_2 in question $A_1|_{V_0 \times S} = A_2|_{V_0 \times S}$ or $A_1|_{V_{0,m} \times S} = A_2|_{V_{0,m} \times S}$ is satisfied, then $A_1 = A_2$.

If we restrict the naturality conditions to local $\mathcal{FM}$-isomorphisms, then natural operators or the associated natural transformations (4) are in the bijective correspondence with $G^r_{m,n}$-equivariant maps

$$V^r_{m,n} \times N_1^m \times N_2^n \to \mathbb{R}^m \times N_1^m \times \mathbb{R}^n \times N_2^n$$

where $G^r_{m,n}$ denotes the group of all r-jets of local $\mathcal{FM}$- isomorphisms of $\mathbb{R}^m \times \mathbb{R}^n$ into itself with source and target zero.

Let $(v_i, v_p) \in V_0$, $(y_i) \in N_1^m$, $(z_p) \in N_2^n$, $(Y_i^0, Y_i) \in \mathbb{R}^m \times N_1^m$ and $(W_p, Z_p) \in \mathbb{R}^n \times N_2^n$. Thus we have

$$\begin{aligned} Y_i^0 &= \widetilde{f}_i(v_j, v_q, y_j, z_q) \\ W_p &= \widetilde{g}_p(v_j, v_q, y_j, z_q) \\ Y_i &= \widetilde{h}_i(v_j, v_q, y_j, z_q) \\ Y_p &= \widetilde{k}_p(v_j, v_q, y_j, z_q) \end{aligned}$$

for suitable smooth functions $\widetilde{f}_i$, $\widetilde{g}_p$, $\widetilde{h}_i$, $\widetilde{k}_p$.

The equivariancy with respect to $G^1_{m,n}$ yields the linearity of $\widetilde{f}_i$, $\widetilde{g}_p$, $\widetilde{h}_i$, $\widetilde{k}_p$. The equivariancy with respect to homotheties on fibers, namely $id_{\mathbb{R}^m} \times kid_{\mathbb{R}^m}$, yields $Y_i^0 = \widetilde{f}_i(v_j, kv_q, y_j, kz_q)$, which implies $Y_i^0 = \widetilde{f}_i(v_j, y_j)$. The exchange of the i-th and j-th axis yields $Y_i^0 = \widetilde{f}(v_i, y_i) = kv_i + f(y_i)$, where $f : N_1 \to \mathbb{R}$ is a linear map. Analogously we deduce $Y_i = g(v_i) + h(y_i)$, $W_p = lv_p + F(z_p)$, $Z_p = G(v_p) + H(z_p)$ for linear maps $g : \mathbb{R} \to N_1$, $h : N_1 \to N_1$, $F : N_2 \to \mathbb{R}$, $G : \mathbb{R} \to N_2$, $H : N_2 \to N_2$ and $l \in \mathbb{R}$. Restricting ourselves to $V_{0,m} \times S$ we obtain

$$\begin{aligned} Y_i^0 &= kv_i + f(y_i), \qquad & Y_i &= g(v_i) + h(y_i) \\ W_p &= F(z_p), & Z_p &= H(z_p) \end{aligned}$$

Let $v_i = 0$. Then we obtain an absolute operator A^1_Y. By Proposition 4, A^1_Y is of the form $(\operatorname{op} D)_Y$ for $D \in \operatorname{Der}(A, \mu, B)$.

For a vector field $X \in V_{0,m}$, $X = (v_i, 0)$ the natural operator $A^2_Y = A_Y - A^1_Y$ satisfies $Y^i_0 = kv_i$, $Y_i = g(v_i)$, $W_p = 0$, $Z_p = 0$. Analogously as in [2] , Theorem 42.12 we obtain

$$(Y_i^0, Y_i) = L(c) \circ \mathcal{T}^\mu, \quad c = k + g(1).$$

If we consider the flow of $\mathcal{T}^\mu X$, which is the product of translations and the identity, we obtain $W_p = 0 = Z_p$. Thus A^2_Y coincides with $L(c) \circ \mathcal{T}^\mu$ on $V_{0,m}$, which completes the proof. □

6. Finally, we disscuss two special important cases of the functor T^μ.

If $B = A$ and $\mu = id_A$, then $F = T^A$ and $FY = T^A Y \to Y \to M$. We obtain the fibered version of Kolář's result from Proposition 2.

The second special case is obtained, if we put $A = \mathbb{R}$, $B = \mathbb{R} \times N$ and $\mu = i : \mathbb{R} \to \mathbb{R} \times N$, where N is the nilpotent part of B and $i(x) = (x, 0)$, $x \in \mathbb{R}$, $0 \in N$. Then $F = T^\mu$ coincides with the vertical Weil bundle V^B and $FY = V^B Y \to Y \to M$. All natural operators in question are of the form

$$k\mathcal{V}^B + \operatorname{op} D$$

for $k \in \mathbb{R}$, $D \in \operatorname{Der} B$. This result was obtained by Slovák, [4] .

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