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NATURAL OPERATORS ON FRAME BUNDLES

MICHAL KRUPKA

ABSTRACT. A basis of zero-order differential operators of the rth order semi-holonomic frame bundles with values in any sth order natural bundle is found. A basis of rth order differential operators of the first order frame bundle with values in any sth order natural bundle is found. An example for $r = 1, 2$, $s = 1$ showing relations to the Lie bracket is given.

1. INTRODUCTION

There is the following problem in the theory of natural bundles and operators: Find to given natural bundle F_1 of order r_1 a natural bundle F_2 of order r_2 and a differential operator $D : F_1 \to F_2$ of order s, such that any other sth order differential operator $\bar{D} : F_1 \to \bar{F}$, where the order of $\bar{F}$ is r_2, can be *factored* through D, i.e., such that there is a zero-order operator $D_0 : F_2 \to \bar{F}$ for which the diagram

(1)
$$\begin{array}{ccc} F_1 & \xrightarrow{D} & F_2 \\ & \underset{\bar{D}}{\searrow} & \downarrow{\scriptstyle D_0} \\ & & \bar{F} \end{array}$$

commutes. The differential operator D is sometimes called *basis* of sth order operators of F_1 with values in bundles of order r_2.

In this paper, we solve this problem in two particular cases. First, for F_1 equal to the semi-holonomic frame bundle $\operatorname{semi} F^{r_1}$, $s = 0$, and $r_2 < r_1$. Second, using the canonical semi-holonomic prolongation of first-order frames into higher order semi-holonomic frames, for F_1 equal to the first order frame bundle F^1, s arbitrary, $r_2 \leq s$. We use the method of *orbit reduction* (Theorem 1), which is based on *factorization* of the type fiber of the prolongation $J^s F_1$ with respect to the canonical Lie group action.

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At the end of the paper, we give concrete results for the case of first and second order and show their relations to the Lie bracket.

The subject has been studied by D. Q. Chau and D. Krupka [2], other related papers are [8], [4], [9]. For information on the Natural bundles and operators theory the reader is referred to [3], [6]. The orbit reduction method was used first in [5]. The theory of semi-holonomic jets is introduced in [1] and reviewed in [7].

2. Conventions

For a multi-index $U = (u_1, \dots, u_t)$ we shall write

$$\chi^l_{J_U} = \chi^l_{j_{u_1} \dots j_{u_t}} . \tag{2}$$

Occasionally, the multi-index U can have the form:

$$U = (u_1, \dots, u_{k-1}, (u_k, u_{k+1}), u_{k+2}, \dots, u_t) .$$

In this case we write

$$\chi^l_{J_U} = \chi^l_{j_{u_1} \dots j_{u_{k-1}} [j_{u_k} j_{u_{k+1}}] j_{u_{k+2}} \dots j_{u_t}} , \tag{3}$$

where the brackets denote antisymmetrization:

$$\begin{aligned} &\chi^l_{j_{u_1} \dots j_{u_{k-1}} [j_{u_k} j_{u_{k+1}}] j_{u_{k+2}} \dots j_{u_t}} \\ &= \chi^l_{j_{u_1} \dots j_{u_{k-1}} j_{u_k} j_{u_{k+1}} j_{u_{k+2}} \dots j_{u_t}} - \chi^l_{j_{u_1} \dots j_{u_{k-1}} j_{u_{k+1}} j_{u_k} j_{u_{k+2}} \dots j_{u_t}} . \end{aligned} \tag{4}$$

If Δ is a decomposition of the set $\{1, 2, \dots, r\}$ then the symbol $|\Delta|$ denotes the number of sets in Δ. The sets $\Delta_1, \Delta_2, \dots, \Delta_{|\Delta|}$ from Δ are ordered by smallest elements. Thus for $t_1 < t_2$, the smallest element of Δ_{t_1} is less than the smallest element of Δ_{t_2}. Further, the symbol $|\Delta_t|$ denotes the number of elements in Δ_t. The elements of Δ_t are denoted by $\Delta_{t1}, \Delta_{t2}, \dots, \Delta_{t|\Delta_t|}$ where $\Delta_{t1} < \Delta_{t2} < \dots < \Delta_{t|\Delta_t|}$.

