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NEW BOSON REALIZATIONS OF QUANTUM GROUPS $U_q(A_n)$

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ABSTRACT. We describe a construction of boson realizations of quantum groups. The realizations are expressed by means of $r(n-r+1)$ q–deformed bosons pairs and generators of certain subalgebra of $U_q(A_n)$.

1. GENERAL CONSTRUCTION

We will study quantum algebras $U_q(\mathcal{L})$, which were defined by [1,2]. Concretely we will use the realization of these algebras which were given by [3].

Let q is independent variable, $\mathcal{A} = \mathbb{C}[q, q^{-1}]$ and $\mathcal{C}(q)$ is its division ring. For $n \in \mathbb{Z}$ and $d \in \mathbb{N}$ we denote $[n]_d = \dfrac{q^{dn} - q^{-dn}}{q^d - q^{-d}} \in \mathcal{A}$ and $[n]_d! = [n]_d \cdot [n-1]_d \cdot \ldots \cdot [1]_d$ and

$$\begin{bmatrix} n \\ j \end{bmatrix}_d = \frac{[n]_d!}{[n-j]_d!\,[j]_d!}\,.$$

If $d = 1$ we omit index d.

Let $\mathcal{L}$ is a simple Lie algebra with Cartan matrix $(a_{ij}) = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$, $i, j = 1, \ldots, k$ and d_i are co–prime integers such that $d_i a_{ij}$ is symmetric matrix. Let $\mathcal{C}(q)$–algebra $U_q(\mathcal{L})$ is generated by E_i, F_i, K_i and K_i^{-1}, where $i = 1, 2, \ldots, k$ which fulfil the relations

$$\begin{aligned}
&K_i K_j = K_j K_i\,, \qquad K_i K_i^{-1} = K_i^{-1} K_i = 1\,,\\
&K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j\,, \qquad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j\,,\\
&E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}\,,\\
&\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{d_i} E_i^{1-a_{ij}-s} E_j E_i^s = 0\,, \quad i \neq j\,,
\end{aligned}$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} F_i^{1-a_{ij}-s} F_j F_i^s = 0\,, \quad i \neq j\,,$$

where $q_i = q^{d_i}$.

The associative algebra $U_q(\mathcal{L})$ with comultiplication $\triangle$, antipode S and counit ϵ, which are given by

$$\begin{array}{lll} \triangle E_i = E_i \otimes 1 + K_i \otimes E_i\,, & \triangle F_i = F_i \otimes K_i^{-1} + 1 \otimes F_i\,, & \triangle K_i = K_i \otimes K_i\,; \\ S(E_i) = -K_i^{-1} E_i\,, & S(F_i) = -F_i K_i\,, & S(K_i) = K_i^{-1}\,; \\ \epsilon(E_i) = 0\,, & \epsilon(F_i) = 0\,, & \epsilon(K_i) = 1\,, \end{array}$$

forms Hopf algebra. This Hopf algebra is denoted as quantum group. Because we understand representation of quantum group homomorphism $U_q(\mathcal{L})$ to $\operatorname{End} V$, we don't interest in these last three operations.

The elements E_i and F_i correspond to the system of positive and negative simple roots of the Lie algebra $\mathcal{L}$. In the paper [4] it is shown how is possible to build up representations of the quantum group $U_q(\mathcal{L})$ by means of Verma modules. But in the case of Lie algebras we can generalize this construction such, that we obtain less boson pairs and the representation of any auxiliary subalgebra of $\mathcal{L}$ [5].

We remind this construction shortly. Let $\mathcal{L}$ is a simple Lie algebra and $\Pi^+ = (\alpha_1, \ldots, \alpha_k)$ system of positive simple roots. In this case it is possible to write any root α in the form

$$\alpha = \sum_{i=1}^{k} n_i \alpha_i\,,$$

where all n_i are positive or negative integers. Let $\Pi_r = \Pi \setminus \{\alpha_r\}$ and Φ_r^+ is a root system which corresponds to Π_r.

Let $\mathcal{L}_r$ is the Lie subalgebra of $\mathcal{L}$ generated by the Cartan subalgebra of $\mathcal{L}$ and by the elements E_α and F_α, where $\alpha \in \Phi_r+$. This subalgebra is reductive.

Let the elements E_β are associated with the roots $\beta \notin \Phi_r^+$. These elements generate the nilpotent subalgebra $\mathcal{N}_+$ and similarly the elements F_β generate algebra $\mathcal{N}_-$. By this way we obtain a decomposition of the Lie algebra $\mathcal{L}$ to the direct sum of vector spaces

$$\mathcal{L} = \mathcal{N}_+ \oplus \mathcal{L}_r \oplus \mathcal{N}_-\,.$$

It is possible to write the universal enveloping algebra $U(\mathcal{L})$ as $U(\mathcal{L}) = U(\mathcal{N}_+) \cdot U(\mathcal{L}_r) \cdot U(\mathcal{N}_-)$.

