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ABOUT DUALITY AND KILLING TENSORS

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ABSTRACT. In this paper the isometries of the dual space were investigated. The dual structural equations of a Killing tensor of order two were found. The general results are applied to the case of the flat space.

1. INTRODUCTION

Killing tensors are indispensable tools in the quest for exact solutions in many branches of general relativity as well as classical mechanics [1]. Killing tensors are important for solving the equations of motion in particular space-times. The notable example here is the Kerr metric which admits a second rank Killing tensor [1]. The Killing tensors give rise to new exact solutions in perfect fluid Bianchi and Katowski-Sachs cosmologies as well in inflationary models with a scalar field sources [2]. Recently the Killing tensors of third rank in $(1+1)$ dimensional geometry were investigated and classified [3]. In a geometrical setting, symmetries are connected with isometries associated with Killing vectors, and more generally, Killing tensors on the configuration space of the system.An example is the motion of a point particle in a space with isometries [4], which is a physicist's way of studying the geodesic structure of a manifold. Any symmetrical tensor $K_{\alpha\beta}$ satisfying the condition

$$K_{(\alpha\beta;\gamma)} = 0\,, \tag{1}$$

is called a Killing tensor. Here the parenthesis denotes a full symmetrization with all indices and coma denotes a covariant derivative. $K_{\alpha\beta}$ will be called redundant if it is equal to some linear combination with constants coefficients of the metric tensor $g_{\alpha\beta}$ and of the form $S_{(\alpha}B_{\beta)}$ where A_α and B_β are Killing vectors. For any Killing vector K_α we have [5]

$$K_{\beta;\alpha} = \omega_{\alpha\beta} = -\omega_{\beta\alpha}\,, \tag{2}$$

$$\omega_{\alpha\beta;\gamma} = R_{\alpha\beta\gamma\delta}K^{\delta}\,. \tag{3}$$

The equations (2) and (3) may be regarded as a system of linear homogeneous first-order equations in the components K_β and $\omega_{\alpha\beta}$. The equations analogous to the above ones for a Killing vector were derived for $K_{\alpha\beta}$ in [5].

The paper is in the final form and no version of it will be submitted elsewhere.

Recently Holten has presented a theorem concerning the reciprocal relation between two local geometries described by metrics which are Killing tensors with respect to one another [6].
In this paper the geometric duality was presented and the structural equations of $K_{\alpha\beta}$ were analyzed.
The plan of the paper is as follows.
In Sec.2 the geometric duality is presented. In Sec.3 the structural equations of $K_{\alpha\beta}$ are investigated. Our comments and concluding remarks are presented in Sec.4.

2. Geometric duality

Let us consider that the space with a metric $g_{\alpha\beta}$ admits a Killing tensor field $K_{\alpha\beta}$.
As it is well known the equation of motion of a particle on a geodesic is derived from the action [7]

$$S = \int d\tau \left(\frac{1}{2} g_{\alpha\beta} \dot{x}^{\alpha} \dot{x}^{\beta}\right). \tag{4}$$

The Hamiltonian is constructed in the following form $H = \frac{1}{2} g_{\alpha\beta} p^{\alpha} p^{\beta}$ and the Poisson brackets are

$$\{x_{\alpha}, p^{\beta}\} = \delta_{\alpha}^{\beta}. \tag{5}$$

The equation of motion for a phase space function $F(x,p)$ can be computed from the Poisson brackets with the Hamiltonian

$$\dot{F} = \{F, H\}, \tag{6}$$

where $\dot{F} = \frac{dF}{d\tau}$. From the covariant component $K_{\alpha\beta}$ of the Killing tensor we can construct a constant of motion K

$$K = \frac{1}{2} K_{\alpha\beta} p^{\alpha} p^{\beta}. \tag{7}$$

We can easy verify that

$$\{H, K\} = 0. \tag{8}$$

The formal similarity between the constants of motion H and K , and the symmetrical nature of the condition implying the existence of the Killing tensor amount to a reciprocal relation between two different models: the model with Hamiltonian H and constant of motion K, and a model with constant of motion H and Hamiltonian K. The relation between the two models has a geometrical interpretation: it implies that if $K_{\alpha\beta}$ are the contravariant components of a Killing tensor with respect to the metric $g_{\alpha\beta}$, then $g_{\alpha\beta}$ must represent a Killing tensor with respect to the metric defined by $K_{\alpha\beta}$. When $K_{\alpha\beta}$ has an inverse we interpret it as the metric of another space and we can define the associated Riemann-Christoffel connection $\hat{\Gamma}_{\alpha\beta}^{\lambda}$ as usual through the metric postulate $\hat{D}_{\lambda} K_{\alpha\beta} = 0$. Here $\hat{D}$ represents the covariant derivative with respect to $K_{\alpha\beta}$. The relation between connections $\hat{\Gamma}_{\alpha\beta}^{\mu}$ and $\Gamma_{\alpha\beta}^{\mu}$ is [8]

$$\hat{\Gamma}_{\alpha\beta}^{\mu} = \Gamma_{\alpha\beta}^{\mu} - K^{\mu\delta} D_{\delta} K_{\alpha\beta}. \tag{9}$$

