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ON C-CONFORMAL CHANGES OF Riemann-Finsler Metrics

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ABSTRACT. In this note we give a coordinate-free characterization of the C-conformality introduced by M. Hashiguchi [4]. In order to illustrate the power of our approach, we prove intrinsically the following result and its three-dimensional analogon:

Let \((M, E)\) and \((\tilde{M}, \tilde{E})\) be two-dimensional Finsler manifolds. Suppose that \(\tilde{g} = \varphi g\) is a C-conformal change of the Riemann-Finsler metric \(g\).

If \((\text{grad} \varphi)(v) \neq 0 (v \in TM)\) then there is a connected neighborhood \(U\) of \(\pi(v)\) such that \((U, \tilde{E} \upharpoonright TU)\) and, consequently, \((U, \tilde{E} \upharpoonright TU)\) are Riemannian manifolds.

1. Preliminaries

1.1. Notations. We employ the terminology and conventions of [7] as far as feasible.

(i) \(M\) is an \(n\)-dimensional \((n > 1)\), \(C^\infty\), connected, paracompact manifold, \(C^\infty(M)\) is the ring of real-valued smooth functions on \(M\).

(ii) \(\pi : TM \to M\) is the tangent bundle of \(M\), \(\pi_0 : TM \to M\) is the bundle of nonzero tangent vectors.

(iii) \(\mathfrak{X}(M)\) denotes the \(C^\infty(M)\)-module of vector fields on \(M\).

(iv) \(\Omega^k(M) (k \in \mathbb{N}^+)\) is the module of (scalar) \(k\)-forms on \(M\), \(\Omega^0(M) := C^\infty(M)\), \(\Omega(M) := \bigoplus_{k=0}^{\infty} \Omega^k(M)\). \(\Omega(M)\) is a graded algebra over \(C^\infty(M)\), with multiplication given by the wedge product \(\wedge\). \(\otimes\) stands for the tensor product.

(v) \(\mathfrak{V}^k(M) (k \in \mathbb{N}^+)\) is the \(C^\infty(M)\)-module of vector \(k\)-forms on \(M\). It can be regarded as the space of \(k\)-linear (over \(C^\infty(M)\)) skew-symmetric maps \(\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathfrak{X}(M)\). \(\mathfrak{V}^0(M) := \mathfrak{X}(M)\), \(\mathfrak{V}(M) := \bigoplus_{k=0}^{\infty} \mathfrak{V}^k(M)\).

(vi) \(i_X, \mathcal{L}_X (X \in \mathfrak{X}(M))\) and \(\mathcal{D}\) are the insertion operator, the Lie-derivative (with respect to \(X\)) and the exterior derivative, respectively.

(vii) We shall apply the Frölicher-Nijenhuis calculus of vector-valued forms and derivatives, for which we refer to [7] again; see also [5], [6], [9]. We recall here two special, but important cases. If \(K \in \mathfrak{V}^1(M), Y \in \mathfrak{V}^0(M) := \mathfrak{X}(M)\) then their Frölicher-Nijenhuis bracket \([K, Y] \in \mathfrak{V}^1(M)\) acts as follows:

\[
(K, Y)(X) = [K(X), Y] - K[X, Y] \quad (X \in \mathfrak{X}(M)).
\]

This paper is in final form and no version of it will be submitted for publication elsewhere.
As for the derivation induced by $K$, we have:

\[ d_K f := df \circ K \quad (f \in C^\infty(M)). \]

1.2. Some basic facts from the differential geometry of the tangent bundle. Let us consider the tangent manifold $TM$ (or the manifold $TM$).

(i) $\mathcal{X}(TM)$ and $\mathcal{X}_{v}(TM)$ denote the $C^\infty(TM)$-module of vertical vector fields on $TM$ and $TM$, respectively. On $TM$ live two canonical objects which play important role among others in Finslerian theory: the Liouville vector field $C \in \mathcal{X}(TM)$ and the vertical endomorphism $J \in \Psi^1(TM)$ (for the definitions see e.g. [6]). We have:

\[ \text{Im } J = \text{Ker } J = \mathcal{X}_{v}(TM), \quad J^2 = 0. \]

The vertical lift ([6], [8]) of a function $f \in C^\infty(M)$ and a vector field $X \in \mathcal{X}(M)$ is denoted by $f^v$ and $X^v$, respectively.

**Lemma 1.** A function $\varphi \in C^\infty(TM)$ (or $C^\infty(TM)$) is a vertical lift iff

\[ \forall X \in \mathcal{X}_{v}(TM) : X \varphi = 0. \]

For a simple proof see [7].