If φ is representation of the Lie algebra $\mathcal{L}_r$ on a vector–space V, for which

$$\varphi(Z) = \varphi\left(\sum_{i=1}^{k} a_i H_i\right) = \lambda = \text{const.}$$

where $Z = \sum_{i=1}^{k} a_i H_i$ is non vanish central element of the Lie algebra $\mathcal{L}_r$. Since $a_r \neq 0$, it is possible to express $\varphi(H_r)$ by means of $\varphi(H_i)$, $i \neq r$, and λ. Therefore the representation φ is given by representations φ_1 and φ_2 of the Lie algebras $\mathcal{L}_1$ and $\mathcal{L}_2$, which have root systems $\Pi_1 = \{\alpha_1, \ldots, \alpha_{r-1}\}$ and $\Pi_2 = \{\alpha_{r+1}, \ldots, \alpha_k\}$. The representation φ is possible to extend on the algebra $\mathcal{L}_r \oplus \mathcal{N}_-$, if we put $\varphi(F_\beta)v = 0$ for all $v \in V$.

The subspace $W \subset U(\mathcal{L}) \otimes V$ generated by relations

$$xz \otimes v - x \otimes \varphi(z)v\,, \quad \text{where } x \in U(\mathcal{L})\,, \quad z \in U(\mathcal{L}_r) \cdot u(\mathcal{N}_-) \quad \text{and} \quad v \in V$$

is invariant for left regular representation. Therefore it is possible to obtain the factor–representation $\mathcal{L}$ on the vector–space $\big((\mathcal{L}) \otimes V\big)/W$. The representation is called induce representation [6].

If we chose basis in $U(\mathcal{N}_+)$ suitably, we are able to rewrite this representation by means of $p = \dim(\mathcal{N}_+)$ bosons operators and representation φ of the auxiliary Lie algebra $\mathcal{L}_r$. By this way we obtain a realization of Lie algebra $\mathcal{L}$.

The above mentioned procedure is possible to apply only with small changes to the quantum group. The first change is that in this case we are not able to express element K_r through central element of $U_q(\mathcal{L}_r)$ and elements K_i, $i \neq r$. Therefore we have to consider representation φ of the whole auxiliary quantum subalgebra $U_q(\mathcal{L}_r)$. Since our aim is to construct whole set of realizations, where we use as φ realization of similar quantum group with less dimension, it is suitable to extent our quantum group by element K_0. This element corresponds in fact imbedding of our quantum group to the quantum group the same type but with rank $k+1$.

The other difference appears, when we construct the induced representations. We don't obtain usual boson representation of Weyl algebra, but representation of its q–deformed version, Hayashi algebra $\mathcal{H}$ [4]. The algebra $\mathcal{H}$ is associative algebra over field $\mathcal{C}(q)$ which is generate by elements a^+, $a^- = a$, q^x and q^{-x}. These operators satisfy the relations

$$\begin{aligned} &aa^+ - q^{-1}a^+a = q^x && aa^+ - qa^+a = q^{-x} \\ &q^x a^+ q^{-x} = qa^+ && q^x a q^{-x} = q^{-1}a \\ &q^x q^{-x} = q^{-x} q^x = 1\,. \end{aligned}$$

This algebra has faithful representation on the space $\{|n\rangle\,,\ n = 0, 1, 2, \ldots\}$

$$\begin{aligned} q^x\,|n\rangle &= q^n\,|n\rangle\,, \\ a^+\,|n\rangle &= |n+1\rangle\,, \\ a^-\,|n\rangle &= [n]\,|n-1\rangle\,. \end{aligned}$$

Exactly these representations appear in induced representations.

For these reasons we use the following

Definition. Let $U_q(\mathcal{L})$ be a quantum group and $U_q(\mathcal{L}_0)$ its subalgebra. A realization of the quantum group $U_q(\mathcal{L})$ is homomorphism

$$\rho : U_q(\mathcal{L}) \longrightarrow \mathcal{H}^n \otimes U_q(\mathcal{L}_0)\,,$$

where $\mathcal{H}^n$ is n–fold tensor product of the Hayashi algebras.

2. Construction of the Realizations $U_q(A_n)$

In papers [7–10] we constructed the realizations of the quantum group for all four infinite series of the classical simple Lie algebras. This realizations correspond to the choice $r = 1$ in our general construction from Section 1. Here we will deal with this construction for the quantum group $U_q(A_n)$ in the case a general r. These realizations correspond to the most simple representations of degenerate series of Lie algebras. The construction of induced representations in this case is much more difficult due to bigger dimension of the factor–space.

In this paper we give our results without proofs only. The proofs we have obtained by slightly modified proofs, which are in [7].

Cartan matrix of the quantum group $U_q(A_n)$ is

$$(a_{ij}) = \begin{pmatrix} 2 & -1 & 0 & \cdots\cdots & & \\ -1 & 2 & -1 & \cdots\cdots & & \\ 0 & -1 & 2 & \cdots\cdots & & \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 2 & -1 \\ \cdots & \cdots & \cdots & \cdots & -1 & 2 \end{pmatrix}.$$

Therefore we have $d_i = 1$ and $q_i = q$ for all $i = 1, \ldots, n$.