We shall also work with decompositions of the set $\{1, \dots, s-1, (s, s+1), s+2, \dots, r\}$. In this case we use the same conventions as in the previous case with $s-1 < (s, s+1) < s+2$.

The sets Δ_t will be used as multi-indices $(\Delta_{t1}, \Delta_{t2}, \dots, \Delta_{t|\Delta_t|})$.

3. Jets

For manifolds X_1, X_2, $\dim X_1 = n_1$, $\dim X_2 = n_2$, and points $x_1 \in X_1$, $x_2 \in X_2$ we fix the following notations:

$$J^r_{x_1, x_2}(X_1, X_2) = \left\{ J^r_{x_1} f \mid f : X_1 \to X_2, f(x_1) = x_2 \right\} , \tag{5}$$

$$J^r_{x_1}(X_1, X_2) = \bigcup_{x \in X_2} J^r_{x_1, x}(X_1, X_2) , \tag{6}$$

$$J^r(X_1, X_2) = \bigcup_{x \in X_1} J^r_x(X_1, X_2) , \tag{7}$$

$$T^r_{n_1} X_2 = J^r_0(R^{n_1}, X_2) . \tag{8}$$

These spaces admit canonical smooth resp. fiber bundle structure. Subsets of these spaces consisting only of jets of maximal rank will be marked by the prefix "reg".

Thus, the r-frame bundle $F^r X_2$ is equal to $\operatorname{reg} T^r_{n_2} X_2$, and the rth differential group L^r_n is equal to $\operatorname{reg} J^r_{0,0}(R^n, R^n)$. For $s \leq r$, we have the canonical projections

$$\pi^{r,s}_{X_1,X_2} \; : \; J^r(X_1, X_2) \to J^s(X_1, X_2)\,, \tag{9}$$

$$\pi^r_{X_1,X_2} \; : \; J^r(X_1, X_2) \to X_1 \times X_2\,. \tag{10}$$

The kernel of the projection $\pi^{r,s}_{R^n,R^n} : L^r_n \to L^s_n$ is denoted by $K^{r,s}_n$.

For a point $x \in X$ we set

$$L^r_x = \operatorname{reg} J^r_{x,x}(X, X)\,, \tag{11}$$

and get a Lie group L^r_x. For $s \leq r$, we denote the kernel of the projection $\pi^{r,s}_{X,X} : L^r_x \to L^s_x$, by $K^{r,s}_x$.

We shall use analogous notations for spaces of semi-holonomic jets with the prefix "semi" added.

On all of these spaces, we shall use induced coordinates as usual. For example, for charts (U, φ) on the manifold X_1 and (V, ψ) on X_2 the induced coordinate system on the semi-holonomic jet space semi $J^r(X_1, X_2)$ is denoted by $(\varphi^j, \psi^k, \psi^k_{j_1}, \psi^k_{j_1 j_2}, \dots, \psi^k_{j_1 j_2 \dots j_r})$ $(j, j_1, \dots, j_r = 1, \dots, n_1,\ k = 1, \dots, n_2)$.

For another manifold X_3 and a chart (W, χ), the coordinate expression of the composition of composable semi-holonomic jets $a_1 \in$ semi $J^r(U, V)$, and $a_2 \in$ semi $J^r(V, W)$ is

$$\begin{aligned} &\chi^i_{j_1 \dots j_s}(a_2 \circ a_1) \\ &\quad = \sum_{\Delta} \chi^i_{k_1 \dots k_{|\Delta|}}(a_2) \psi^{k_1}_{J_{\Delta_1}}(a_1) \cdot \ldots \cdot \psi^{k_{|\Delta|}}_{J_{\Delta_{|\Delta|}}}(a_1)\,, \end{aligned} \tag{12}$$

where $s \in \{1, \dots, r\}$ and Δ runs through all decompositions of $\{1, 2, \dots, s\}$.

