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LAGRANGE FUNCTIONS GENERATING POISSON MANIFOLDS OF GEODESIC ARCS

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ABSTRACT. Necessary and sufficient conditions are found under which a given Lagrange function generates a Poisson manifold of geodesic arcs. These conditions are framed in terms of tangent Frobenius algebras.

1. BASIC NOTIONS

In this paper notions of geodesics, Lagrangian mechanics, linear connections, Poisson manifolds, Frobenius algebras and homogeneous functions are used in the usual sense (see, e.g. [1], [2], [4], [5] and [6]). In all local expressions we use the standard summation convention.

Let us consider the closed interval $[0,1] \subset \mathbb{R}$, a smooth finite-dimensional manifold X, the tangent bundle TX, the canonical projection $\pi : TX \to X$, and a smooth symmetric linear connection Γ on TX. A geodesic $[0,1] \to X$ of the connection Γ is called a *geodesic arc*. Let $W_\Gamma(X)$ be the set of all geodesic arcs. It is well known that there exists a bijective mapping $\beta_\Gamma : W_\Gamma(X) \ni \gamma \to \dot{\gamma}(0) \in \operatorname{codom} \beta_\Gamma$, where $\dot{\gamma}$ is the prolongation of the geodesic arc γ on tangent bundle TX. The subset $\operatorname{codom} \beta_\Gamma \subset TX$ is open and contains the zero section. The set $W_\Gamma(X)$ equipped with a structure of smooth fibered manifold such that β_Γ is an isomorphism of smooth fibered manifolds is called a *manifold of geodesic arcs*.

Let M be the set of all polynomial mappings $[0,1] \to [0,1]$ of degree ≤ 1. Then it is known that for any $\mu \in M$ there exists the smooth mapping $R_\mu : W_\Gamma(X) \ni \gamma \to \gamma \circ \mu \in W_\Gamma(X)$. A *Poisson manifold of geodesic arcs* is a manifold of geodesic arcs $W_\Gamma(X)$ equipped with a Poisson structure such that all mappings R_μ, where $\mu \in M$, are endomorphisms of $W_\Gamma(X)$. General Poisson manifolds of geodesic arcs are the subject of the paper [3].

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A mapping whose codomain is $\mathbb{R}$ will be called a *function*. Let us consider a smooth regular Lagrange function L, where $\operatorname{dom} L \subset TX$ is an open submanifold equipped with the canonical symplectic structure. Any mapping $[0,1] \to X$ satisfying the corresponding Euler-Lagrange equations is called an *extremal arc* of the Lagrange function L. Let $W_L(X)$ be the set of all extremal arcs of L. The set $W_L(X)$ equipped with a symplectic structure such that the bijective mapping $\beta_L : W_L(X) \ni \gamma \to \dot{\gamma}(0) \in \operatorname{codom} \beta_L \subset \operatorname{dom} L$ is an isomorphism of symplectic manifolds is called a *symplectic manifold of extremal arcs*. We say that the Lagrange function L *generates* a Poisson manifold of geodesic arcs $W_\Gamma(X)$ if and only if

(1) $W_L(X) \subset W_\Gamma(X)$ is a symplectic submanifold,

(2) $W_L(X) \ni \gamma \to \gamma(0) \in X$ is a surjective mapping.

Let us remark that using local expressions (see [3]) we get the following two assertions: No Poisson manifold of geodesic arcs is symplectic, so $W_L(X) \neq W_\Gamma(X)$. If L is a Lagrange function satisfying (2), then there exists at most one Poisson manifold of geodesic arcs satisfying (1).

Let $Z \to X$ be a fibered manifold, f be a function such that $\operatorname{dom} f \subset Z$. We say that f is X-*projectable* if there exists a function $\tilde{f}$ such that $f = \tilde{f} \circ \tilde{\pi}$, where $\tilde{\pi} : Z \to X$ is the canonical projection.

2. Fibrations of algebras

Throughout this paper an *algebra* A is a finite-dimensional $\mathbb{R}$-module A together with a bilinear multiplication $A \times A \to A$ which makes A into an associative ring with the unity element. A *structure tensor* of A is the tensor of the type $(2,1)$ associated with this multiplication. An algebra A is called *commutative* if A is a commutative ring. Any algebra A is a left A-module. The dual $\mathbb{R}$-module A^* equipped with the multiplication $A \times A^* \ni (a, \alpha) \to (A \ni b \to \alpha(ba) \in \mathbb{R}) \in A^*$ is a left A-module as well. An algebra A is a *Frobenius algebra* if and only if the left A-modules A and A^* are isomorphic. An algebra A^* is a *dual Frobenius algebra of* A if and only if the following conditions hold: (i) A is a Frobenius algebra; (ii) there exists an isomorphism of algebras $A \to A^*$; (iii) the isomorphism of algebras $A \to A^*$ is an isomorphism of left A-modules. The unity element in the dual Frobenius algebra A^* will be denoted by $\langle \cdot \rangle : A \ni a \to \langle a \rangle \in \mathbb{R}$.