(ii) A mapping $S : TM \to TTM$ is said to be a semispray on $M$ if it satisfies the following conditions:

(SPR1) $S$ is a vector field of class $C^1$ on $TM$.

(SPR2) $S$ is smooth on $TM$.

(SPR3) $JS = C$.

A semispray $S$ is called a spray if it is homogeneous of degree 2, i.e.

(SPR4) $[C, S] = S$

also holds.

(iii) Let $\varphi = f \circ \pi$ ($f \in C^\infty(M)$) be a vertical lift. If $S$ and $\overline{S}$ are semisprays on $M$ then $\overline{S} - S$ is vertical because of (SPR3). According to Lemma 1 the function

\[ f^c := S \varphi = S(f \circ \pi) \]

is well-defined; it is called the complete lift of $f$.

Now the complete lift $X^c$ of a vector field $X \in \mathcal{X}(M)$ can be introduced as in [8]:

\[ \forall f \in C^\infty(M) : X^c f^c := (Xf)^c. \]

The derivation of the following well-known formulas is straightforward:

\[ \forall X, Y \in \mathcal{X}(M), \quad f \in C^\infty(M) : \]

\[ (4) \quad X^v f^c = X^c f^v = (Xf)^v, \]

\[ (5) \quad [X^c, Y^c] = [X, Y]^c, \quad [X^v, Y^v] = [X, Y]^v, \]

\[ (6) \quad [C, X^c] = 0 \quad (\text{i.e. } X^c \text{ is homogeneous of degree } 1), \]

\[ (7) \quad JX^c = X^v, \quad [J, X^v] = 0, \quad [J, X^c] = 0. \]
Lemma 2. A vector field \( X \in \mathfrak{X}^v(TM) \) (or \( \mathfrak{X}^v(TM) \)) is a vertical lift iff \( \forall Y \in \mathfrak{X}(M) : [X, Y^v] = 0 \).

Proof. It is easy to check that the following assertions are equivalent:

\begin{itemize}
  \item \( \forall Y \in \mathfrak{X}(M) : [X, Y^v] = 0 \),
  \item \( \forall Y \in \mathfrak{X}(M), f \in C^\infty(M) : [X, Y^v]f^c = 0 \),
  \item \( \forall Y \in \mathfrak{X}(M), f \in C^\infty(M) : 0 = X(Y^v f) - Y^v(Xf^c) \) \( \xrightarrow{\text{Lemma 1, (4)}} \) \( Y^v(Xf^c) = 0 \),
  \item \( \forall f \in C^\infty(M) : Xf^c \) is a vertical lift,
  \item \( X \) is a vertical lift.
\end{itemize}

Remark 1. In the sequel we consider forms over \( TM \) or \( \mathcal{T}M \). Differentiability of vector (and scalar) \( k \)-forms (\( k \in \mathbb{N}^+ \)) is required only over \( \mathcal{T}M \), unless otherwise stated.

(iv) A vector \( 1 \)-form \( h \in \mathfrak{X}^1(TM) \) is said to be a horizontal endomorphism on \( M \) if it satisfies the following conditions:

(HE1) \( h \) is smooth over \( \mathcal{T}M \).

(HE2) \( h \) is a projector, i.e. \( h^2 = h \).

(HE3) \( \text{Ker} h = \mathcal{X}^v(TM) \).

The horizontal lift of a vector field \( X \in \mathfrak{X}(M) \) (with respect to \( h \)) is \( X^h := hX^c \).

\begin{itemize}
  \item \( H := [h, C] \) is the tension of \( h \),
  \item \( t := [J, h] \) is the weak torsion of \( h \),
  \item \( T := i_{st} + H \) (\( S \) is an arbitrary semispray on \( M \)) is the strong torsion of \( h \) (cf. 1.1. Notations/(vii)).
\end{itemize}

Any horizontal endomorphism \( h \) determines a canonical almost complex structure \( F \in \mathfrak{X}^1(TM) \) \( (F^2 = -1, F \) is smooth on \( \mathcal{T}M \)\) such that

\[ F \circ h = -J, \quad F \circ J = h; \]

it is called the almost complex structure associated with \( h \) (see [2]).

1.3. Finsler manifolds. Let a function \( E : TM \to \mathbb{R} \), called energy, be given. The pair \( (M, E) \), or simply \( M \), is said to be a Finsler manifold if the energy function satisfies the following conditions:

(F0) \( E(v) > 0 \) \( (v \in TM) \), \( E(0) = 0 \).