Let $1 \leq r \leq n$ and $U_q(\mathcal{L}_r)$ is the auxiliary algebra generated by elements E_j, F_j, K_j, K_j^{-1}, $j \neq r$, and K_r, K_r^{-1}. Let representation φ of this algebra has the property $\varphi(Z)v = q^\lambda v$, where Z is central element of the algebra $\mathcal{L}_r$

$$Z = K_1^n \cdot K_2^{n-1} \cdot \ldots \cdot K_{n-1}^2 \cdot K_n \,.$$

We will denote

$$\begin{aligned} X_{s,s+1} &= E_s & s &= 1, 2, \ldots, n \\ X_{s,t+1} &= E_t X_{s,t} - q^{-1} X_{s,t} E_t \,, & 1 &\leq s < t \leq n \,. \end{aligned}$$

For the construction of the induced representation we will need relations between this elements, which are given by the following Lemmas

Lemma 1. *The following commutation relations hold in quantum group $U_q(A_n)$*

$$\begin{aligned} E_s X_{t,r} &= X_{t,r} E_s & s+1 &< t < r \\ E_s X_{s+1,t} &= q X_{s+1,t} E_s - q X_{s,t} & s+1 &< t \\ E_s X_{s,t} &= q^{-1} X_{s,t} E_s & s+1 &< t \\ E_s X_{t,r} &= X_{t,r} E_s & t &< r < s \\ E_s X_{t,s} &= q^{-1} X_{t,s} E_s + X_{t,s+1} & t &< s \\ E_s X_{t,s+1} &= q X_{t,s+1} E_s & t &< s \\ E_s X_{t,r} &= X_{t,r} E_s & t &< s < r-1 \end{aligned}$$

Lemma 2. *The elements $X_{r,s}$ and $X_{t,u}$ of the quantum group $U_1(A_n)$ fulfil the following relations:*

$$
\begin{aligned}
&X_{r,s}X_{t,u} = X_{t,u}X_{r,s} && \text{for } r < s < t < u\,,\\
&X_{r,s}X_{s,u} = qX_{s,u}X_{r,s} - qX_{r,u} && \text{for } r < s < u\\
&X_{r,s}X_{t,s} = q^{-1}X_{t,s}X_{r,s} && \text{for } r < t < s\\
&X_{r,s}X_{t,u} = X_{t,u}X_{r,s} && \text{for } r < t < u < s\\
&X_{r,s}X_{r,u} = q^{-1}X_{r,u}X_{r,s} && \text{for } r < s < u\\
&X_{r,s}X_{t,u} - X_{t,u}X_{r,s} = \left(q - q^{-1}\right)X_{t,s}X_{r,u} && \text{for } t < r < u < s
\end{aligned}
$$

Lemma 3. *Non trivial commutation relations between elements K_s and $X_{t,u}$ are*

$$
\begin{aligned}
&K_sX_{t,s} = q^{-1}X_{t,s}K_s && t < s\\
&K_sX_{t,s+1} = qX_{t,s+1}K_s && t < s\\
&K_sX_{s,t} = qX_{s,t}K_s && t > s + 1\\
&K_sX_{s+1,t} = q^{-1}X_{s+1,t}K_s && t > s + 1\\
&K_sX_{s,s+1} = q^2X_{s,s+1}K_s &&
\end{aligned}
$$

Lemma 4. *The following relations hold in the quantum group $U_q(A_n)$*

$$
\begin{aligned}
&F_sX_{t,u} = X_{t,u}F_s && \text{for } s < t\,,\\
&F_sX_{s,s+1} = X_{s,s+1}F_s - \frac{K_s - K_s^{-1}}{q - q^{-1}} &&\\
&F_sX_{s,t} = X_{s,t}F_s - K_sX_{s+1,t} && \text{for } s + 1 < t\,,\\
&F_sX_{t,u} = X_{t,u}F_s && \text{for } t < s\,,\ u \neq s + 1\,,\\
&F_sX_{t,s+1} = X_{t,s+1}F_s + X_{t,s}K_s^{-1} && \text{for } t < s\,.
\end{aligned}
$$

It is very easy to prove by means of induction from this Lemmas following relations for general powers

Lemma 5. *The relations*

$$
\begin{aligned}
&E_sX^k_{s+1,t} = q^kX_{s+1,t}E_s - q^k[k]X_{s,t}X^{k-1}_{s+1,t} && s + 1 < t\\
&E_sX^k_{t,s} = q^{-k}X^k_{t,s}E_s + [k]X^{k-1}_{t,s}E_s && t < s\\
&F_sX^k_{s,t} = X^k_{s,t}F_s - q^{k-2}[k]X^{k-1}_{s,t}X_{s+1,t}K_s && s + 1 < t\\
&F_sX^k_{t,s+1} = X^k_{t,s+1}F_s + [k]X_{t,s}X^{k-1}_{t,s+1}K_s^{-1} && t < s\\
&F_sX^k_{s,s+1} = X^k_{s,s+1}F_s - \frac{[k]}{q - q^{-1}}X^{k-1}_{s,s+1}\left(q^{k-1}K_s - q^{-k+1}K_s^{-1}\right) &&
\end{aligned}
$$

$$X_{t,s}X^k_{u,s} = q^k X^k_{u,s}X_{t,s} \qquad u<t$$
$$X_{s,t}X^k_{s,u} = q^k X^k_{s,u}X_{s,t} \qquad u<t$$
$$X_{r,s}X^k_{t,u} = X^k_{t,u}X_{r,s} - \big(q-q^{-1}\big)[k]X_{r,u}X^{k-1}_{t,u}X_{t,s} \qquad r<t<s<u$$
$$X_{s,t}X^k_{t,u} = q^k X^k_{t,u}X_{s,t} - q^k[k]X_{s,u}X^{k-1}_{t,u} \qquad s<t<u$$

are valid for $k = 1, 2, \ldots$.