4. Bases of differential operators

From the natural differential operators theory point of view, we shall work mostly in the category $\mathcal{D}_n$ of n-dimensional second countable Hausdorff manifolds and their embeddings, where n will be a fixed integer.

The following theorem shows that the problem of finding bases of differential operators formulated in Introduction, can be in some cases solved by computing orbit space of some Lie group action. See [5], [3].

Theorem 1. *Let F_1, F_2 be natural bundles of orders r_1, r_2, P_1, P_2 their type fibers. Let $D : F_1 \to F_2$ be an sth order differential operator, $g : T^s_n P_1 \to P_2$ its fiber representation. Suppose that g is a surjective submersion and quotient projection with respect to the action of K^{r_1+s,r_2}_n on $T^s_n P_1$. Then for any other differential operator $\bar{D} : F_1 \to \bar{F}$ with order of $\bar{F}$ equal to r_2, there is a unique zero-order operator $D_0 : F_2 \to \bar{F}$ such that the diagram (1) commutes.*

5. Note on factorization of tensor spaces

For an integer $r > 1$ and vector space V, consider an affine space H, with the associated vector space $\bigotimes^r V^*$. Let us fix a basis in V and choose compatible coordinates in H. For $s = 1, \dots, r-1$ we have the mappings

$$\begin{array}{lrcl} (13) & A_{s,s+1} : H & \to & \bigotimes^{r} V^{*} , \\ & u_{j_1 \dots j_r} & \to & u_{j_1 \dots j_{s-1}[j_s j_{s+1}] j_{s+2} \dots j_r} . \end{array}$$

Denote by pr the quotient projection $pr : H \to H / \bigodot^r V^*$ ($\odot$ means the symmetric tensor product) and set $A = (A_{1,2}, \dots, A_{r-1,r})$. We have $A : H \to (\bigotimes^r V^*)^{r-1}$. We have the following simple lemma:

Lemma 2. *There is a unique injection*

$$(14) \qquad \iota : H / \bigodot^r V^* \to (\bigotimes^r V^*)^{r-1} ,$$

such that the diagram

$$(15) \qquad \begin{array}{ccc} H & \xrightarrow{pr} & H / \bigodot^r V^* \\ & {}_{A}\searrow & \downarrow \iota \\ & & (\bigotimes^{r} V^*)^{r-1} \end{array}$$

commutes.

6. Factorization of semi-holonomic jets

Let X_1, and X_2 be two manifolds with $\dim X_1 = n_1$, $\dim X_2 = n_2$. It is a well-known fact [7], that the bundle $\operatorname{semi} J^r(X_1, X_2) \to \operatorname{semi} J^{r-1}(X_1, X_2)$ is an affine bundle with associated vector bundle $(\pi^{r,r-1}_{X_1,X_2})^*(TX_2 \otimes \bigotimes^r T^* X_1)$.

The quotient space $\operatorname{semi} J^r(X_1, X_2) / (\pi^{r,r-1}_{X_1,X_2})^*(TX_2 \otimes \bigodot^r T^* X_1)$ is also an affine bundle over $\operatorname{semi} J^{r-1}(X_1, X_2)$, with associated vector bundle

$$(\pi^{r,r-1}_{X_1,X_2})^* \, (TX_2 \otimes (\bigotimes^r T^* X_1 / \bigodot^r T^* X_1)) .$$

Let

$$(16) \qquad \begin{array}{l} \tau^r_{X_1,X_2} : \operatorname{semi} J^r_{x_1,x_2}(X_1, X_2) \\ \quad \to \ \operatorname{semi} J^r(X_1, X_2) / (\pi^{r,r-1}_{X_1,X_2})^*(TX_2 \otimes \bigodot^{r} T^* X_1) \end{array}$$

be the quotient projection.

