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ON $\mathcal{C}$-CONFORMAL CHANGES OF RIEMANN-FINSLER METRICS

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ABSTRACT. In this note we give a coordinate-free characterization of the $\mathcal{C}$-conformality introduced by M. Hashiguchi [4]. In order to illustrate the power of our approach, we prove intrinsically the following result and its three-dimensional analogon:

Let (M, E) and $(M, \overline{E})$ be two-dimensional Finsler manifolds. Suppose that $\overline{g} = \varphi g$ is a $\mathcal{C}$-conformal change of the Riemann-Finsler metric g.

If $(\operatorname{grad}\varphi)(v) \neq 0$ $(v \in \mathcal{T}M)$ then there is a connected neighborhood $\mathcal{U}$ of $\pi(v)$ such that $(\mathcal{U}, E \restriction T\mathcal{U})$ and, consequently, $(\mathcal{U}, \overline{E} \restriction T\mathcal{U})$ are Riemannian manifolds.

1. Preliminaries

1.1. Notations. We employ the terminology and conventions of [7] as far as feasible.

(i) M is an n-dimensional $(n > 1)$, C^∞, connected, paracompact manifold, $C^\infty(M)$ is the ring of real-valued smooth functions on M.

(ii) $\pi : TM \to M$ is the tangent bundle of M, $\pi_0 : \mathcal{T}M \to M$ is the bundle of nonzero tangent vectors.

(iii) $\mathfrak{X}(M)$ denotes the $C^\infty(M)$-module of vector fields on M.

(iv) $\Omega^k(M)$ $(k \in \mathbb{N}^+)$ is the module of (scalar) k-forms on M, $\Omega^0(M) := C^\infty(M)$, $\Omega(M) := \bigoplus_{k=0}^{n} \Omega^k(M)$. $\Omega(M)$ is a graded algebra over $C^\infty(M)$, with multiplication given by the wedge product $\wedge$. $\otimes$ stands for the tensor product.

(v) $\Psi^k(M)$ $(k \in \mathbb{N}^+)$ is the $C^\infty(M)$-module of vector k-forms on M. It can be regarded as the space of k-linear (over $C^\infty(M)$) skew-symmetric maps $\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathfrak{X}(M)$. $\Psi^0(M) := \mathfrak{X}(M)$, $\Psi(M) := \bigoplus_{k=0}^{n} \Psi^k(M)$.

(vi) i_X, $\mathcal{L}_X$ $(X \in \mathfrak{X}(M))$ and d are the *insertion operator*, the *Lie-derivative* (with respect to X) and the *exterior derivative*, respectively.

(vii) We shall apply the Frölicher-Nijenhuis calculus of vector-valued forms and derivations, for which we refer to [7] again; see also [5], [6], [9]. We recall here two special, but important cases. If $K \in \Psi^1(M)$, $Y \in \Psi^0(M) := \mathfrak{X}(M)$ then their *Frölicher-Nijenhuis bracket* $[K, Y] \in \Psi^1(M)$ acts as follows:

$$(1) \qquad [K,Y](X) = [K(X), Y] - K[X,Y] \qquad (X \in \mathfrak{X}(M)).$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

As for the derivation induced by K, we have:

$$d_K f := df \circ K \quad (f \in C^\infty(M)). \tag{2}$$

1.2. Some basic facts from the differential geometry of the tangent bundle. Let us consider the tangent manifold TM (or the manifold $\mathcal{T}M$).

(i) $\mathfrak{X}^v(TM)$ and $\mathfrak{X}^v(\mathcal{T}M)$ denote the $C^\infty(TM)$-module of vertical vector fields on TM and $\mathcal{T}M$, respectively. On TM live two canonical objects which play important role among others in Finslerian theory: the *Liouville vector field* $C \in \mathfrak{X}^v(TM)$ and the *vertical endomorphism* $J \in \Psi^1(TM)$ (for the definitions see e.g. [6]). We have:

$$\operatorname{Im} J = \operatorname{Ker} J = \mathfrak{X}^v(TM), \quad J^2 = 0. \tag{3}$$

The *vertical lift* ([6], [8]) of a function $f \in C^\infty(M)$ and a vector field $X \in \mathfrak{X}(M)$ is denoted by f^v and X^v, respectively.