According to PBW theorem the basis of the factor–space $\mathcal{N}_+$ is formed by elements

$$|x\rangle = X^{x_{1,r+1}}_{1,r+1} \cdot \ldots \cdot X^{x_{1,n+1}}_{1,n+1} \cdot X^{x_{2,r+1}}_{2,r+1} \cdot \ldots X^{x_{2,n+1}}_{2,n+1} \cdot \ldots \cdot X^{x_{r,r+1}}_{r,r+1} \cdot \ldots \cdot X^{x_{r,n+1}}_{r,n+1},$$

where $x_{s,t} = 0, 1, 2, \ldots$. Moreover we denote special elements for which are $x_{u,k} = 0$ for $u < s$ and $u > t$ as $|x_s, \ldots, x_t\rangle$

Let φ is a representation of the auxiliary quantum group $\mathcal{L}_r$ which is generated by elements E_j, F_j, K_j, K_j^{-1}, $j \neq r$, K_r and K_r^{-1} on a vector space V. If we put $\varphi(F_r)v = 0$ for all $v \in V$, we can extend this representation to the quantum group $U_q(\mathcal{L}_r) \cdot U_q(\mathcal{N}_-)$. It is not difficult to derive from the previous Lemmas

Theorem 1. *For the factor–representation on the vector space, which is generated by the elements* $|x\rangle \otimes v$ *the relations*

for $s < r$

$$E_s\,|x\rangle \otimes v = -\sum_{t=r+1}^{n+1} q^{X^t_{r+1}(s+1)-X^t_{r+1}(s)}\big[x_{s+1,t}\big]\,|x+1_{s,t}-1_{s+1,t}\rangle \otimes v + \\ + q^{X^{n+1}_{r+1}(s+1)-X^{n+1}_{r+1}(s)} \otimes \varphi(E_s)v$$

$$K_s\,|x\rangle \otimes v = q^{X^{n+1}_{r+1}(s)-X^{n+1}_{r+1}(s+1)}\,|x\rangle \otimes \varphi(K_s)v$$

$$F_s\,|x\rangle \otimes v = -\sum_{t=r+1}^{n+1} q^{X^{n+1}_t(s)-X^{n+1}_t(s+1)-2}\big[x_{s,t}\big]\,|x-1_{s,t}+1_{s+1,t}\rangle \otimes \varphi\big(K_s\big)v + \\ + |x\rangle \otimes \varphi(F_s)v$$

for $s > r$

$$E_s\,|x\rangle \otimes v = \sum_{t=1}^{r} q^{X^{t-1}_1(s+1)-X^{t-1}_1(s)}\big[x_{t,s}\big]\,|x-1_{t,s}+1_{t,s+1}\rangle \otimes v + \\ + q^{X^r_1(s+1)-X^r_1(s)}\,|x\rangle \otimes \varphi(E_s)v$$

$$K_s\,|x\rangle \otimes v = q^{X^r_1(s)-X^r_1(s+1)}\,|x\rangle \otimes \varphi(K_s)v$$

$$F_s\,|x\rangle \otimes v = \sum_{t=1}^{r} q^{X^r_{t+1}(s)-X^r_{t+1}(s+1)}\big[x_{t,s+1}\big]\,|x+1_{t,s}-1_{t,s+1}\rangle \otimes \varphi\big(K_s^{-1}\big)v + \\ + |x\rangle \otimes \varphi(F_s)v$$

for $s = r$

$$E_r|x\rangle\otimes v = q^{X_1^{r-1}(r+1)}|x+1_{r,r+1}\rangle\otimes v$$
$$K_r|x\rangle\otimes v = q^{X_1^r(r+1)+X_{r+1}^{n+1}(r)}|x\rangle\otimes\varphi(K_r)v$$
$$\begin{aligned}F_r|x\rangle\otimes v =& -\frac{[x_{r,r+1}]}{q-q^{-1}}|x-1_{r,r+1}\rangle\otimes\left(q^{X_{r+1}^{n+1}(r)-1}\varphi(K_r)-q^{1-X_{r+1}^{n+1}(r)}\varphi(K_r^{-1})\right)v-\\&-\sum_{t=r+2}^{n+1}q^{X_t^{n+1}(r)-2}[x_{r,t}]|x-1_{r,t}\rangle\otimes\varphi(X_{r+1,t}K_r)v+\\&+\sum_{t=1}^{r-1}q^{-X_{t+1}^r(r+1)-X_{r+1}^{n+1}(t)-X_{r+1}^{n+1}(r)+1}[x_{t,r+1}]|x_1,\dots,x_t-1_{t,r+1}\rangle\times\\&\qquad\times X_{t,r}|x_{t+1},\dots,x_r\rangle\otimes\varphi(K_r^{-1})v\,,\end{aligned}$$

is valid. For abbreviation we use $X_s^t(u)=\sum_{k=s}^t x_{u,k}$ *for* $u\le r$ *and* $X_s^t(u)=\sum_{k=s}^t x_{k,u}$ *for* $u>r$ *respectively.*