Let A be an algebra. Denote by exp the mapping that takes each point $a \in A$ to $y(1) \in A$, where $y : \mathbb{R} \to A$ is the solution of the differential equation $dy/d\tau = ay$ under the condition $y(0) = 1$. The mapping exp exists and the solution y is given by $y : \mathbb{R} \ni \tau \to \exp(\tau a) \in A$. Moreover, the mapping exp is a local diffeomorphism. This means that for any $a_0 \in A$ there is a neighborhood $U \ni a_0$ such that the mapping $U \ni a \to \exp a \in \exp U$ is a diffeomorphism. Therefore, we can locally define a smooth mapping $\ln : \exp U \to U$ by the formula $\ln \circ \exp|_U = \mathrm{id}_U$.

Let A be an commutative algebra. By the above for each pair $a, b \in A$ it follows that $\exp(a+b) = \exp(a)\exp(b)$ and so

$$\left. \frac{d \exp(a + b\tau)}{d\tau} \right|_{\tau=0} = b \exp(a)\,. \tag{3}$$

A vector bundle $Z \to X$ is called a *fibration of algebras* if the following conditions hold: (i) any fiber of Z is an algebra; (ii) the structure tensor field is smooth. If $Z \to X$ is a fibration of algebras, then the mapping $Z \ni z \to \exp(z) \in Z$ is a local diffeomorphism. Over the manifold X we shall consider partly a fibration of tangent algebras TX, partly a fibration of cotangent algebras T^*X.

If g_k^{ij} are components of a cotangent algebra structure tensor field, then the commutativity gives

$$g_k^{ij} = g_k^{ji}\,, \tag{4}$$

and the associativity gives

$$g_i^{ml} g_m^{jk} = g_i^{jm} g_m^{kl}\,. \tag{5}$$

There exists a differential invariant of a structure tensor field. This invariant is a tensor field of the type $(3,2)$. Its components are

$$\begin{aligned} J_{jk}^{ilm} = {} & g_s^{il}\frac{\partial g_j^{sm}}{\partial x^k} + g_s^{im}\frac{\partial g_j^{sl}}{\partial x^k} + g_k^{si}\frac{\partial g_s^{lm}}{\partial x^j} + g_j^{si}\frac{\partial g_k^{lm}}{\partial x^s} + g_j^{sl}\frac{\partial g_k^{im}}{\partial x^s} + g_j^{sm}\frac{\partial g_k^{il}}{\partial x^s} \\ & - g_s^{il}\frac{\partial g_k^{sm}}{\partial x^j} - g_s^{im}\frac{\partial g_k^{sl}}{\partial x^j} - g_j^{si}\frac{\partial g_s^{lm}}{\partial x^k} - g_k^{si}\frac{\partial g_j^{lm}}{\partial x^s} - g_k^{sl}\frac{\partial g_j^{im}}{\partial x^s} - g_k^{sm}\frac{\partial g_j^{il}}{\partial x^s}\,. \end{aligned} \tag{6}$$

It is easy to prove that (4), (5), (6) imply $J_{jk}^{ilm} = J_{jk}^{lim} = J_{jk}^{iml} = -J_{kj}^{ilm}$.

3. Local expressions

On TX we shall use standard local fiber coordinates x^i, v^i. The canonical symplectic structure on $\operatorname{codom}\beta_L \subset \operatorname{dom} L$ is defined by the relations

$$\left\{x^i, x^j\right\} = 0, \quad \left\{x^i, \frac{\partial L}{\partial v^j}\right\} = \delta_j^i, \quad \left\{\frac{\partial L}{\partial v^i}, \frac{\partial L}{\partial v^j}\right\} = 0\,. \tag{7}$$