As it is well known for a given metric $g_{\alpha\beta}$ the conformal transformation is defined as $\hat{g}_{\alpha\beta} = e^{2U(x)} g_{\alpha\beta}$ and the relation between the corresponding connections is

$$\hat{\Gamma}^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\alpha\beta} + 2\delta^{\lambda}_{(\alpha} U_{\beta)'} - g_{\alpha\beta} U'^{\lambda}, \tag{10}$$

where $U'^{\lambda} = \frac{dU^{\lambda}}{dx}$. After some calculations we conclude that the dual transformation (9) is not a conformal transformation .

For this reason is interesting to investigate the case when the manifold and its dual have the same isometries. Let us denote by χ_α a Killing vector corresponding to $g_{\alpha\beta}$ and by $\hat{\chi}_\alpha$ a Killing vector corresponding to $K_{\alpha\beta}$.Then the following Proposition holds.

Proposition. *The manifold and its dual have the same Killing vectors iff*

$$(D_\delta K_{\alpha\beta}) \hat{\chi}^\delta = 0 . \tag{11}$$

Proof. Let us consider χ_σ a vector satisfies

$$D_\alpha \chi_\beta + D_\beta \chi_\alpha = 0 . \tag{12}$$

Using (9) the corresponding dual Killing vector's equations are

$$D_\alpha \hat{\chi}_\beta + D_\beta \hat{\chi}_\alpha + 2K^{\delta\sigma}(D_\delta K_{\alpha\beta}) \hat{\chi}_\sigma = 0 . \tag{13}$$

If we assume that $\hat{\chi}_\alpha = \chi_\alpha$, then using (12) and (13) we obtain

$$(D_\delta K_{\alpha\beta}) \hat{\chi}^\delta = 0 . \tag{14}$$

Conversely if we suppose that (11) holds, then from (13) we can deduce immediately that $\chi_\alpha = \hat{\chi}_\alpha$. □

As an example we mention here the separable coordinates in $(1 + 1)$ dimensions because in this case $D_\lambda K_{\alpha\beta} = 0$.

3. The structural equations

The following two vectors play roles analogous to that of a bivector $\omega_{\alpha\beta}$

$$L_{\alpha\beta\gamma} = K_{\beta\gamma;\alpha} - K_{\alpha\gamma;\beta} , \tag{15}$$

$$M_{\alpha\beta\gamma\delta} = \frac{1}{2}(L_{\alpha\beta[\gamma;\delta]} + L_{\gamma\delta[\alpha;\beta]}) . \tag{16}$$

The properties of the tensors $L_{\alpha\beta\gamma}$ and $M_{\alpha\beta\gamma\delta}$ were derived in [5]. $M_{\alpha\beta\gamma\delta}$ has the same symmetries as the Riemannian tensor and the covariant derivatives of $K_{\alpha\beta}$ and $L_{\alpha\beta\gamma}$ satisfy relations reminiscent of those satisfied by Killing vectors.

From (15) and (16) we found that $M_{\alpha\beta\gamma\delta} = \frac{1}{2}(K_{\beta\gamma;(\alpha\delta)} + K_{\alpha\delta;(\beta\gamma)} - K_{\alpha\gamma;(\beta\delta)} - K_{\beta\delta;(\alpha\gamma)})$ and $M_{\alpha\beta\gamma\delta} + M_{\gamma\alpha\beta\delta} + M_{\beta\gamma\alpha\delta} = 0$ [5].