Fix two points $x_1 \in X_1$, $x_2 \in X_2$ and consider the space $\operatorname{reg} \operatorname{semi} J^r_{x_1,x_2}(X_1, X_2)$. For elements $g_1 \in K^{r,r-1}_{x_1}$, $g_2 \in K^{r,r-1}_{x_2}$, and $p \in \operatorname{reg} \operatorname{semi} J^r_{x_1,x_2}(X_1, X_2)$ we set

$$(17) \qquad g_1 \cdot p = p \circ g_1^{-1} ,$$

$$(18) \qquad g_2 \cdot p = g_2 \circ p ,$$

$$(19) \qquad (g_1, g_2) \cdot p = g_2 \circ p \circ g_1^{-1} ,$$

and get left actions of the groups $K^{r,r-1}_{x_1}$,$K^{r,r-1}_{x_2}$, and $K^{r,r-1}_{x_1} \times K^{r,r-1}_{x_2}$ on the manifold $\operatorname{reg} \operatorname{semi} J^r_{x_1,x_2}(X_1, X_2)$. For these actions, we have the following result:

Theorem 3. *The equivalence of the action (19), and the equivalence of the projection $\tau^r_{X_1,X_2}$, coincide on* $\operatorname{reg} \operatorname{semi} J^r_{x_1,x_2}(X_1, X_2)$. *If $n_1 \geq n_2$, then the same holds for the action (17), if $n_1 \leq n_2$, then the same holds for the action (18).*

Proof. The group $K^{r,r-1}_{x_1}$ can be identified with $T_{x_1}X_1 \otimes \bigodot^r T^*_{x_1}X_1$ and the manifold $J^1_{x_1,x_2}(X_1,X_2)$ with $T_{x_2}X_2 \otimes T^*_{x_1}X_1$. Using (12) we easily obtain that elements $g_1 \in K^{r,r-1}_{x_1}$ act on semi $J^r_{x_1,x_2}(X_1,X_2)$ as translations $g_1 \cdot p = p - \pi^{r,1}_{X_1,X_2}(p) \cdot g_1$, where $\pi^{r,1}_{X_1,X_2}(p) \cdot g_1 = \operatorname{trace}(\pi^{r,1}_{X_1,X_2}(p) \otimes g_1)$.

Similarly, the action of $K^{r,r-1}_{x_2}$ on semi $J^r_{x_1,x_2}(X_1,X_2)$ can be viewed as the translation $g_2 \cdot p = p + g_2 \cdot \pi^{r,1}_{X_1,X_2}(p)$, where $g_2 \cdot \pi^{r,1}_{X_1,X_2}(p)$ is the image of $(g_2, \pi^{r,1}_{X_1,X_2}(p))$ with the canonical mapping

$$(20) \qquad \begin{aligned}&(T_{x_2}X_2 \otimes \overset{r}{\bigodot} T^*_{x_2}X_2) \times (T_{x_2}X_2 \otimes T^*_{x_1}X_1)\\ &\quad \to (T_{x_2}X_2 \otimes \overset{r}{\bigodot} T^*_{x_1}X_1)\end{aligned}$$

(r tensor products and r traces).

This shows that the tensors $p \circ g_1^{-1} - p$, $g_2 \circ p - p$, and $g_2 \circ p \circ g_1^{-1} - p$ are always symmetric (i.e., elements of $TX_2 \otimes \bigodot^r T^*X_1$), which means that the orbits of the actions (17,18,19) are subsets of the equivalence classes of $\tau^r_{X_1,X_2}$.

From the other hand, for any $p, \bar{p} \in \operatorname{reg\,semi} J^r_{x_1,x_2}(X_1,X_2)$ such that $\tau^r_{X_1,X_2}(p) = \tau^r_{X_1,X_2}(\bar{p})$, the tensor $\bar{p} - p$ is symmetric. If $n_1 \geq n_2$, then there is a right inverse $b \in J^1_{x_2,x_1}(X_2,X_1)$ of the jet $\pi^{r,1}_{X_1,X_2}(\bar{p}) = \pi^{r,1}_{X_1,X_2}(p)$. Now, the equation $\bar{p} = p - \pi^{r,1}_{X_1,X_2}(p) \cdot g_1$ has a solution $g_1 = b \cdot (p - \bar{p})$. If $n_1 \leq n_2$, then the jet $\pi^{r,1}_{X_1,X_2}(\bar{p}) = \pi^{r,1}_{X_1,X_2}(p)$ has a left inverse $b \in J^1_{x_2,x_1}(X_2,X_1)$, and the equation $\bar{p} = p + g_2 \cdot \pi^{r,1}_{X_1,X_2}(p)$ has a solution $g_2 = (\bar{p} - p) \cdot b$. This proves the converse. □