Lemma 1. A function $\varphi \in C^\infty(TM)$ (or $C^\infty(\mathcal{T}M)$) is a vertical lift iff $\forall X \in \mathfrak{X}^v(TM) : X\varphi = 0$.

For a simple *proof* see [7].

(ii) A mapping $S : TM \to TTM$ is said to be a *semispray* on M if it satisfies the following conditions:

(SPR1) S is a vector field of class C^1 on TM.

(SPR2) S is smooth on $\mathcal{T}M$.

(SPR3) $JS = C$.

A semispray S is called a *spray* if it is homogeneous of degree 2, i.e.

(SPR4) $[C, S] = S$

also holds.

(iii) Let $\varphi = f \circ \pi$ ($f \in C^\infty(M)$) be a vertical lift. If S and $\overline{S}$ are semisprays on M then $\overline{S} - S$ is vertical because of (SPR3). According to Lemma 1 the function

$$f^c := S\varphi = S(f \circ \pi)$$

is well-defined; it is called the *complete lift* of f.

Now the *complete lift* X^c of a vector field $X \in \mathfrak{X}(M)$ can be introduced as in [8]:

$$\forall f \in C^\infty(M) : X^c f^c := (Xf)^c.$$

The derivation of the following well-known formulas is straightforward:

$\forall X, Y \in \mathfrak{X}(M), \quad f \in C^\infty(M)$:

$$X^v f^c = X^c f^v = (Xf)^v, \tag{4}$$

$$[X^c, Y^c] = [X, Y]^c, \quad [X^v, Y^c] = [X, Y]^v, \tag{5}$$

$$[C, X^c] = 0 \quad \text{(i.e. } X^c \text{ is homogeneous of degree 1)}, \tag{6}$$

$$JX^c = X^v, \quad [J, X^v] = 0, \quad [J, X^c] = 0. \tag{7}$$

Lemma 2. A vector field $X \in \mathfrak{X}^v(TM)$ (or $\mathfrak{X}^v(\mathcal{T}M)$) is a vertical lift iff $\forall Y \in \mathfrak{X}(M) : [X, Y^v] = 0$.

Proof. It is easy to check that the following assertions are equivalent:

- $\forall Y \in \mathfrak{X}(M) : [X, Y^v] = 0$,
- $\forall Y \in \mathfrak{X}(M),\ f \in C^\infty(M) : [X, Y^v]f^c = 0$,
- $\forall Y \in \mathfrak{X}(M),\ f \in C^\infty(M) :$
  $0 = X(Y^v f^c) - Y^v(Xf^c) \overset{\text{Lemma 1, (4)}}{\iff} Y^v(Xf^c) = 0$,
- $\forall f \in C^\infty(M) : Xf^c$ is a vertical lift,
- X is a vertical lift. □

Remark 1. In the sequel we consider forms over TM or $\mathcal{T}M$. *Differentiability of vector (and scalar) k-forms* ($k \in \mathbb{N}^+$) *is required only over* $\mathcal{T}M$, unless otherwise stated.

(iv) A vector 1-form $h \in \Psi^1(TM)$ is said to be a *horizontal endomorphism* on M if it satisfies the following conditions:

(HE1) h is smooth over $\mathcal{T}M$.

(HE2) h is a projector, i.e. $h^2 = h$.

(HE3) $\operatorname{Ker} h = \mathfrak{X}^v(TM)$.

The *horizontal lift* of a vector field $X \in \mathfrak{X}(M)$ (with respect to h) is $X^h := hX^c$.

- $H := [h, C]$ is the *tension* of h,
- $t := [J, h]$ is the *weak torsion* of h,
- $T := i_S t + H$ (S is an arbitrary semispray on M) is the *strong torsion* of h (cf. 1.1. Notations/(vii)).