Remark. The relation for $F_r|x\rangle\otimes v$ in the Theorem 1 is not the representation of this element on our vector space. We need commute element $X_{r,t}$ to the end and arrange the outcome of the commutation. The resulting formula is very complicated. It contain all possible terms of the form

$$\left|x-1_{t,r+1}+\sum_{k=1}^{s}(1_{t_k,u_k}-1_{t_{k+1},u_k})\right\rangle X_{u_{s+1},r}\,,$$

where $t_1=t$, $t_k<t_{k+1}$, and we put $X_{r,r}=1$. In the case $r=1$ these terms vanished and therefore we can be able to find induced representation [7].

Therefore we give this representation for special r only.

When $r=n$ the element F_n has representation

$$\begin{aligned}F_n|x\rangle\otimes v =& -\frac{[x_{n,n+1}]}{q-q^{-1}}q^{-X_1^{n-1}(n+1)}|x-1_{n,n+1}\rangle\otimes\\&\otimes\left(q^{X_1^n(n+1)-1}\varphi(K_n)-q^{-X_1^n(n+1)+1}\varphi(K_n^{-1})\right)v+\\&+\sum_{k=1}^{n-1}q^{-X_1^{k-1}(n+1)-X_1^n(n+1)+1}[x_{k,n+1}]|x-1_{k,n+1}\rangle\otimes\varphi(X_{k,n}K_n^{-1})v\,.\end{aligned}$$

When $r=2$ the element F_2 has representation

$$\begin{aligned}F_2|x\rangle\otimes v =& -\frac{[x_{2,3}]}{q-q^{-1}}q^{-x_{1,3}}|x-1_{2,3}\rangle\otimes\\&\otimes\left(q^{x_{1,3}+X_3^{n+1}(2)-1}\varphi(K_2)-q^{-x_{1,3}-X_3^{n+1}(2)+1}\varphi(K_2^{-1})\right)v+\\&+q^{-X_3^{n+1}(1)-x_{2,3}+1}[x_{1,3}]|x-1_{1,3}\rangle\otimes\varphi(X_{1,2}K_2^{-1})v-\end{aligned}$$

$$
\begin{aligned}
&-\sum_{t=4}^{n+1} q^{X_t^{n+1}(2)-2}\left[x_{2,t}\right]|x-1_{2,t}\rangle\,\varphi(X_{3,t})K_2)v-\\
&-\sum_{t=4}^{n+1} q^{-X_3^t(1)-X_{t+1}^{n+1}(2)-x_{2,3}+1}\left[x_{1,3}\right]\left[x_{2,t}\right]\times\\
&\qquad\times|x-1_{1,3}+1_{1,t}-1_{2,t}\rangle\otimes\varphi(K_2^{-1})v
\end{aligned}
$$

When $r=3$ the element F_3 has representation

$$
\begin{aligned}
F_3\,|x\rangle\otimes v=&-\frac{\left[x_{3,4}\right]}{q-q^{-1}}\,q^{-X_1^2(4)}\,|x-1_{3,4}\rangle\otimes\\
&\otimes\left(q^{X_1^2(4)+X_4^{n+1}(3)-1}\varphi(K_3)-q^{-X_1^2(4)-X_4^{n+1}(3)+1}\varphi(K_3^{-1})\right)v-\\
&-\sum_{t=5}^{n+1} q^{X_t^{n+1}(3)-2}\left[x_{3,t}\right]|x-1_{3,t}\rangle\otimes\varphi(X_{4,t}K_3)v+\\
&+q^{-X_4^{n+1}(1)-X_2^3(4)+1}\left[x_{1,4}\right]|x-1_{1,4}\rangle\otimes\varphi(X_{1,3}K_3^{-1})v+\\
&+q^{-2x_{1,4}-X_4^{n+1}(2)-x_{3,4}+1}\left[x_{2,4}\right]|x-1_{2,4}\rangle\otimes\varphi(X_{2,3}K_3^{-1})v-\\
&-(q-q^{-1})\sum_{t=5}^{n+1} q^{-X_4^t(1)-X_{t+1}^{n+1}(2)-X_2^3(4)+1}\left[x_{1,4}\right]\left[x_{2,t}\right]\times\\
&\qquad\times|x-1_{1,4}+1_{1,t}-1_{2,t}\rangle\otimes\varphi(X_{2,3}K_3^{-1})v-\\
&-\sum_{t=5}^{n+1} q^{-X_4^t(1)-x_{2,t}-X_{t+1}^{n+1}(3)-X_2^3(4)+1}\left[x_{1,4}\right]\left[x_{3,t}\right]\times\\
&\qquad\times|x-1_{1,4}+1_{1,t}-1_{3,t}\rangle\otimes\varphi(K_3^{-1})v-\\
&-\sum_{t=5}^{n+1} q^{-2x_{1,4}-X_4^t(2)-X_{t+1}^{n+1}(3)-x_{3,4}+1}\left[x_{2,4}\right]\left[x_{3,t}\right]\times\\
&\qquad\times|x-1_{2,4}+1_{2,t}-1_{3,t}\rangle\otimes\varphi(K_3^{-1})v+\\
&+(q-q^{-1})\sum_{t=6}^{n+1}\sum_{s=5}^{t-1} q^{-X_4^s(1)-X_{s+1}^t(2)-X_{t+1}^{n+1}(3)-X_2^3(4)+1}\left[x_{1,4}\right]\left[x_{2,s}\right]\left[x_{3,t}\right]\times\\
&\qquad\times|x-1_{1,4}+1_{1,s}-1_{2,s}+1_{2,t}-1_{3,t}\rangle\otimes\varphi(K_3^{-1})v
\end{aligned}
$$