7. Factorization of semi-holonomic frames

We shall apply the previous results to the natural bundle semi F^r. We have for any n-dimensional manifold X (i.e., object of the category $\mathcal{D}_n$) the semi-holonomic frame bundle $\pi^r_X : \operatorname{semi} F^r X \to X$. The bundle $\operatorname{semi} F^r X \to \operatorname{semi} F^{r-1} X$ has a structure of affine bundle with associated vector space $(\pi^{r-1}_X)^*(TX \otimes \bigotimes^r R^{n*})$. We have the projection $\tau^r_X : \operatorname{semi} F^r X \to \operatorname{semi} F^r X / (\pi^{r-1}_X)^*(TX \otimes \bigodot^r R^{n*})$.

The system $\tau^r = (\tau^r_X)$ is a natural transformation of the functors semi F^r and $\operatorname{semi} F^r/(\pi^{r-1})^*(T \otimes \bigodot^r R^*)$; the latter functor is of the order $r-1$. In the following theorem we use the canonical identification of τ^r and some zero-order differential operator.

Theorem 4. *Any zero-order differential operator of* semi F^r *with values in a bundle of order $r-1$ can be factored through τ^r.*

Proof. The type fiber of semi F^r is equal to semi $F^r_0 R^n$ (action of L^r_n is given by jet composition). According to Theorem 3, the projection τ_{R^n}, restricted to this type fiber, is the quotient projection of the action of $K^{r,r-1}_n \subset L^r_n$. Thus, the result follows from Theorem 1. □

Let us now consider the mapping A from Par. 5. We have $A_{s,s+1} : \operatorname{semi} F^r X \to (\pi^{r-1}_X)^*(TX \otimes \bigotimes^r R^{n*})$. From (12) there easily follow local transformation properties of the mapping $A_{s,s+1}$. Namely, for $x \in X$, $p \in \operatorname{semi} F^r_x X$, and $g \in L^r_x$ it holds

$$(21) \qquad \chi^i_{j_1 \dots j_r}(A_{s,s+1}(g \circ p)) = \sum_{\Delta} \chi^i_{l_1 \dots l_{|\Delta|}}(g) \chi^{l_1}_{J_{\Delta_1}}(p) \cdot \dots \cdot \chi^{l_{|\Delta|}}_{J_{\Delta_{|\Delta|}}}(p)\,,$$

where Δ runs through all decompositions of the set $\{1,\dots,s-1,\{s,s+1\},s+2,\dots,r\}$.

Hence we obtain equations of the left action of L^{r-1}_x on the image $A_{s,s+1}(\operatorname{semi} F^r_x X)$: For $t < r-1$,

$$
(22)\qquad
\begin{aligned}
\chi^i_{j_1\dots j_t}(g\cdot q) &= \sum_{\Delta} \chi^i_{k_1\dots k_{|\Delta|}}(g)\chi^{k_1}_{J_{\Delta_1}}(q)\cdot\dots\cdot\chi^{k_{|\Delta|}}_{J_{\Delta_{|\Delta|}}}(q)\\
\chi^i_{j_1\dots j_r}(g\cdot q) &= \sum_{\bar\Delta} \chi^i_{l_1\dots l_{|\bar\Delta|}}(g)\chi^{l_1}_{J_{\bar\Delta_1}}(q)\cdot\dots\cdot\chi^{l_{|\bar\Delta|}}_{J_{\bar\Delta_{|\bar\Delta|}}}(q)\,.
\end{aligned}
$$

(Δ runs through all decompositions of the set $\{1,\dots,t\}$ and $\bar\Delta$ through all decompositions of the set $\{1,\dots,s-1,\{s,s+1\},s+2,\dots,r\}$).