Any horizontal endomorphism h determines a canonical *almost complex structure* $F \in \Psi^1(TM)$ ($F^2 = -1$, F is smooth on $\mathcal{T}M$) such that

(8) $$F \circ h = -J, \quad F \circ J = h;$$

it is called the almost complex structure associated with h (see [2]).

1.3. Finsler manifolds. Let a function $E : TM \to \mathbb{R}$, called *energy*, be given. The pair (M, E), or simply M, is said to be a *Finsler manifold* if the energy function satisfies the following conditions:

(F0) $E(v) > 0$ ($v \in \mathcal{T}M$), $E(0) = 0$.

(F1) E is of class C^1 on TM and smooth on $\mathcal{T}M$.

(F2) $CE = 2E$, i.e. E is homogeneous of degree 2.

(F3) The *fundamental form* $\omega := dd_J E \in \Omega^2(\mathcal{T}M)$ is symplectic.

The mapping

(9) $$g : \mathfrak{X}^v(\mathcal{T}M) \times \mathfrak{X}^v(\mathcal{T}M) \to C^\infty(\mathcal{T}M), \quad (JX, JY) \to g(JX, JY) := \omega(JX, Y)$$

is a well-defined, nondegenerate symmetric bilinear form, which we call the *Riemann-Finsler metric* of the Finsler manifold (M, E). If the Riemann-Finsler metric is positive definite then we speak of a *positive definite* Finsler manifold.

On any Finsler manifold there is a spray $S : TM \to TTM$, which is uniquely determined on $\mathcal{T}M$ by the formula

$$i_S\omega = -dE. \tag{10}$$

This spray is called the *canonical spray* of the Finsler manifold.

The fundamental lemma of Finsler geometry [2]. On a Finsler manifold (M, E) there is a unique horizontal endomorphism $h \in \Psi^1(TM)$ such that

(B1) $d_hE = 0$ ("h is conservative").

(B2) $T = 0$ (the strong torsion of h vanishes).

h is called the *Barthel endomorphism* of M. It is given by the formula

$$h = \frac{1}{2}(1 + [J, S]),$$

where S is the canonical spray.

Let us suppose that (M, E) is a Finsler manifold with Riemann-Finsler metric g. There exists a unique (symmetric) tensor $\mathcal{C} : \mathfrak{X}(\mathcal{T}M) \times \mathfrak{X}(\mathcal{T}M) \to \mathfrak{X}(\mathcal{T}M)$, satisfying the following conditions:

(CAR1) $J \circ \mathcal{C} = 0$.

(CAR2) $\forall X, Y, Z \in \mathfrak{X}(\mathcal{T}M) : g(\mathcal{C}(X, Y), JZ) = \frac{1}{2}(\mathcal{L}_{JX}J^*g)\ (Y, Z)$, where J^* is the adjoint operator of J (see [6]). $\mathcal{C}$ is called the *Cartan tensor* of the Finsler manifold (cf. [3]).

(It is a well-known fundamental fact that the vanishing of $\mathcal{C}$ characterizes the Riemannian manifolds!)

The Cartan connection on a Finsler manifold [3]. Let a Finsler manifold (M, E) be given and let denote h the Barthel endomorphism on M. If $\nu := 1 - h$ then the mapping

$$\begin{aligned} g_h : \mathfrak{X}(\mathcal{T}M) \times \mathfrak{X}(\mathcal{T}M) &\to C^\infty(\mathcal{T}M), \\ (X, Y) \to g_h(X, Y) &:= g(JX, JY) + g(\nu X, \nu Y) \end{aligned} \tag{11}$$

is a (pseudo-) Riemannian metric on $\mathcal{T}M$, which we call the *prolonged metric* of g.