(F1) \( E \) is of class \( C^1 \) on \( TM \) and smooth on \( \mathcal{T}M \).

(F2) \( CE = 2E \), i.e. \( E \) is homogeneous of degree 2.

(F3) The fundamental form \( \omega := ddJ E \in \Omega^2(TM) \) is symplectic.

The mapping

\[ g : \mathfrak{X}^v(TM) \times \overline{\mathfrak{X}}^v(TM) \to C^\infty(TM), \quad (JX, JY) \to g(JX, JY) := \omega(JX, Y) \]
is a well-defined, nondegenerate symmetric bilinear form, which we call the Riemann-Finsler metric of the Finsler manifold \((M, E)\). If the Riemann-Finsler metric is positive definite then we speak of a positive definite Finsler manifold.

On any Finsler manifold there is a spray \(S : TM \to TTM\), which is uniquely determined on \(TM\) by the formula

\[
i_S \omega = -dE.
\]

This spray is called the canonical spray of the Finsler manifold.

The fundamental lemma of Finsler geometry [2]. On a Finsler manifold \((M, E)\) there is a unique horizontal endomorphism \(h \in \mathcal{U}^1(TM)\) such that

- \(d_hE = 0\) ("\(h\) is conservative").
- \(T = 0\) (the strong torsion of \(h\) vanishes).

\(h\) is called the Barthel endomorphism of \(M\). It is given by the formula

\[
h = \frac{1}{2}(1 + [J,S]),
\]

where \(S\) is the canonical spray.

Let us suppose that \((M, E)\) is a Finsler manifold with Riemann-Finsler metric \(g\). There exists a unique (symmetric) tensor \(C : \mathfrak{X}(TM) \times \mathfrak{X}(TM) \to \mathfrak{X}(TM)\), satisfying the following conditions:

1. \(J \circ C = 0\).
2. \(\forall X, Y, Z \in \mathfrak{X}(TM) : g(C(X,Y), JZ) = \frac{1}{2}(L_X J^* g) (Y, Z)\), where \(J^*\) is the adjoint operator of \(J\) (see [6]). \(C\) is called the Cartan tensor of the Finsler manifold (cf. [3]).

(It is a well-known fundamental fact that the vanishing of \(C\) characterizes the Riemannian manifolds!)

The Cartan connection on a Finsler manifold [3]. Let a Finsler manifold \((M, E)\) be given and let denote \(h\) the Barthel endomorphism on \(M\). If \(\nu := 1 - h\) then the mapping

\[
g_h : \mathfrak{X}(TM) \times \mathfrak{X}(TM) \to C^\infty(TM),
\]

\[
(X, Y) \to g_h(X,Y) := g(JX, JY) + g(\nu X, \nu Y)
\]
is a (pseudo-) Riemannian metric on \(TM\), which we call the prolonged metric of \(g\).

There is a unique linear connection \(D\) on \(TM\) such that

- \(Dh = 0\) (\(D\) is reducible),
- \(DF = 0\) (\(D\) is almost complex with respect to the almost complex structure associated with \(h\)),

is a (pseudo-) Riemannian metric on \(TM\), which we call the prolonged metric of \(g\).
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- \( Dg_h = 0 \) (\( D \) is metrical),

and \( \forall X, Y \in \mathfrak{X}(TM) \):

- \( \nu T(\nu X, \nu Y) = 0 \) (the \( \nu(\nu) \)-torsion of \( D \) vanishes),
- \( hT(hX, hY) = 0 \) (the \( h(h) \)-torsion of \( D \) vanishes),

where \( T \) is the classical torsion tensor of \( D \).

**Proposition 1.** (Brickell's theorem, [1]). Let \((M, E)\) be a positive definite Finsler manifold of dimension \( n \geq 3 \) and let us suppose that the energy function is symmetric, i.e. \( \forall v \in TM : E(v) = E(-v) \).

If the third curvature tensor \( Q := J^*K \) of the Cartan connection \( D \) (where \( K \) is the classical curvature tensor of \( D \)) vanishes then the Finsler manifold \((M, E)\) is Riemannian.

The gradient operator on the tangent bundle of a Finsler manifold [7]. Let \((M, E)\) be a Finsler manifold with fundamental form \( \omega \). Consider a smooth function \( \varphi : TM \to \mathbb{R} \). Nondegeneracy of \( \omega \) guarantees the existence and unicity of a vector field \( \text{grad} \varphi \in \mathfrak{X}(TM) \) characterized by the formula

\[
d\varphi = \iota_{\text{grad} \varphi} \omega.
\]

This vector field is called the gradient of \( \varphi \).