From these equations, one can deduce that this action can be extended from the group L^{r-1}_x to $\operatorname{semi} L^{r-1}_x$. Equations of the new action will be the same, just forgetting the condition of symmetry of $\chi^i_{j_1\dots j_t}(g)$ $(t=1,\dots,r-1)$ in subscripts. From these considerations it follows that elements from the image $q\in A_{s,s+1}(\operatorname{semi} F^r_x X)$ can be multiplied from the left by any regular semi-holonomic $(r-1)$-jet $a\in \operatorname{reg}\operatorname{semi} J^{r-1}_{x,\bar x}(X,\bar X)$, where $\dim \bar X = n$, with $a\cdot q\in A_{s,s+1}(\operatorname{semi} F^r_{\bar x}\bar X)$. This multiplication is associative.

Let us consider for $p\in \operatorname{semi} F^r_x X$ the point $\pi_X(p^{-1})\cdot A_{s,s+1}(p)$. Evidently, this point belongs to the fiber of $(\pi^{r-1}_{R^n})^*(R^n\otimes\bigotimes^r R^{n*})\to \operatorname{semi} F^{r-1}_0 R^n$ over the point $J^{r-1}_0 \operatorname{id}_{R^n}$. This fiber is canonically isomorphic to $R^n\otimes\bigotimes^r R^{n*}$, so we shall write simply $\pi^{r,r-1}_X(p^{-1})\cdot A_{s,s+1}(p)\in R^n\otimes\bigotimes^r R^{n*}$.

Finally, we have $\pi^{r,1}_X(p)\in T_xX\otimes R^{n*}$ and we set

$$
(23)\qquad \bar\tau^{r,s}_X(p) = \operatorname{trace}(\pi^{r,1}_X(p)\otimes(\pi^{r,r-1}_X(p^{-1})\cdot A_{s,s+1}(p)))\,.
$$

We set $\bar\tau^r_X : \operatorname{semi} F^r X\to \operatorname{Im}\bar\tau^r_X$, $\bar\tau^r_X(p) = (\bar\tau^{r,1}_X(p),\dots,\bar\tau^{r,r-1}_X(p))$. The image of the mapping $\bar\tau^r_X$ is an affine sub-bundle of the bundle $TX\otimes(\bigotimes^r R^{n*})^r$.

Lemma 5. *There is a bijection $\iota : \operatorname{Im}\tau^r_X \to \operatorname{semi} F^{r-1}X\times \operatorname{Im}\bar\tau^r_X$ such that the diagram*

$$
(24)\qquad
\begin{array}{ccc}
 & \operatorname{semi} F^r X & \\
 \swarrow^{\tau^r_X} & & \searrow^{(\pi^{r,r-1}_X,\,\bar\tau^r_X)} \\
\operatorname{Im}\tau^r_X & \xrightarrow{\ \iota\ } & \operatorname{semi} F^{r-1}X\times \operatorname{Im}\bar\tau^r_X
\end{array}
$$

commutes.

Proof. This follows from Lemma 2 and invertibility of the assignment $A_{s,s+1}(p)\to(\pi^{r,r-1}_X(p),\bar\tau^{r,s}_X(p))$ (23). □

We have natural transformations $\bar\tau^r = (\bar\tau^r_X)$, $\pi^{r,t} = (\pi^{r,t}_X)$, whose codomains are first and tth order bundles, respectively.

Theorem 6. *Let $0 < t < r$. Any zero-order differential operator of* semi F^r *with values in a tth-order bundle can be factored through $(\pi^{r,t},\bar\tau^{t+1}\circ\pi^{r,t+1},\dots,\bar\tau^{r-1}\circ\pi^{r,r-1},\bar\tau^r)$.*

Proof. For $t = r-1$ the result follows from Theorem 4 and Lemma 5. The general result is easily obtained by induction. □