There is a unique linear connection D on $\mathcal{T}M$ such that

- $Dh = 0$ (D is *reducible*),
- $DF = 0$ (D is *almost complex* with respect to the almost complex structure associated with h),

- $Dg_h = 0$ (D is *metrical*),

and $\forall\, X, Y \in \mathfrak{X}(\mathcal{T}M)$:

- $\nu\mathbb{T}(\nu X, \nu Y) = 0$ (the $\nu(\nu)$-torsion of D vanishes),
- $h\mathbb{T}(hX, hY) = 0$ (the $h(h)$-torsion of D vanishes),

where $\mathbb{T}$ is the classical torsion tensor of D.

Proposition 1. (*Brickell's theorem*, [1]). Let (M, E) be a positive definite Finsler manifold of dimension $n \geq 3$ and let us suppose that the energy function is symmetric, i.e. $\forall\, v \in \mathcal{T}M : E(v) = E(-v)$.

If the third curvature tensor $\mathbb{Q} := J^*\mathbb{K}$ of the Cartan connection D (where $\mathbb{K}$ is the classical curvature tensor of D) vanishes then the Finsler manifold (M, E) is Riemannian.

The gradient operator on the tangent bundle of a Finsler manifold [7]. Let (M, E) be a Finsler manifold with fundamental form ω. Consider a smooth function $\varphi : TM \to \mathbb{R}$. Nondegeneracy of ω guarantees the existence and unicity of a vector field grad $\varphi \in \mathfrak{X}(\mathcal{T}M)$ charactérized by the formula

$$d\varphi = i_{\operatorname{grad}\varphi}\omega.$$

This vector field is called the *gradient* of φ.

Proposition 2. [7] If φ is a vertical lift (i.e. $\varphi = f \circ \pi$, $f \in C^\infty(M)$) then the gradient vector field of φ has the following properties

(i) $\operatorname{grad}\varphi \in \mathfrak{X}^v(\mathcal{T}M)$.

(ii) $[C, \operatorname{grad}\varphi] = -\operatorname{grad}\varphi$, i.e. $\operatorname{grad}\varphi$ is homogeneous of degree 0.

(iii) $\operatorname{grad}\varphi(E) = f^c$.

2. C-conformal changes of Riemann-Finsler metrics

Definition. Consider the Finsler manifolds (M, E) and $(M, \overline{E})$. Their Riemann-Finsler metrics g and $\overline{g}$ are *conformally equivalent*, if there exists a positive smooth function $\varphi : TM \to \mathbb{R}$ such that $\overline{g} = \varphi g$. In this case we also speak of a *conformal change* of the metric g. The function φ is called the *scale function.* If φ is constant then the conformal change is *homothetic.*

Lemma 3. (*Knebelman's observation*) The scale function between conformally equivalent Riemann-Finsler metrics is a vertical lift.

For a simple coordinate-free *proof* see [7].

Theorem 1. [7] Suppose that g and $\overline{g}$ are conformally equivalent Riemann-Finsler metrics on M:

$$\overline{g} = \varphi g; \quad \varphi = \exp\circ\alpha\circ\pi, \quad \alpha\in C^\infty(M).$$

Then the canonical sprays and the Barthel endomorphisms are related as follows:

$$\overline{S} = S - \alpha^c C + E\,\mathrm{grad}\,\alpha^v, \tag{12}$$

$$\overline{h} = h - \frac{1}{2}(\alpha^c J + d\alpha^v\otimes C) + \frac{1}{2}E[J,\mathrm{grad}\,\alpha^v] + \frac{1}{2}d_J E\otimes\mathrm{grad}\,\alpha^v. \tag{13}$$

Definition. Let g and $\overline{g}$ be Riemann-Finsler metrics on M. The conformal change $\overline{g}=\varphi g$ is *$\mathcal{C}$-conformal* if the scale function satisfies the following conditions:

($\mathcal{C}1$) the change $\overline{g}=\varphi g$ is not homothetic.

($\mathcal{C}2$) $i_{F\,\mathrm{grad}\,\varphi}\mathcal{C}=0$.