**Proposition 2.** [7] If \( \varphi \) is a vertical lift (i.e. \( \varphi = f \circ \pi, f \in C^\infty(M) \)) then the gradient vector field of \( \varphi \) has the following properties

(i) \( \text{grad} \varphi \in \mathfrak{X}^v(TM) \).

(ii) \( [C, \text{grad} \varphi] = -\text{grad} \varphi \), i.e. \( \text{grad} \varphi \) is homogeneous of degree 0.

(iii) \( \text{grad} \varphi(E) = f^c \).

2. C-conformal changes of Riemann-Finsler metrics

**Definition.** Consider the Finsler manifolds \((M, E)\) and \((M, \overline{E})\). Their Riemann-Finsler metrics \( g \) and \( \overline{g} \) are conformally equivalent, if there exists a positive smooth function \( \varphi : TM \to \mathbb{R} \) such that \( \overline{g} = \varphi g \). In this case we also speak of a conformal change of the metric \( g \). The function \( \varphi \) is called the scale function. If \( \varphi \) is constant then the conformal change is homothetic.

**Lemma 3.** (Knebelman's observation) The scale function between conformally equivalent Riemann-Finsler metrics is a vertical lift.

For a simple coordinate-free proof see [7].
Theorem 1. [7] Suppose that $g$ and $\bar{g}$ are conformally equivalent Riemann-Finsler metrics on $M$: 
$$\bar{g} = \varphi g; \quad \varphi = \exp \circ \alpha \circ \pi, \quad \alpha \in C^\infty(M).$$

Then the canonical sprays and the Barthel endomorphisms are related as follows:

(12) 
$$\overline{S} = S - \alpha^c C + E \text{grad} \alpha^v,$$

(13) 
$$\overline{h} = h - \frac{1}{2}(\alpha^c J + d\alpha^v \otimes C) + \frac{1}{2}E[J, \text{grad} \alpha^v] + \frac{1}{2}d_J E \otimes \text{grad} \alpha^v.$$

**Definition.** Let $g$ and $\bar{g}$ be Riemann-Finsler metrics on $M$. The conformal change $\bar{g} = \varphi g$ is $C$-conformal if the scale function satisfies the following conditions:

(C1) the change $\bar{g} = \varphi g$ is not homothetic.

(C2) $i_{F \text{grad} \varphi} C = 0.$

**Proposition 3.** If $\varphi$ is a vertical lift (i.e. $\varphi = f \circ \pi$, $f \in C^\infty(M)$) then the following assertions are equivalent:

(i) $\text{grad} \varphi$ is smooth on the whole tangent manifold $TM$.

(ii) $\text{grad} \varphi = X^v (X \in \mathfrak{X}(M)$, i.e. $\text{grad} \varphi$ is a vertical lift).

(iii) $i_{F \text{grad} \varphi} C = 0.$

**Proof.** (i) $\iff$ (ii) This follows immediately from Proposition 2/(ii).

(ii) $\iff$ (iii) $\forall Y, Z \in \mathfrak{X}(M)$:

$$2g(C(F \text{grad} \varphi, Y^c), Z^v) = 2g(C(Y^c, F \text{grad} \varphi), Z^v) = (CY^c J^* g)(F \text{grad} \varphi, Z^c) =$$

$$= Y^v g(\text{grad} \varphi, Z^c) - g(J[Y^v, F \text{grad} \varphi], Z^v) - g(\text{grad} \varphi, J[Y^v, Z^c]) \overset{(3),(5)}{=}$$

$$= Y^v g(\text{grad} \varphi, Z^c) - g(J[Y^v, F \text{grad} \varphi], Z^v) =$$

$$= Y^v (Z^c \varphi) - g(J[Y^v, F \text{grad} \varphi], Z^v) \overset{1, (4)}{=}$$

$$= -g(J[Y^v, F \text{grad} \varphi], Z^v) \overset{(1),(7)}{=} g([\text{grad} \varphi, Y^v], Z^v).$$

Thus we have:

$$\forall Y \in \mathfrak{X}(M) : i_{F \text{grad} \varphi} C(Y^c) = \frac{1}{2}[\text{grad} \varphi, Y^v].$$

In view of Lemma 2 this implies that (ii) $\iff$ (iii).