We obtain the realizations of $U_q(A_n)$ from these representations by the standard way. The realizations are given by the following

Theorem 2. *Let $r=2$, 3 or n. Let $U_q(\mathcal{L}_r)$ is the quantum subalgebra $U_q(A_n)$ generated by elements E_j, F_j, K_i and K_i^{-1}, where $i,j=1,2,\ldots,n$, $j\neq r$. Then the mapping $\rho: U_q(A_n)\to\mathcal{H}^{r(n-r+1)}\otimes U_q(\mathcal{L}_r)$ given by formulae*

for $s<r$

$$\rho(E_s) = -\sum_{t=r+1}^{n+1} q^{X_{r+1}^{t}(s+1)-X_{r+1}^{t}(s)+2} a_{s+1,t}a_{s,t}^{+} - q^{X_{r+1}^{n+1}(s+1)-X_{r+1}^{n+1}(s)} E_s$$

$$\rho(K_s) = q^{X_{r+1}^{n+1}(s+1)-X_{r+1}^{n+1}(s)} K_s$$

$$\rho(F_s) = -\sum_{t=r+1}^{n+1} q^{X_t^{n+1}(s)-X_t^{n+1}(s+1)} a_{s,t}a_{s+1,t}^{+} K_s + F_s$$

for $\underline{s > r}$

$$\rho(E_s) = \sum_{t=1}^{r} q^{X_1^{t-1}(s+1)-X_1^{t-1}(s)} a_{t,s}a_{t,s+1}^{+} + q^{X_1^{r}(s+1)-X_1^{r}(s)} E_s$$

$$\rho(K_s) = q^{X_1^{r}(s)-X_1^{r}(s+1)} K_s$$

$$\rho(F_s) = \sum_{t=1}^{r} q^{X_{t+1}^{r}(s)-X_{t+1}^{r}(s+1)} a_{t,s+1}a_{t,s}^{+} K_s^{-1} + F_s$$

for $\underline{s = r}$

$$\rho(E_r) = q^{X_1^{r-1}(r+1)} a_{r,r+1}^{+}$$

$$\rho(K_r) = q^{X_1^{r}(r+1)+X_{r+1}^{n+1}(r+1)} K_r$$

and $\rho(F_r)$ *is for* $\underline{r = n}$

$$\rho(F_n) = -\left(q-q^{-1}\right)^{-1} q^{-X_1^{n-1}(n+1)} \left(q^{X_1^{n}(n+1)} K_n - q^{-X_1^{n}(n+1)} K_n^{-1}\right) a_{n,n+1} + \\ + \sum_{k=1}^{n-1} q^{-X_1^{k}(n+1)-X_1^{n}(n+1)} a_{k,n+1} X_{k,n} K_n^{-1}$$

for $\underline{r = 2}$

$$\rho(F_2) = -\left(q-q^{-1}\right)^{-1} q^{-x_{1,3}} \left(q^{x_{1,3}+X_3^{n+1}(2)} K_2 - q^{-x_{1,3}-X_3^{n+1}(2)} K_2^{-1}\right) a_{2,3} + \\ + q^{-X_3^{n+1}(1)-x_{2,3}} a_{1,3} - \sum_{t=4}^{n+1} q^{X_t^{n+1}(2)} a_{2,t} K_2 X_{3,t} - \\ - \sum_{t=4}^{n+1} q^{-X_3^{t}(1)-X_{t+1}^{n+1}(2)-x_{2,3}+1} a_{1,3} a_{2,t} a_{1,t}^{+} K_2^{-1}$$

and for $\underline{r = 3}$

$$\rho(F_3) = -\left(q-q^{-1}\right)^{-1} q^{-X_1^{2}(4)} \left(q^{X_1^{2}(4)+X_4^{n+1}(3)} K_3 - q^{-X_1^{2}(4)-X_4^{n+1}(3)} K_3^{-1}\right) a_{3,4} - \\ - \sum_{t=5}^{n+1} q^{X_t^{n+1}(3)} a_{3,t} K_3 X_{4,t} + \\ + q^{-X_4^{n+1}(1)-X_2^{3}(4)} a_{1,4} X_{1,3} K_3^{-1} + q^{-2x_{1,4}-X_4^{n+1}(2)-x_{3,4}} a_{2,4} X_{2,3} K_3^{-1} - \\ - \left(q-q^{-1}\right) \sum_{t=5}^{n+1} q^{-X_4^{t}(1)-X_{t+1}^{n+1}(2)-X_2^{3}(4)+1} a_{1,4} a_{2,t} a_{1,t}^{+} X_{2,3} K_3^{-1} -$$