Proposition 3. If φ is a vertical lift (i.e. $\varphi = f\circ\pi$, $f\in C^\infty(M)$) then the following assertions are equivalent:

(i) $\mathrm{grad}\,\varphi$ is smooth on the whole tangent manifold TM.

(ii) $\mathrm{grad}\,\varphi = X^v$ ($X\in\mathfrak{X}(M)$, i.e. $\mathrm{grad}\,\varphi$ is a vertical lift).

(iii) $i_{F\,\mathrm{grad}\,\varphi}\mathcal{C}=0$.

Proof. (i) $\iff$ (ii) This follows immediately from Proposition 2/(ii).
(ii) $\iff$ (iii) $\forall Y,Z\in\mathfrak{X}(M)$:

$$\begin{aligned}
2g(\mathcal{C}(F\,\mathrm{grad}\,\varphi,Y^c),Z^v) &= 2g(\mathcal{C}(Y^c,F\,\mathrm{grad}\,\varphi),Z^v) = (\mathcal{L}_{Y^v}J^*g)(F\,\mathrm{grad}\,\varphi,Z^c) = \\
&= Y^v g(\mathrm{grad}\,\varphi,Z^v) - g(J[Y^v,F\,\mathrm{grad}\,\varphi],Z^v) - g(\mathrm{grad}\,\varphi,J[Y^v,Z^c]) \overset{(3),(5)}{=} \\
&= Y^v g(\mathrm{grad}\,\varphi,Z^v) - g(J[Y^v,F\,\mathrm{grad}\,\varphi],Z^v) = \\
&= Y^v(Z^c\varphi) - g(J[Y^v,F\,\mathrm{grad}\,\varphi],Z^v) \overset{\text{Lemma 1, (4)}}{=} \\
&= -g(J[Y^v,F\,\mathrm{grad}\,\varphi],Z^v) \overset{(1),(7)}{=} g([\mathrm{grad}\,\varphi,Y^v],Z^v).
\end{aligned}$$

Thus we have:

$$\forall Y\in\mathfrak{X}(M): i_{F\,\mathrm{grad}\,\varphi}\mathcal{C}(Y^c) = \frac{1}{2}[\mathrm{grad}\,\varphi,Y^v].$$

In view of Lemma 2 this implies that (ii) $\iff$ (iii). □

Corollary 1. Under the $\mathcal{C}$-conformal change $\overline{g}=\varphi g$ ($\varphi=\exp\circ\alpha\circ\pi$, $\alpha\in C^\infty(M)$), the Barthel endomorphisms are related as follows:

$$\overline{h} = h - \frac{1}{2}(\alpha^c J + d\alpha^v\otimes C) + \frac{1}{2}d_J E\otimes\mathrm{grad}\,\alpha^v. \tag{14}$$

3. Applications to two- and three-dimensional Finsler manifolds

Proposition 4. Let (M, E) and $(M, \overline{E})$ be two-dimensional Finsler manifolds. Suppose that $\overline{g} = \varphi g$ is a $\mathcal{C}$-conformal change of the Riemann-Finsler metric g.

If $(\operatorname{grad}\varphi)\ (v) \neq 0$ $(v \in \mathcal{T}M)$ then there is a connected neighborhood $\mathcal{U}$ of $\pi(v)$ such that $(\mathcal{U}, E \restriction T\mathcal{U})$ and, consequently, $(\mathcal{U}, \overline{E} \restriction T\mathcal{U})$ are Riemannian manifolds.

Proof. It is easy to check that the Cartan tensor $\mathcal{C}$ of the Finsler manifold (M, E) is semibasic and $i_S\mathcal{C} = 0$ (S is an arbitrary semispray on M).