The Hamilton function is defined by the relation

$$H = v^i\,\frac{\partial L}{\partial v^i} - L\,. \tag{8}$$

Lemma 1. *A given smooth Lagrange function L, where* $\operatorname{dom} L \subset TX$, $\pi(\operatorname{codom}\beta_L) = X$, *generates a Poisson manifold of geodesic arcs if and only if on a neighborhood of every point $v_0 \in \operatorname{codom}\beta_L$ there exist X-projectable functions $g_l^{ik} = g_l^{ki}$, $\Gamma_{kl}^j = \Gamma_{lk}^j$ such that*

$$g_l^{ik}\,v^l\,\frac{\partial^2 L}{\partial v^k\,\partial v^j} = \delta_j^i\,, \tag{9}$$

$$\frac{\partial^2 L}{\partial v^i\,\partial v^j}\,\Gamma_{kl}^j\,v^k\,v^l - \frac{\partial^2 L}{\partial v^i\,\partial x^j}\,v^j + \frac{\partial L}{\partial x^i} = 0\,. \tag{10}$$

Proof. Let us suppose that L generates a Poisson manifold of geodesic arcs $W_\Gamma(X)$. Then from (1) we get $\operatorname{codom}\beta_L \subset \operatorname{codom}\beta_\Gamma$. On $\operatorname{codom}\beta_L$, relations (7) imply

$$\{x^i, x^j\} = 0 . \tag{11}$$

Since x^i are X-projectable functions, (11) can be extended to $\operatorname{codom}\beta_\Gamma$. Let us consider a Poisson structure on $\operatorname{codom}\beta_\Gamma$ such that β_Γ is an isomorphism of Poisson manifolds. If $k \in [0,1]$, $v \in \operatorname{codom}\beta_\Gamma$, $\kappa : [0,1] \ni \tau \to k\tau \in [0,1]$, then

$$\beta_\Gamma \circ R_\kappa \circ \beta_\Gamma^{-1}(v) = kv.$$

Let x^i, v^k be standard fiber coordinates on a neighborhood of a point $v_0 \in \operatorname{codom}\beta_\Gamma$. Since $\beta_\Gamma \circ R_\kappa \circ \beta_\Gamma^{-1}$ is an endomorphism of the Poisson manifold $\operatorname{codom}\beta_\Gamma$, $\{x^i, v^k\}$ are homogeneous functions of degree 1 in v^i. Because $\operatorname{codom}\beta_\Gamma$ contains the zero section of TX, these homogeneous functions are polynomials (see, e.g. [5]). Then there exist X-projectable functions g_l^{ik} such that

$$\{x^i, v^j\} = g_k^{ij} v^k . \tag{12}$$

Relations (1), (7), (12) imply (9), relation (9) implies $g_k^{ij} = g_k^{ji}$. Denoting by $\Gamma^i_{jk} = \Gamma^i_{kj}$ components of the connection Γ, from (1) we get (10).

Conversely, let $g_l^{ik} = g_l^{ki}$, $\Gamma^j_{kl} = \Gamma^j_{lk}$ be X-projectable functions on a neighborhood of a point $v_0 \in \operatorname{codom}\beta_L$ such that (9) and (10) hold. Then from (7), (9), (10) we have (11), (12) and

$$\{v^i, v^j\} = (g_k^{im}\,\Gamma^j_{lm} - g_k^{jm}\,\Gamma^i_{lm})\, v^k\, v^l . \tag{13}$$

Consider the Hamilton vector field ξ_H generated by the Hamilton function (8) on $\operatorname{codom}\beta_L$. Its components are given, according to (7) and (10), by the relations

$$\{x^i, H\} = v^i, \quad \{v^i, H\} = -\Gamma^i_{jk} v^j v^k . \tag{14}$$

Since x^i, g_l^{ik}, Γ^j_{kl} are X-projectable functions, v^i are globally defined on any fiber, and $\pi(\operatorname{codom}\beta_L) = X$, we can extend the Poisson structure defined by the relations (11), (12), (13), the linear symmetric connection Γ defined by the components Γ^i_{jk}, and the Hamilton vector field ξ_H defined by the relations (14) from the symplectic manifold $\operatorname{codom}\beta_L$ to the whole manifold TX. The set $W_\Gamma(X)$ of all geodesic arcs of the connection Γ can be equipped with a Poisson structure such that β_Γ is an isomorphism of Poisson manifolds. Hence, for all $\mu \in M$ we have

$$R_\mu = \beta_\Gamma^{-1} \circ \theta_{\mu(1)-\mu(0)} \circ \exp(\mu(0)\xi_H) \circ \beta_\Gamma,$$

where $\theta_{\mu(1)-\mu(0)}$ is the mapping $TX \ni v \to (\mu(1)-\mu(0))v \in TX$ and $\exp(\mu(0)\xi_H)$ is the flow of the vector field $\mu(0)\xi_H$. Since $R_\mu : W_\Gamma(X) \to W_\Gamma(X)$ is the composition of four homomorphisms, it is an endomorphism of the Poisson manifold and so $W_\Gamma(X)$ is the Poisson manifold of geodesic arcs. From (9), (10) we get (1) and from $\pi(\operatorname{codom}\beta_L) = X$ we get (2). Thus, the Lagrange function L generates the Poisson manifold of geodesic arcs $W_\Gamma(X)$. This completes the proof.