$$
-\sum_{t=5}^{n+1} q^{-X_4^t(1)-x_{2,t}-X_{t+1}^{n+1}(3)-X_2^3(4)+1} a_{1,4}a_{3,t}a_{1,t}^{+}K_3^{-1}-
$$

$$
-\sum_{t=5}^{n+1} q^{-2x_{1,4}-X_4^t(2)-X_{t+1}^{n+1}(3)-x_{3,4}+1} a_{2,4}a_{3,t}a_{2,t}^{+}K_3^{-1}+
$$

$$
+\left(q-q^{-1}\right)\sum_{t=6}^{n+1}\sum_{s=5}^{t-1} q^{-X_4^s(1)-X_{s+1}^t(2)-X_{t+1}^{n+1}(3)-X_2^3(4)+2} a_{1,4}a_{2,s}a_{3,t}a_{1,s}^{+}a_{2,t}^{+}K_3^{-1}
$$

is realizatian of quantum group $U_q(A_n)$.

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SOME REMARKS ON THE PLÜCKER RELATIONS

MICHAEL G. EASTWOOD AND PETER W. MICHOR

1. The Plücker relations

Let V denote a finite-dimensional vector space. An s-vector $P \in \Lambda^s V$ is called *decomposable* or *simple* if it can be written in the form

$$P = u \wedge v \wedge \cdots \wedge w \quad \text{for } u, v, \dots, w \in V.$$

We shall use in the following both Penrose's abstract index notation and exterior calculus with the conventions of [3].

Theorem 1. *Let $P \in \Lambda^s V$ be an s-vector. Then P is decomposable if and only if one of the following conditions holds:*

1. $i(\Phi)P \wedge P = 0$ *for all* $\Phi \in \Lambda^{s-1}V^*$. *In index notation* $P_{[abc\cdots d}P_{e]fg\cdots h} = 0$.
2. $i(i_P\Psi)P = 0$ *for all* $\Psi \in \Lambda^{s+1}V^*$.
3. $i_{\alpha_1\wedge\cdots\wedge\alpha_{s-k}}P$ *is decomposable for all* $\alpha_i \in V^*$, *for any fixed* $k \geq 2$.
4. $i(\Psi)P \wedge P = 0$ *for all* $\Psi \in \Lambda^{s-2}V^*$ *In index notation* $P_{[abc\cdots d}P_{ef]g\cdots h} = 0$.
5. $i(i_P\Psi)P = 0$ *for all* $\Psi \in \Lambda^{s+2}V^*$.

Proof. (1) These are the well known classical Plücker relations. For completeness' sake we include a proof. Let $P \in \Lambda^n V$ and consider the induced linear mapping $\sharp_P : \Lambda^{s-1}V^* \to V$. Its image, W, is contained in each linear subspace U of V with $P \in \Lambda^s U$. Thus W is the minimal subspace with this property. P is decomposable if and only if $\dim W = s$, and this is the case if and only if $w \wedge P = 0$ for each $w \in W$. But $i_\Phi P$ for $\phi \in \Lambda^{s-1}V^*$ is the typical element in W.

(2) This well known variant of the Plücker relations follows by duality (see [4]):

$$\begin{aligned}\langle P \wedge i(\Phi)P, \Psi\rangle = \langle i(\Phi)P, i_P\Psi\rangle = \langle P, \Phi \wedge i_P\Psi\rangle = \\ = (-1)^{(s-1)}\langle P, i_P\Psi \wedge \Phi\rangle = (-1)^{(s-1)}\langle i(i_P\Psi)P, \Phi\rangle.\end{aligned}$$

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(3) This is due to [6]. There it is proved using exterior algebra. Apparently, this result is included in formula (4), page 116 of [7].

(4) Another proof using representation theory will be given below. Here we prove it by induction on s. Let $s = 3$. Suppose that $i_\alpha P \wedge P = 0$ for all $\alpha \in V^*$. Then for all $\beta \in V^*$ we have $0 = i_\beta(i_\alpha P \wedge P) = i_{\alpha\wedge\beta}P \wedge P + i_\alpha P \wedge i_\beta P$. Interchange α and β in the last expression and add it to the original, then we get $0 = 2i_\alpha P \wedge i_\beta P$ and in turn $i_{\alpha\wedge\beta}P \wedge P = 0$ for all α and β, which are the original Plücker relations, so P is decomposable. Now the induction step. Suppose that $P \in \Lambda^s V$ and that $i_{\alpha_1\wedge\cdots\wedge\alpha_{s-2}}P \wedge P = 0$ for all $\alpha_i \in V^*$. Then we have

$$0 = i_{\alpha_1}(i_{\alpha_1\wedge\cdots\wedge\alpha_{s-2}}P \wedge P) = i_{\alpha_1\wedge\cdots\wedge\alpha_{s-2}}P \wedge i_{\alpha_1}P = i_{\alpha_2\wedge\cdots\wedge\alpha_{s-2}}(i_{\alpha_1}P) \wedge (i_{\alpha_1}P)$$

for all α_i, so that by induction we may conclude that $i_{\alpha_1}P$ is decomposable for all α_1, and then by (3) P is decomposable.