**Corollary 1.** Under the $C$-conformal change $\bar{g} = \varphi g$ ($\varphi = \exp \circ \alpha \circ \pi$, $\alpha \in C^\infty(M)$), the Barthel endomorphisms are related as follows:

(14) 
$$\overline{h} = h - \frac{1}{2}(\alpha^c J + d\alpha^v \otimes C) + \frac{1}{2}d_J E \otimes \text{grad} \alpha^v.$$
3. Applications to two- and three-dimensional Finsler manifolds

Proposition 4. Let $(M, E)$ and $(\overline{M}, \overline{E})$ be two-dimensional Finsler manifolds. Suppose that $\overline{g} = \varphi g$ is a $C$-conformal change of the Riemann-Finsler metric $g$.

If $(\text{grad} \varphi)(v) \neq 0 \ (v \in TM)$ then there is a connected neighborhood $U$ of $\pi(v)$ such that $(U, E \upharpoonright TU)$ and, consequently, $(\overline{U}, \overline{E} \upharpoonright TU)$ are Riemannian manifolds.

Proof. It is easy to check that the Cartan tensor $C$ of the Finsler manifold $(M, E)$ is semibasic and $i_S C = 0$ ($S$ is an arbitrary semispray on $M$).

Since the change is not homothetic there is a tangent vector $v \in TM$ satisfying the condition $(\text{grad} \varphi)(v) \neq 0$. According to Proposition 3, grad $\varphi$ is a vertical lift: grad $\varphi = X^u$, $X \in \mathfrak{X}(M)$. Thus there is a connected neighborhood $U$ of $\pi(v)$ such that

- $\forall w \in \pi_0^{-1}(U) : X^u(w) := (\text{grad} \varphi)(w) \neq 0$.

Let $\Delta := \{ z \in \pi_0^{-1}(U) \mid (X^u(z), C(z)) \text{ is linearly dependent in } T_zTM \}$. Then $\forall p \in U$:

$$\Delta_p := \Delta \cap T_pM = \{ rX(p) \mid r \in \mathbb{R} \setminus \{0\} \},$$

and thus int$\Delta$ is empty in $\pi_0^{-1}(U)$.

Since $FC = S$ ($S$ is the canonical spray) and $i_S C = 0$, (C2) implies the vanishing of $C$ over $\pi_0^{-1}(U) \setminus \Delta$. Therefore $C \upharpoonright \pi_0^{-1}(U) = 0$, i.e. $(U, E \upharpoonright TU)$ is a Riemannian manifold.

Proposition 5. Let $(M, E)$ and $(\overline{M}, \overline{E})$ be three-dimensional, positive definite Finsler manifolds with symmetric energy functions. Suppose that $\overline{g} = \varphi g$ is a $C$-conformal change of the Riemann-Finsler metric $g$.

If $(\text{grad} \varphi)(v) \neq 0 \ (v \in TM)$ then there is a connected neighborhood $U$ of $\pi(v)$ such that $(U, E \upharpoonright TU)$ and, consequently, $(\overline{U}, \overline{E} \upharpoonright TU)$ are Riemannian manifolds.

Proof. Let us choose a tangent vector $v \in TM$ satisfying the condition $(\text{grad} \varphi)(v) \neq 0$. Since grad $\varphi$ is a vertical lift there is a connected neighborhood $U$ of $\pi(v)$ such that

- $\forall w \in \pi_0^{-1}(U) : X^u(w) := (\text{grad} \varphi)(w) \neq 0$.

Consider the third curvature tensor $Q$ of the Cartan connection of $(M, E)$. In view of Brickell's theorem it is sufficient to show that $Q \upharpoonright \pi_0^{-1}(U) = 0$.

Applying the explicit formulas of [3] which describe the covariant derivatives with respect to the Cartan connection, we get:

$$Q(X,Y)Z = C(FC(X,Z),Y) - C(X,FC(Y,Z)) \quad (X,Y,Z \in \mathfrak{X}(TM)).$$

Therefore

(i) $Q(X,Y)S = Q(X,S)Y = Q(S,X)Y = 0$ ($S$ is an arbitrary semispray on $M$),
(ii) $Q(X,Y)F \text{ grad } \varphi = Q(X,F \text{ grad } \varphi)Y = Q(F \text{ grad } \varphi,X)Y = 0$,
(iii) $Q(X,X)Y = 0$. 


Let $\Delta := \{ z \in \pi_0^{-1}(U) | (X^*(z), C(z)) \text{ is linearly dependent in } T_zTM \}$. Then (i)–(iii) imply the vanishing of $Q$ over the set $\pi_0^{-1}(U) \setminus \Delta$.

Thus we obtain, as in the proof of Proposition 4, that $Q \mid \pi_0^{-1}(U) = 0$. □

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