Lemma 2. *Let us suppose that a Lagrange function L, $\operatorname{dom} L \subset TX$, $\pi(\operatorname{codom}\beta_L) = X$, satisfies condition (9) of Lemma 1. Then condition (10) of Lemma 1 is satisfied if and only if on a neighborhood of every point $v_0 \in \operatorname{codom}\beta_L$ there exist X-projectable functions g_i such that*

$$J^{ilm}_{jk} = 0\,, \tag{15}$$

$$H = g_i\, v^i - \text{const}, \tag{16}$$

$$g^{ik}_j g_k = \delta^i_j\,. \tag{17}$$

Proof. Let (9), (10) hold. According to (9), $\partial^2 L/\partial v^k\,\partial v^j$ are homogeneous functions of degree -1 in v^i. Hence,

$$v^i\,\frac{\partial^3 L}{\partial v^i\,\partial v^k\,\partial v^j} + \frac{\partial^2 L}{\partial v^k\,\partial v^j} = 0\,. \tag{18}$$

From (8), (18) we obtain $\partial^2 H/\partial v^k\,\partial v^j = 0$. Therefore the Hamilton function is a polynomial of degree less than 2 in v^i. Differentiating (10) with respect to v^m we see that $\partial^2 L/\partial x^i\,\partial v^m - \partial^2 L/\partial x^m\,\partial v^i$ are homogeneous function of degree 0 in v^i. Thus, according to (8), (10), $\partial H/\partial x^i$ must be a homogeneous function of degree 1 in v^i. We have proved (16). From (9), (10) we get

$$g^{ij}_l\, v^l \left(\frac{\partial^2 L}{\partial v^j\,\partial x^k}\, v^k - \frac{\partial L}{\partial x^j}\right) = \Gamma^i\,, \tag{19}$$

where

$$\Gamma^i = \Gamma^i_{jk}\, v^j\, v^k\,. \tag{20}$$

Differentiating (19) with respect to v^j, v^k, v^l, according to (6), (9) we obtain

$$g^{im}_n v^n \frac{\partial^2 L}{\partial v^j\,\partial v^p}\,\frac{\partial^2 L}{\partial v^k\,\partial v^q}\,\frac{\partial^2 L}{\partial v^l\,\partial v^r}\, J^{pqr}_{ms}\, v^s = \frac{\partial^3 \Gamma^i}{\partial v^j\,\partial v^k\,\partial v^l}\,. \tag{21}$$

Differentiating (8) with respect to v^j, according to (16) we obtain

$$\frac{\partial^2 L}{\partial v^j\,\partial v^i}\, v^i = g_j\,. \tag{22}$$

Combining (9), (20), (21), (22) we obtain (15), (17).

Conversely, suppose that (9), (15), (16), (17) hold and Γ^i is given by (19). Relations (8), (16), (19) imply

$$v^j \frac{\partial \Gamma^i}{\partial v^j} - 2\Gamma^i = g^{ij}_k \left(\frac{\partial g_j}{\partial x^l} - \frac{\partial g_l}{\partial x^j}\right) v^k\, v^l\,. \tag{23}$$

From (6), (17) we get

$$\frac{\partial g_j}{\partial x^k} - \frac{\partial g_k}{\partial x^j} = \frac{1}{2} J^{ilm}_{jk}\, g_i g_l g_m\,. \tag{24}$$

According to (15), (23), (24), Γ^i are homogeneous functions of degree 2 in v^i. Further, according to (15), (21), Γ^i are polynomials in v^i. Hence, there exist X-projectable functions $\Gamma^i_{jk} = \Gamma^i_{kj}$ such that (20) holds. Finally from (9), (19), (20) we get (10). This completes the proof.

4. Lagrange functions

Theorem. *A given smooth Lagrange function* L*, where* $\operatorname{dom} L \subset TX$*,* $\operatorname{codom} L = \mathbb{R}$*, generates a Poisson manifold of geodesic arcs if and only if the three following conditions hold:*

1. *there exists a fibration of tangent commutative Frobenius algebras* TX *such that for every* $v \in \operatorname{codom} \beta_L$
$$L(v) = \langle\, v\,(\ln v - 1)\,\rangle + \text{const}\,,$$
2. *there exists a fibration of dual Frobenius algebras* T^*X *such that the differential invariant* (6) *is zero,*
3. $\pi(\operatorname{codom} \beta_L) = X$.