Let us define a tensor $H^{\mu}_{\alpha\beta}$ as

$$H^{\mu}_{\alpha\beta} = \hat{\Gamma}^{\mu}_{\alpha\beta} - \Gamma^{\mu}_{\alpha\beta} . \tag{17}$$

Taking into account (9) we found

$$H^{\delta}_{\alpha\beta} K_{\delta\gamma} = -D_\gamma K_{\alpha\beta} . \tag{18}$$

Using (15) and (18) we obtain

$$L_{\alpha\beta\gamma} = H^{\delta}_{\alpha\gamma} K_{\delta\beta} - H^{\delta}_{\beta\gamma} K_{\delta\alpha} . \tag{19}$$

From (19) we conclude that $L_{\alpha\beta\gamma}$ looks like an angular momentum. This result is in agreement with those presented in [5]. Taking into account (16) and (18) the expression of $M_{\alpha\beta\gamma\delta}$ becomes

$$\begin{aligned} M_{\alpha\beta\gamma\delta} &= \frac{K_{\sigma\alpha}}{2}[-D_\delta H^\sigma_{\beta\gamma} + D_\gamma H^\sigma_{\beta\delta} + H^\sigma_{\theta\gamma}H^\theta_{\delta\beta} - H^\theta_{\gamma\beta}H^\sigma_{\theta\delta}] \\ &+ \frac{K_{\sigma\beta}}{2}[D_\delta H^\sigma_{\alpha\gamma} - D_\gamma H^\sigma_{\alpha\delta} + H^\theta_{\delta\alpha}H^\sigma_{\theta\gamma} - H^\theta_{\gamma\alpha}H^\sigma_{\theta\delta}] \\ &+ \frac{K_{\sigma\gamma}}{2}[-D_\beta H^\sigma_{\delta\alpha} + D_\alpha H^\sigma_{\delta\beta} - H^\theta_{\beta\delta}H^\sigma_{\theta\alpha} + H^\theta_{\alpha\delta}H^\sigma_{\theta\beta}] \\ &+ \frac{K_{\sigma\delta}}{2}[-D_\alpha H^\sigma_{\gamma\beta} + D_\beta H^\sigma_{\gamma\alpha} + H^\theta_{\beta\gamma}H^\sigma_{\theta\alpha} - H^\theta_{\alpha\gamma}H^\sigma_{\theta\beta}]\,. \end{aligned} \tag{20}$$

We will investigate now the dual structural equations. Using (17) and (18) we found the dual expressions of $L_{\alpha\beta\gamma\delta}$ and $M_{\alpha\beta\gamma\delta}$ as

$$\begin{aligned} \hat{L}_{\alpha\beta\gamma} &= \hat{D}_\alpha g_{\beta\gamma} - \hat{D}_\beta g_{\alpha\gamma} = -H^\delta_{\alpha\gamma}g_{\delta\beta} + H^\delta_{\beta\gamma}g_{\delta\alpha}\,, \\ \hat{M}_{\alpha\beta\gamma\delta} &= \frac{1}{2}(\hat{L}_{\alpha\beta[\gamma;\delta]} + \hat{L}_{\gamma\delta[\alpha;\beta]}) = -g_{\sigma\alpha}R'^\sigma_{\beta\gamma\delta} - g_{\sigma\beta}R'^\sigma_{\alpha\delta\gamma} - g_{\sigma\gamma}R'^\sigma_{\delta\alpha\beta} - g_{\sigma\delta}R'^\sigma_{\gamma\beta\alpha} \\ &- g_{\sigma\chi}(H^\chi_{\beta\delta}\hat{G}^\sigma_{\alpha\gamma} - H^\chi_{\alpha\delta}\hat{G}^\sigma_{\beta\gamma} - H^\chi_{\beta\gamma}\hat{G}^\sigma_{\alpha\delta} + H^\chi_{\alpha\gamma}\hat{G}^\sigma_{\beta\delta}) \\ &- H^\sigma_{\beta\delta}g_{\alpha\gamma,\sigma} - H^\sigma_{\alpha\gamma}g_{\beta\delta,\sigma} + H^\sigma_{\beta\gamma}g_{\alpha\delta,\sigma} + H^\sigma_{\alpha\delta}g_{\beta\gamma,\sigma}\,. \end{aligned} \tag{21}$$

Here $\hat{G}^\sigma_{\alpha\delta} = H^\sigma_{\alpha\delta} + \hat{\Gamma}^\sigma_{\alpha\delta}$, the semicolon denotes the dual covariant derivative and $R'^\beta_{\nu\rho\sigma}$ has the following form

$$R'^\beta_{\nu\rho\sigma} = H^\beta_{\nu\sigma,\rho} - H^\beta_{\nu\rho,\sigma} + H^\alpha_{\nu\sigma}H^\beta_{\alpha\rho} - H^\alpha_{\nu\rho}H^\beta_{\alpha\sigma}\,. \tag{22}$$

From (22) we conclude that $R'^\beta_{\nu\rho\sigma}$ looks like as the curvature tensor $R^\beta_{\nu\rho\sigma}$ [9].