8. Natural operators on frame bundles

Let $s : X \to F^1X$ be a section of a first-order frame bundle. The well-known semi-holonomic prolongations of this section can be defined using semi-holonomic jet composition:

$$s^{(0)} = s, \tag{25}$$

$$s^{(r)}(x) = (J^r_x s) \circ (s^{(r-1)}(x)). \tag{26}$$

In the second equation, if $s^{(r-1)}(x)$ belongs to $\mathrm{semi}\, F^r X$, then

$$s^{(r)}(x) \in \mathrm{semi}\, J^r_0(R^n, F^1X).$$

Using translations in R^n, we can identify this element with an element of $\mathrm{semi}\, F^{r+1}X$. Thus, $s^{(r)}$ is a section of the bundle $\mathrm{semi}\, F^{r+1}X \to X$. We get an injection $\iota^r : J^r F^1 X \to \mathrm{semi}\, F^{r+1}X$. For any $p \in \mathrm{Im}(\iota^r)$ we can consider p as an element of the manifold $\mathrm{semi}\, J^r_0(R^n, F^1X)$, and get $(\iota^r)^{-1}(p) = p \circ \pi^{r+1,r}_X(p^{-1})$.

Since the image $\iota^r(J^rF^1X)$ is a subspace of $\mathrm{semi}\, F^{r+1}X$, we can compute rth-order operators of F^1X by means of zero-order operators of $\mathrm{semi}\, F^{r+1}X$ (in Theorem 1, quotient projection of an invariant subspace is equal to the restriction of projection of whole space to this subspace). Thus, we can use Theorem 6:

Theorem 7. *Let $0 < t \leq r$. Any rth-order differential operator of F^1 with values in a tth-order bundle can be factored through the operator $s \to (\pi^{r,t}, \bar{\tau}^{t+1} \circ \pi^{r,t+1}, \ldots, \bar{\tau}^{r-1} \circ \pi^{r,r-1}, \bar{\tau}^r) \circ \iota^r \circ J^r s$, defined for any section $s : X \to F^1X$.*

9. Examples

Let p be an element of $\mathrm{semi}\, F^2X$. Then, in local coordinates,

$$\chi^i_j(\pi^{2,1}_X(p)) = \chi^i_j(p), \tag{27}$$

$$\chi^i_{j_1 j_2}(\bar{\tau}^{2,1}_X(p)) = \chi^i_{[j_1 j_2]}(p), \tag{28}$$

with

$$(\pi^{2,1}_X, \bar{\tau}^{2,1}_X) : \mathrm{semi}\, F^2X \to (TX \otimes R^{n*}) \times (TX \otimes R^{n*} \wedge R^{n*}) \tag{29}$$

being surjective. The inclusion $\iota^1 : J^1F^1X \to \mathrm{semi}\, F^2X$ is given by

$$\chi^i_j(\iota^1(q)) = \chi^i_j(q), \tag{30}$$

$$\chi^i_{j_1 j_2}(\iota^1(q)) = \chi^i_{j_1,l}(q)\chi^l_{j_2}(q), \tag{31}$$

and is also surjective. Thus, for the composition $\bar{\tau}^{2,1}_X \circ \iota^1$ we get

$$\chi^i_{j_1 j_2}(\bar{\tau}^{2,1}_X \circ \iota^1(q)) = \chi^i_{j_1,l}(q)\chi^l_{j_2}(q) - \chi^i_{j_2,l}(q)\chi^l_{j_1}(q). \tag{32}$$

Consider a section $s : X \to F^1X$ and denote by s_j its jth vector field. From the above equations we obtain

$$(\pi^{2,1}_X, \bar{\tau}^{2,1}_X) \circ \iota^1 \circ J^1 s = (s, [s_{j_1}, s_{j_2}]), \tag{33}$$

where we have the system of all Lie brackets of vector fields s_j, on the right-hand side. Thus, we have a result: *For $r = 1$, the operator from Theorem 7 assigns to each frame field $s : X \to F^1X$ the vector fields s_j, $[s_{j_1}, s_{j_2}]$.*