(5) Again this follows by duality. □

Let us note that the following result (Lemma 1 in [2]), a version of the 'three plane lemma' also implies (3):

Let $\{P_i : i \in I\}$ be a family of decomposable non-zero k-vectors in V such that each $P_i + P_j$ is again decomposable. Then

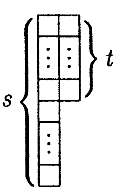
(a) either the linear span W of the linear subspaces $W(P_i) = \operatorname{Im}(\sharp_{P_i})$ is at most $(k+1)$-dimensional
(b) or the intersection $\bigcap_{i\in I} W(P_i)$ is at least $(k-1)$-dimensional.

Finally note that (1) and (4) are both invariant under $\mathrm{GL}(V)$. In the next section we shall decompose (1) into its irreducible components in this representation.

If $\dim V$ is high enough in comparison with s, then (4) seemingly comprises less equations.

2. Representation theory

In order efficiently to analyse (1) and (4) it is necessary to take a small excursion through representation theory. An extensive discussion of Young tableau may be found in [1]. Here we shall just need

$Y^{s,t} \equiv$ 

regarded as irreducible representations of $\mathrm{GL}(V)$. Then, as special cases of the Littlewood Richardson rules, we have

$$\begin{aligned}
\Lambda^s V \otimes \Lambda^s V &= Y^{s,s} \oplus Y^{s+1,s-1} \oplus Y^{s+2,s-2} \oplus Y^{s+3,s-3} \oplus \cdots \oplus Y^{2s,0}\\
\Lambda^{s+1} \otimes \Lambda^{s-1} V &= Y^{s+1,s-1} \oplus Y^{s+2,s-2} \oplus Y^{s+3,s-3} \oplus \cdots \oplus Y^{2s,0}\\
\Lambda^{s+2} \otimes \Lambda^{s-2} V &= Y^{s+2,s-2} \oplus Y^{s+3,s-3} \oplus \cdots \oplus Y^{2s,0}
\end{aligned}$$

and from the first two of these (1) says that $P \otimes P \in Y^{s,s}$. In fact,

$$(\star\star) \qquad \begin{array}{rclcccccc} \Lambda^s V \odot \Lambda^s V & = & Y^{s,s} & \oplus & & Y^{s+2,s-2} & \oplus & \cdots \\ \Lambda^s V \wedge \Lambda^s V & = & & Y^{s+1,s-1} & \oplus & Y^{s+3,s-3} & \oplus & \cdots \end{array}$$

so we can also see the equivalence of (1) and (4) without any calculation. Having decomposed $\Lambda^s V \odot \Lambda^s V$ into irreducibles, it behoves one to investigate the consequences of having each irreducible component of $P \otimes P$ vanish separately. The first of these gives us another improvement on the classical Plücker relations:

Theorem 2. *An s-form P is simple if and only if the component of $P \otimes P$ in $Y^{s+2,s-2}$ vanishes.*

Proof. The representation $Y^{s+2,s-2}$ may be realised as those tensors

$$T_{a_1 b_1 a_2 b_2 \ldots a_{s-2} b_{s-2} cdef}$$

which are symmetric in the pairs $a_j b_j$ for $j = 1, 2, \ldots s-2$, skew in $cdef$, and have the property that symmetrising over any three indices gives zero. The corresponding Young projection of

$$P_{a_1 a_2 \ldots a_{s-2} cd} P_{b_1 b_2 \ldots b_{s-2} ef}$$

is obtained by skewing over $cdef$ and symmetrising over each of the pairs $a_j b_j$ for $j = 1, 2, \ldots, s-2$. Its vanishing, therefore, is equivalent to the vanishing of

$$Q_{[cd} Q_{ef]} \quad \text{where } Q_{cd} = \alpha^{a_1} \beta^{a_2} \cdots \gamma^{a_{s-2}} P_{a_1 a_2 \ldots a_{s-2} cd}$$

for all $\alpha^a, \beta^a, \ldots, \gamma^a \in V^*$. According to (4), this means that Q_{cd} is simple. Therefore, the theorem is equivalent to criterion (3) of Theorem 1. □

Notice that this generally cuts down further the number of equations needed to characterise the simple s-vectors. The simplest instance of this is for 4-forms: P is simple if and only if

$$P_{[abcd} P_{ef]gh} = P_{[abcd} P_{efgh]}.$$

Written in this way, it is slightly surprising that one can deduce the vanishing of each side of this equation separately. Theorem 2 is optimal in the sense that the vanishing of any other component or components in the irreducible decomposition $(\star\star)$ of $P \otimes P$ is either insufficient to force simplicity or causes P to vanish. In the case of four-forms, for example,

$$P_{[abcd} P_{efgh]} = 0$$

if $P = v \wedge Q$ for some vector v and three-form Q. On the other hand, if the $Y^{4,4}$ component of $P \otimes P$ vanishes, then arguing as in the proof of Theorem 2 shows that $P = 0$.

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