Taking into account (17) we found a new identity for $K_{\alpha\beta}$

$$K^\sigma_\beta D_\sigma K_{\nu\lambda} + K^\sigma_\lambda D_\sigma K_{\beta\nu} + K^\sigma_\nu D_\sigma K_{\beta\lambda} = 0\,. \tag{23}$$

By duality we get from (23) another identity

$$g^\sigma_\beta \hat{D}_\sigma g_{\nu\lambda} + g^\sigma_\lambda \hat{D}_\sigma g_{\beta\nu} + g^\sigma_\nu \hat{D}_\sigma g_{\beta\lambda} = 0\,. \tag{24}$$

In the flat space case the general solution of eq. (1) has the form

$$K_{\beta\gamma} = s_{\beta\gamma} + \frac{2}{3}B_{\alpha(\beta\gamma)}x^\alpha + \frac{1}{3}A_{\alpha\beta\gamma\delta}x^\alpha x^\delta\,. \tag{25}$$

Here $s_{\beta\gamma}$, $B_{\alpha\beta\gamma}$ and $A_{\alpha\beta\gamma\delta}$ are constant tensors having the same symmetries as $K_{\beta\gamma}$, $L_{\alpha\beta\gamma}$ and $M_{\alpha\beta\gamma\delta}$ respectively. From (19) and (25) we get

$$\begin{aligned} L_{\alpha\beta\gamma} &= H^\delta_{\alpha\gamma}(s_{\delta\beta} + \frac{2}{3}B_{\sigma(\delta\beta)}x^\sigma + \frac{1}{3}A_{\sigma\delta\beta\lambda}x^\sigma x^\lambda) \\ &\qquad - H^\delta_{\beta\gamma}(s_{\delta\alpha} + \frac{2}{3}B_{\sigma(\delta\alpha)}x^\sigma + \frac{1}{3}A_{\sigma\delta\alpha\lambda}x^\sigma x^\lambda)\,. \end{aligned} \tag{26}$$

Using (20) and (25) the expression of $M_{\alpha\beta\gamma\delta}$ becomes

$$M_{\alpha\beta\delta\gamma} = \frac{1}{2}(K_{\sigma\alpha}R'^\sigma_{\beta\gamma\delta} + K_{\sigma\beta}R'^\sigma_{\alpha\delta\gamma} + K_{\sigma\gamma}R'^\sigma_{\delta\alpha\beta} + K_{\sigma\delta}R'^\sigma_{\gamma\beta\alpha})\,. \tag{27}$$

Let us suppose now that $K_{\alpha\beta}$ given by (25) is non-degenerate [10]. In this case we found that

$$\hat{L}_{\alpha\beta\gamma} = -H_{\beta\alpha\gamma} + H_{\alpha\beta\gamma}\,, \tag{28}$$

and

$$\hat{M}_{\alpha\beta\gamma\delta} = -R'_{\alpha\beta\gamma\delta} - R'_{\beta\alpha\delta\gamma} - R'_{\gamma\delta\alpha\beta} - R'_{\delta\gamma\beta\alpha} - H^{\sigma}_{\beta\delta}\hat{G}^{\sigma}_{\alpha\gamma} + H^{\sigma}_{\alpha\delta}\hat{G}^{\sigma}_{\beta\gamma} + H^{\sigma}_{\beta\gamma}\hat{G}^{\sigma}_{\alpha\delta} - H^{\sigma}_{\alpha\gamma}\hat{G}^{\sigma}_{\beta\delta}\,. \quad (29)$$

4. Conclusions

The geometric duality between local geometries described by $g_{\alpha\beta}$ and by $K_{\alpha\beta}$ was presented. We found the relation between connections corresponding to $g_{\alpha\beta}$ and $K_{\alpha\beta}$ respectively and we have shown that the dual transformation is not a conformal transformation. The manifold and its dual have the same isometries if $D_\lambda K_{\alpha\beta} = 0$. We have shown that $L_{\alpha\beta\gamma}$ looks like an angular momentum. The dual structural equations were analyzed and the expressions of $\hat{L}_{\alpha\beta\gamma}$ and $\hat{M}_{\alpha\beta\gamma\delta}$ were calculated. In the flat space case the general forms of $(L_{\alpha\beta\gamma}, \hat{L}_{\alpha\beta\gamma})$ and $(M_{\alpha\beta\gamma\delta}, \hat{M}_{\alpha\beta\gamma\delta})$ were found.

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