Let us turn to the case $r = 2$. For $p \in \operatorname{semi} F^3X$ we have

$$
(34)\qquad
\begin{aligned}
\chi^i_j(\pi^{3,1}_X(p)) &= \chi^i_j(p),\\
\chi^i_{j_1j_2}(\bar\tau^{2,1}_X \circ \pi^{3,2}_X(p)) &= \chi^i_{[j_1j_2]}(p),\\
\chi^i_{j_1j_2j_3}(\bar\tau^{3,1}_X(p)) &= \chi^i_l(p)(\chi^l_u(p^{-1})\chi^u_{[j_1j_2]j_3}(p)\\
&\quad + \chi^l_{u_1u_2}(p^{-1})\chi^{u_1}_{[j_1j_2]}(p)\chi^{u_2}_{j_3}(p)),\\
\chi^i_{j_1j_2j_3}(\bar\tau^{3,2}_X(p)) &= \chi^i_l(p)(\chi^l_u(p^{-1})\chi^u_{j_1[j_2j_3]}(p)\\
&\quad + \chi^l_{u_1u_2}(p^{-1})\chi^{u_1}_{j_1}(p)\chi^{u_2}_{[j_2j_3]}(p)).
\end{aligned}
$$

The mapping $(\pi^{3,1}_X, \bar\tau^2_X \circ \pi^{3,2}_X, \bar\tau^3_X)$ is not surjective. To determine the image, we shall prove the following lemma:

Lemma 8. *In Lemma 2 with $r = 3$, the image of the mapping A consists of all tensors (v, w), satisfying*

$$
(35)\qquad
\begin{aligned}
v_{jkl} + v_{kjl} &= 0,\\
w_{jkl} + w_{jlk} &= 0,\\
v_{jkl} + v_{klj} + v_{ljk} &= w_{jkl} + w_{klj} + w_{ljk}.
\end{aligned}
$$

Proof. Direct computation gives that for any $u \in H$, $v = A_{1,2}(u)$, $w = A_{2,3}(u)$, the equations (35) hold. From the other hand, given (v, w) such that the equations are true, we can set $u_{jkl} = \frac{1}{6}(5v_{jkl} + 3v_{klj} + v_{ljk} - 4c_{klj} - 2c_{ljk})$. □

Now we can see that the image of the mapping $(\pi^{3,1}_X, \bar\tau^2_X, \bar\tau^3_X)$ is an affine bundle with associated vector bundle $TX \otimes (R^{n*} \times (R^{n*} \wedge R^{n*}) \times W)$, where W is the subspace of $(\bigotimes^3 R^{n*})^2$, given by the equations (35).

Consider the inclusion $\iota^2 : J^2F^1X \to \operatorname{semi} F^3X$. We have

$$
(36)\qquad
\begin{aligned}
\chi^i_j(\iota^2(q)) &= \chi^i_j(q),\\
\chi^i_{j_1j_2}(\iota^2(q)) &= \chi^i_{j_1,l}(q)\chi^l_{j_2}(q),\\
\chi^i_{j_1j_2j_3}(\iota^2(q)) &= \chi^i_{j_1,l}(q)\chi^l_{j_2j_3}(q) + \chi^i_{j_1,l_1l_2}(q)\chi^{l_1}_{j_2}(q)\chi^{l_2}_{j_3}(q).
\end{aligned}
$$

This inclusion is not surjective; its image is determined by $\chi^i_{j_1,l_1l_2}(q) = \chi^i_{j_1,l_2l_1}(q)$. Direct computation gives that $\operatorname{Im}\iota^2$ is exactly the subset of $\operatorname{semi} F^3X$, consisting of elements p such that

$$
(37)\qquad \bar\tau^{3,2}_X(p) = 0.
$$

After some calculations, we get for the composition $\bar\tau^{3,1}_X \circ \iota^2$ and a section $s : X \to F^1X$,

$$
(38)\qquad \bar\tau^{3,1}_X \circ \iota^2 \circ J^2s = [[s_{j_1}, s_{j_2}], s_{j_3}],
$$

and the third equation of (35) together with (37) gives the Jacobi identity. We can formulate our result: *For $r = 2$, $t = 1$, the operator from Theorem 7 assigns to each frame field $s : X \to F^1X$ the vector fields s_j, $[s_{j_1}, s_{j_2}]$, $[[s_{j_1}, s_{j_2}], s_{j_3}]$.*

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