Since the change is not homothetic there is a tangent vector $v \in \mathcal{T}M$ satisfying the condition $(\operatorname{grad}\varphi)\ (v) \neq 0$. According to Proposition 3, $\operatorname{grad}\varphi$ is a vectical lift: $\operatorname{grad}\varphi = X^v$, $X \in \mathfrak{X}(M)$. Thus there is a connected neighborhood $\mathcal{U}$ of $\pi(v)$ such that

- $\forall w \in \pi_0^{-1}(\mathcal{U}) : X^v(w) := (\operatorname{grad}\varphi)(w) \neq 0.$

  Let $\Delta := \{z \in \pi_0^{-1}(U) \mid (X^v(z), C(z))$ is linearly dependent in $T_zTM\}$.

  Then $\forall p \in \mathcal{U}$:

$$\Delta_p := \Delta \cap T_pM = \{rX(p) \mid r \in \mathbf{R} \setminus \{0\}\},$$

and thus $int\Delta$ is empty in $\pi_0^{-1}(\mathcal{U})$.

Since $FC = S$ (S is the canonical spray) and $\iota_S\mathcal{C} = 0$, $(\mathcal{C}2)$ implies the vanishing of $\mathcal{C}$ over $\pi_0^{-1}(\mathcal{U}) \setminus \Delta$. Therefore

$\mathcal{C} \restriction \pi_0^{-1}(\mathcal{U}) = 0$, i.e. $(\mathcal{U}, E \restriction T\mathcal{U})$ is a Riemannian manifold. □

Proposition 5. Let (M, E) and $(M, \overline{E})$ be three-dimensional, positive definite Finsler manifolds with symmetric energy functions. Suppose that $\overline{g} = \varphi g$ is a $\mathcal{C}$-conformal change of the Riemann-Finsler metric g.

If $(\operatorname{grad}\varphi)(v) \neq 0$ $(v \in \mathcal{T}M)$ then there is a connected neighborhood $\mathcal{U}$ of $\pi(v)$ such that $(\mathcal{U}, E \restriction T\mathcal{U})$ and, consequently, $(\mathcal{U}, \overline{E} \restriction T\mathcal{U})$ are Riemannian manifolds.

Proof. Let us choose a tangent vector $v \in \mathcal{T}M$ satisfying the condition $(\operatorname{grad}\varphi)(v) \neq 0$. Since $\operatorname{grad}\varphi$ is a vertical lift there is a connected neighborhood $\mathcal{U}$ of $\pi(v)$ such that

- $\forall w \in \pi_0^{-1}(\mathcal{U}) : X^v(w) := (\operatorname{grad}\varphi)(w) \neq 0$

Consider the third curvature tensor $\mathbb{Q}$ of the Cartan connection of (M, E). In view of Brickell's theorem it is sufficient to show that $\mathbb{Q} \restriction \pi_0^{-1}(\mathcal{U}) = 0$.

Applying the explicit formulas of [3] which describe the covariant derivatives with respect to the Cartan connection, we get:

$$\mathbb{Q}(X,Y)Z = \mathcal{C}(F\mathcal{C}(X,Z),Y) - \mathcal{C}(X, F\mathcal{C}(Y,Z)) \quad (X,Y,Z \in \mathfrak{X}(TM)). \tag{15}$$

Therefore

(i) $\mathbb{Q}(X,Y)S = \mathbb{Q}(X,S)Y = \mathbb{Q}(S,X)Y = 0$ (S is an arbitrary semispray on M),

(ii) $\mathbb{Q}(X,Y)F\operatorname{grad}\varphi = \mathbb{Q}(X, F\operatorname{grad}\varphi)Y = \mathbb{Q}(F\operatorname{grad}\varphi, X)Y = 0,$

(iii) $\mathbb{Q}(X,X)Y = 0.$

Let $\Delta := \{z \in \pi_0^{-1}(\mathcal{U}) \mid (X^v(z), C(z))$ is linearly dependent in $T_zTM\}$. Then (i)–(iii) imply the vanishing of $\mathbf{Q}$ over the set $\pi_0^{-1}(\mathcal{U}) \setminus \Delta$.

Thus we obtain, as in the proof of Proposition 4, that $\mathbf{Q} \restriction \pi_0^{-1}(\mathcal{U}) = 0$. $\square$

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