Proof. Let us suppose that L generates a Poisson manifold of geodesic arcs $W_\Gamma(X)$. Then from (2) we get $\pi(\operatorname{codom} \beta_L) = X$. Hence, (4), (9), (10) follow from Lemma 1 and (15), (16), (17) follow from Lemma 2. The functions g_l^{ik} are components of a tensor field. Its type is $(2, 1)$. Since $\pi(\operatorname{codom} \beta_L) = X$, this field is defined on the whole manifold X. Consider the associated bilinear multiplication $T_x^*X \times T_x^*X \to T_x^*X$ for any $x \in X$. According to (4), this multiplication is commutative. Differentiating (9) with respect to v^m, and multiplying by $(g_r^{pm} g_s^{qj} - g_r^{qm} g_s^{pj})\, v^r\, v^s$, we obtain (5). Therefore, the multiplication $T_x^*X \times T_x^*X \to T_x^*X$ is associative. Since by (17) there exists the unity element, the cotangent space T_x^*X is a commutative algebra. Put $g_{ij} = \partial \exp_i \circ \lambda(v)/\partial v^j$, where $\lambda : T_xX \cap \operatorname{codom} \beta_L \to T_x^*X$ is the Legendre transformation $\lambda_i(v) = \partial L(v)/\partial v^i$. Since (3) implies

$$\frac{\partial \exp_i(p)}{\partial p_j} = g_i^{jk} \exp_k(p)\,, \tag{25}$$

we obtain

$$g_{ij} = \frac{\partial^2 L(v)}{\partial v^j\, \partial v^k}\, g_i^{kl} \exp_l \circ \lambda(v). \tag{26}$$

According to (5), (9), (25), all second derivatives of the mapping $\exp \circ \lambda$ are zeros. Whence, g_{ij}'s are independent of v^j's. Multiplying (26) by v^i, according to (9) we obtain

$$\exp_j \circ \lambda(v) = g_{ij}\, v^i\,. \tag{27}$$

Therefore, there exists a linear mapping $\varphi : T_xX \to T_x^*X$ such that

$$\varphi|_{T_xX \cap \operatorname{codom} \beta_L} = \exp \circ \lambda\,. \tag{28}$$

Because λ, $\exp$ are local diffeomorphisms, φ is a linear isomorphism. Since (9), (26), (27) imply $g_{ij}\, g_l^{kj} = g_{lj}\, g_i^{kj}$, we have $(\varphi(a)\varphi(b))(c) = (\varphi(c)\varphi(a))(b)$ for all $a, b, c \in T_xX$. If we consider the structure of algebra on T_xX such that φ is an isomorphism of algebras, we get $\varphi(ab)(c) = \varphi(ca)(b)$. If $a = 1$, then $\varphi(b)(c) = \varphi(c)(b)$. Hence, $\varphi(ab)(c) = \varphi(b)(ca)$, and so $\varphi(ab) = a\varphi(b)$. Therefore, φ is an isomorphism of left T_xX-modules, T_xX is a Frobenius algebra, and T_x^*X is a dual Frobenius algebra of T_xX. Since $x \in X$ is arbitrary, we get a fibration of tangent commutative Frobenius algebras TX and a fibration of dual Frobenius algebras T^*X. From (15) it follows that the corresponding differential invariant (6) is zero. Suppose that $v \in \operatorname{codom} \beta_L$. Then $v \in T_xX \cap \operatorname{codom} \beta_L$, where $x = \pi(v)$. Since φ is an isomorphism of Frobenius algebras and isomorphism of left modules, from (28) we obtain $v^i \partial L(v)/\partial v^i = \lambda(v)(v) = \ln(\varphi(v))(v) = \varphi(\ln v)(v) = (\ln v\, \varphi(1))(v) = \varphi(1)(v \ln v) = \langle v \ln v \rangle$. Since (16), (17) imply $H(v) = \langle v \rangle -$ const, from (8) we get condition 1 of the Theorem.

Conversely, suppose that the conditions 1–3 of the Theorem are satisfied. Denoting by g_k^{ij} and g_i components of the cotangent algebra structure tensor field and the cotangent algebra unity element field, we have (15) and (17). From (8) and condition 1 of the Theorem by a straightforward computation we get (9), (16). Thus, from Lemmas 1 and 2 it follows that the Lagrange function L generates a Poisson manifold of geodesic arcs. This completes the proof.

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